# Multiple Solutions for Singular Systems with Sign-Changing Weight, Nonlinear Singularities and Critical Exponent 

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This paper is an attempt to establish the existence and multiplicity results of nontrivial solutions to singular systems with signchanging weight, nonlinear singularities, and critical exponent. By using variational methods, the Nehari manifold, and under sufficient conditions on the parameter $\eta$ which represent some physical meanings, we prove some existing results by researching the critical points as the minimizers of the energy functional associated with the proposed problem (2) on the constraint defined by the Nehari manifold, which are solutions of our system, under some sufficient conditions on the parameters $\alpha, \beta, \mu$, and $\eta$. To the best of our knowledge, this paper is one of the first contributions to the study of singular systems with sign-changing weight, nonlinear singularities, and critical exponent.

## 1. Introduction

The proposed problem (2) is important in many fields of sciences, and it arises in biological applications (e.g., population dynamics) or physical applications (e.g., models of a nuclear reactor) and has drawn a lot of attention; see [1, 2] and references therein.

A natural question that arises in concert applications is to see what happens if these elliptic problems (degenerate or nondegenerate) are affected by certain singular perturbations.

The degeneracy and singularity occur in system (2); thus, standard variational methods are not applied which means that in our work, we research the critical points as the minimizers of the energy functional associated with the proposed problem (2) on the constraint defined by the Nehari manifold, which are solutions of our system, under some sufficient conditions on the parameters $\alpha, \beta, \mu$, and $\eta$.

In recent years, much attention has been paid to the existence of nontrivial solutions for problems $\left(\mathscr{P}_{a, \eta, \mu}\right)$ of the type as follows:

$$
\begin{cases}-\operatorname{div}\left(|x|^{-2 a} \nabla u\right)-\mu|x|^{-2(a+1)} u=h(x)|x|^{-2_{*} b}|u|^{2_{*}-2} u+n f(x), & \text { in } \Omega  \tag{1}\\ u=0, & \text { on } \partial \Omega\end{cases}
$$

Wang and Zhou [3] have proved that $\left(\mathscr{P}_{0, \mu, 1}\right)$, for $f(x) \equiv h(x) \equiv 1$ and $a=0$, has at least two distinct solutions when $0 \leq \mu<\bar{\mu}_{0}:=((N-2) / 2)^{2}$ and under some sufficient conditions on $f$. In [4], Bouchekif and Matallah have shown the existence of two nontrivial solutions of $\left(\mathscr{P}_{a, \eta, \mu}\right)$ when $0<\mu \leq \bar{\mu}_{a},-\infty<a<(N-2) / 2, \quad a \leq b<a+1, \quad$ and
$\eta \in\left(0, \Lambda_{*}\right)$ with $\Lambda_{*}$ a positive constant and under some appropriate conditions on functions $f$ and $h$.

Many existing results are available for regular and critical problems that arise from potentials; see, for example, [5-7]. However, to our knowledge, there are few results for singular systems (see [8, 9]).

This paper is organized as follows. In Section 2, we give our system and main results. In Section 3, we cite some preliminaries. Conclusion is presented in Section 4.

## 2. The Mathematical Model and Main Results

This paper deals with the existence and multiplicity of nontrivial solutions to the following proposed problem:

$$
\begin{cases}-\operatorname{div}\left(|x|^{-2 a} \nabla u\right)-\mu \frac{u}{|x|^{2(a+1)}}=(\alpha+1) h(x) \frac{|u|^{\alpha-1} u|v|^{\beta+1}}{|x|^{2} b}+\eta \frac{|u|^{-1-\theta}}{|x|^{\gamma}} u, & \text { in } \Omega  \tag{2}\\ -\operatorname{div}\left(|x|^{-2 a} \nabla v\right)-\mu \frac{v}{|x|^{2(a+1)}}=(\beta+1) h(x) \frac{|u|^{\alpha+1}|v|^{\beta-1} v}{|x|^{2 *} b}+\eta \frac{|v|^{-1-\theta}}{|x|^{\gamma}} v, & \text { in } \Omega \\ u=v=0, & \text { on } \partial \Omega\end{cases}
$$

where $\Omega$ is a bounded regular domain in $\mathbb{R}^{N}(N \geq 3)$ containing 0 in its interior, $-\infty<a<(N-2) / 2, a \leq b<$ $a+1, \quad 0 \leq \gamma<N(\alpha+\beta+1+\theta) /(\alpha+\beta+2), \quad 0<\theta<1, \quad 2_{*}=$ $2 N /(N-2+2(b-a))$ is the critical Caffarelli-KohnNirenberg exponent, $-\infty<\mu<\bar{\mu}_{a}:=((N-2(a+1)) / 2)^{2}, \alpha$
and $\beta$ are positive real such that $\alpha+\beta=2_{*}-2, \eta$ is a real parameter, and $h$ is a function defined on $\bar{\Omega}$.

By $\mathscr{H}_{\mu}:=\mathscr{H}_{\mu}(\Omega)$, we denote the completion of the space $\mathscr{C}_{0}^{\infty}\left(\mathbb{R}^{N}\right)$ with respect to the norm as follows:

$$
\begin{equation*}
\|u\|_{\mu, a}=\left(\int_{\Omega}\left(|x|^{-2 a}|\nabla u|^{2}-\mu|y|^{-2(a+1)}|u|^{2}\right) \mathrm{d} x\right)^{1 / 2}, \quad \text { for }-\infty<\mu<\bar{\mu}_{a} \tag{3}
\end{equation*}
$$

Using the Hardy inequality, this norm is equivalent to $\|u\|_{0, a}$. More explicitly, we have

$$
\begin{equation*}
\sqrt{\left(\frac{1-\max (\mu, 0)}{\bar{\mu}_{a}}\right)}\|u\|_{0, a} \leq\|u\|_{\mu, a} \leq \sqrt{\left(\frac{1-\min (\mu, 0)}{\bar{\mu}_{a}}\right)}\|u\|_{0, a} \tag{4}
\end{equation*}
$$

where
The space $\mathscr{H}:=\mathscr{H}_{\mu} \times \mathscr{H}_{\mu}$ is endowed with the norm

$$
\begin{align*}
& P(u, v):=\int_{\Omega} h(x) \frac{|u|^{\alpha+1}|v|^{\beta+1}}{|x|^{2 *} b} \mathrm{~d} x, \\
& Q(u, v):=\frac{\eta}{1-\theta} \int_{\Omega}\left[\frac{(u+\omega)^{1-\theta}-\omega^{1-\theta}}{|x|^{\gamma}}+\frac{(v+\omega)^{1-\theta}-\omega^{1-\theta}}{|x|^{\gamma}}\right] \mathrm{d} x . \tag{7}
\end{align*}
$$

A couple $(u, v) \in \mathscr{H}$ is a weak solution of the proposed problem (2) if it satisfies

$$
\begin{align*}
\left\langle I^{\prime}(u, v),(\varphi, \psi)\right\rangle & :=R(u, v)(\varphi, \psi)-S(u, v)(\varphi, \psi)+-T(u, v)(\varphi, \psi)  \tag{8}\\
& =0, \quad \text { for all }(\varphi, \psi) \in \mathscr{H}
\end{align*}
$$

with

$$
\begin{align*}
& R(u, v)(\varphi, \psi):=\int_{\Omega}\left(\frac{\nabla u \nabla \varphi+\nabla v \nabla \psi}{|x|^{2 a}}-\mu \frac{u \varphi+v \psi}{|x|^{2(a+1)}}\right) \\
& S(u, v)(\varphi, \psi):=\int_{\Omega} h(x) \frac{(\alpha+1)|u|^{\alpha}|v|^{\beta+1} \varphi+(\beta+1)|u|^{\alpha+1}|v|^{\beta} \psi}{|x|^{2} b},  \tag{9}\\
& T(u, v)(\varphi, \psi):=\eta \int_{\Omega}\left[\frac{\varphi}{(u+\omega)^{\theta}|x|^{\gamma}}+\frac{\psi}{(v+\omega)^{\theta}|x|^{\gamma}}\right] \mathrm{d} x .
\end{align*}
$$

Here, $\langle\ldots .$,$\rangle denotes the product in the duality \mathscr{H}^{\prime}, \mathscr{H}$. which ensures the existence of a positive constant $C_{a, b}$ such We list here a few integral inequalities. The first one that that we need is the Caffarelli-Kohn-Nirenberg inequality [10],

$$
\begin{equation*}
\left(\int_{\mathbb{R}^{N}}|x|^{-2 * b}|v|^{2_{*}} \mathrm{~d} x\right)^{2 / 2_{*}} \leq C_{a, b} \int_{\mathbb{R}^{N}}|x|^{-2 a}|\nabla v|^{2} \mathrm{~d} x, \text { for all } v \in \mathscr{C}_{0}^{\infty}\left(\mathbb{R}^{N}\right) \tag{10}
\end{equation*}
$$

In (10), as $b=a+1$, then $2_{*}=2$, and we have the following weighted Hardy inequality [11]:

$$
\begin{equation*}
\int_{\mathbb{R}^{N}}|x|^{-2(a+1)} v^{2} \mathrm{~d} x \leq \frac{1}{\bar{\mu}_{a}} \int_{\mathbb{R}^{N}}|x|^{-2 a}|\nabla v|^{2} \mathrm{~d} x, \text { for all } v \in \mathscr{C}_{0}^{\infty}\left(\mathbb{R}^{N}\right) \tag{11}
\end{equation*}
$$

Let

$$
\begin{align*}
& S_{\mu}:=\inf _{u \in \mathscr{\mathscr { R } _ { \mu } \backslash \{ 0 \}}} \frac{\|u\|_{\mu, a}^{2}}{\left(\int_{\Omega}|u|^{2_{*}} /|x|^{2 *} b \mathrm{~d} x\right)^{2 / 2}}  \tag{12}\\
& \widetilde{S}_{\mu}:=\inf _{(u, v) \in \mathscr{H}\{(0,0)\}} \frac{\|(u, v)\|_{\mu, a}^{2}}{\left(\int_{\Omega}|u|^{\alpha+1}|v|^{\beta+1} /|x|^{2, b} \mathrm{~d} x\right)^{2 / 2 *}}
\end{align*}
$$

From [12], $S_{\mu}$ is achieved.
Lemma 1. Let $\Omega$ be a domain (not necessarily bounded), $-\infty<\mu<\bar{\mu}_{a}$, and $\alpha+\beta \leq 2_{*}-2$. Then, we have

$$
\begin{equation*}
\widetilde{S}_{\mu}:=\left[\left(\frac{\alpha+1}{\beta+1}\right)^{(\beta+1) / 2_{*}}+\left(\frac{\beta+1}{\alpha+1}\right)^{(\alpha+1) / 2_{*}}\right] S_{\mu} \tag{13}
\end{equation*}
$$

For simplicity of writing, let us note the quantity $\left[(\alpha+1 / \beta+1)^{(\beta+1) / 2_{*}}+(\beta+1 / \alpha+1)^{(\alpha+1) / 2_{*}}\right]$ by $K(\alpha, \beta)$.

Proof. The proof is essentially given in [1] with minor modifications.

We set assumptions on the function $h$ which is somewhere positive but which may change sign in $\bar{\Omega}$
(H1) $h \in C(\bar{\Omega})$ and $h^{+}=\max \{h, 0\} \neq 0$ in $\Omega$
(H2) There exists $\eta_{0}>0$ such that $\left|h^{+}\right|_{\infty}=h(0)=$ $\max _{x \in \Omega} h(x)>\eta_{0}$
As regards, problems containing the weight function $h(x)$ change sign; see [13-15] and references therein.

Here, we can address some background works on the critical points; see, for example, [16, 17].

Let $\Lambda_{0}$ be a positive number such that

$$
\begin{equation*}
\Lambda_{0}:=\frac{2_{*}\left(2_{*}-2\right)}{\left(\left|h^{+}\right|_{\infty}\right)^{\left(1 / 2_{*}-2\right)} A\left[2_{*}\left(2_{*}-1\right)\right]^{\left(2_{*}-1\right) /\left(2_{*}-2\right)}}[K(\alpha, \beta)]^{2_{*} / 2\left(2_{*}-2\right)}\left(S_{\mu}\right)^{2_{*} / 2\left(2_{*}-2\right)}, \tag{14}
\end{equation*}
$$

where

$$
\begin{equation*}
A=\left[\frac{2 \pi^{(N / 2)}(\alpha+\beta+\gamma)}{N \Gamma(N / 2)(\alpha+\beta+\gamma)-\theta(\alpha+\beta+1)}\right]^{\alpha+\beta+\gamma / \alpha+\beta+1} R_{0}^{N / \alpha+\beta+1(\alpha+\beta+\gamma)-\theta}>0 \tag{15}
\end{equation*}
$$

with

$$
\begin{equation*}
0 \leq \theta<\frac{N}{\alpha+\beta+1}(\alpha+\beta+\gamma) \tag{16}
\end{equation*}
$$

Then, we obtain the following results.
Theorem 2. Assume that $-\infty<a<(N-2) / 2, a \leq b<a+1$, $0 \leq \gamma<N(\alpha+\beta+1+\theta) /(\alpha+\beta+2), \quad 0<\theta<1, \quad-\infty<\mu<$ $\bar{\mu}_{a}:=((N-2(a+1)) / 2)^{2}, \alpha+\beta+2=2_{*}$ (H1), and $\eta$ real parameter satisfying $0<\eta<\Lambda_{0}$, then (2) has at least one nontrivial solution.

Theorem 3. In addition to the assumptions of Theorem 2, if (H2) holds and $\eta$ verifies $0<\eta<(1 / 2) \Lambda_{0}$, then (2) has at least two nontrivial solutions.

## 3. Preliminaries

Definition 4. Let $c \in \mathbb{R}, E$ be a Banach space and $I \in C^{1}$ $(E, \mathbb{R})$.
(i) $\left(u_{n}, v_{n}\right)_{n}$ is a Palais-Smale sequence at level $c$ (in short $(P S)_{c}$ ) in $E$ for $I$ if
$I\left(u_{n}, v_{n}\right)=c+o_{n}(1)$ and $I^{\prime}\left(u_{n}, v_{n}\right)=o_{n}(1)$,
where $o_{n}(1)$ tends to 0 as $n$ goes at infinity.
(ii) We say that $I$ satisfies the $(P S)_{c}$ condition if any $(P S)_{c}$ sequence in $E$ for $I$ has a convergent subsequence.
3.1. Nehari Manifold. It is well known that $J$ is of class $C^{1}$ in $\mathscr{H}$ and the solutions of (2) are the critical points of $I$ which is not bounded below on $\mathscr{H}$. Consider the following Nehari manifold:

$$
\begin{equation*}
\mathscr{M}=\left\{(u, v) \in \mathscr{H} \backslash\{0,0\}:\left\langle I^{\prime}(u, v),(u, v)\right\rangle=0\right\} . \tag{18}
\end{equation*}
$$

Thus, $(u, v) \in \mathscr{M}$ if and only if

$$
\begin{equation*}
\|(u, v)\|_{\mu, a}^{2}-2_{*} P(u, v)-Q(u, v)=0 . \tag{19}
\end{equation*}
$$

Note that $\mathscr{M}$ contains every nontrivial solution of problem (2). Moreover, we have the following results.

Lemma 5. I is coercive and bounded from below on $\mathscr{M}$.
Proof. Let $R_{0}>0$ such that $\Omega \subset B\left(0, R_{0}\right)=\left\{x \in \mathbb{R}^{N}:|x|<\right.$ $\left.R_{0}\right\}$. If $u \in \mathscr{M}$, then, by (19) and the Hölder inequality, we obtain

$$
\begin{align*}
\int_{\Omega} \frac{\left(u^{+}\right)^{1-\theta}}{|x|^{\gamma}} \mathrm{d} x & \leq C_{1}\|u\|^{1-\theta}  \tag{20}\\
\quad \text { with } C_{1} & =A S_{1}^{-(1-\theta) / p}
\end{align*}
$$

where

$$
\begin{equation*}
S_{1}:=\inf _{u \in \mathscr{H} \backslash\{0\}} \frac{\|u\|^{p}}{\left(\int_{\Omega}|u|^{1-\theta} /|x|^{\gamma} \mathrm{d} x\right)^{p /(1-\theta)}}, \tag{21}
\end{equation*}
$$

and we deduce that

$$
\begin{equation*}
I(u, v) \geq\left(\frac{(\alpha+\beta-1)}{2(1+\alpha+\beta)}\right)\|(u, v)\|_{\mu, a}^{2}+-2 \eta\left(\frac{(\alpha+\beta+\theta)}{(\alpha+\beta+1)(1-\theta)}\right) A\|(u, v)\|_{\mu, a}^{1-\theta} K(\alpha, \beta) S_{1}^{-(1-\theta) / 2} \tag{22}
\end{equation*}
$$

for $0 \leq \gamma<N / \alpha+\beta+1(\alpha+\beta+\theta)$.
Thus, $I$ is coercive and bounded from below on $\mathscr{M}$. Define

$$
\begin{equation*}
\phi(u, v)=\left\langle I^{\prime}(u, v),(u, v)\right\rangle . \tag{23}
\end{equation*}
$$

Then, for $(u, v) \in \mathscr{M}$,

$$
\begin{align*}
\left\langle\phi^{\prime}(u, v),(u, v)\right\rangle & =2\|(u, v)\|_{\mu, a}^{2}-\left(2_{*}\right)^{2} P(u, v)-\eta Q(u, v) \\
& =\|(u, v)\|_{\mu, a}^{2}-2_{*}\left(2_{*}-\eta\right) P(u, v) \\
& =\left(2_{*}-\eta\right) Q(u, v)-\left(2_{*}-2\right)\|(u, v)\|_{\mu, a}^{2} \tag{24}
\end{align*}
$$

Now, we split $\mathscr{M}$ into the following three parts:

$$
\begin{align*}
& \mathscr{M}^{+}=\left\{(u, v) \in \mathscr{M}:\left\langle\phi^{\prime}(u, v),(u, v)\right\rangle>0\right\},  \tag{25}\\
& \mathscr{M}^{0}=\left\{(u, v) \in \mathscr{M}:\left\langle\phi^{\prime}(u, v),(u, v)\right\rangle=0\right\},
\end{align*}
$$

and $\mathscr{M}^{-}=\left\{(u, v) \in \mathscr{M}:\left\langle\phi^{\prime}(u, v),(u, v)\right\rangle<0\right\}$.
We have the following results.

Lemma 6. Suppose that $\left(u_{0}, v_{0}\right)$ is a local minimizer for Ion $\mathscr{M}$. Then, if $\left(u_{0}, v_{0}\right) \notin \mathscr{M}^{0},\left(u_{0}, v_{0}\right)$ is a critical point of $I$.

Proof. If $\left(u_{0}, v_{0}\right)$ is a local minimizer for $I$ on $\mathscr{M}$, then ( $u_{0}, v_{0}$ ) is a solution of the optimization problem

$$
\begin{equation*}
\min _{\{(u, v) / \phi(u, v)=0\}} I(u, v) . \tag{26}
\end{equation*}
$$

Hence, there exists a Lagrange multiplier $\xi \in \mathbb{R}$ such that

$$
\begin{equation*}
I^{\prime}\left(u_{0}, v_{0}\right)=\xi \phi^{\prime}\left(u_{0}, v_{0}\right) \text { in } \mathscr{H}^{\prime}(\text { dual of } \mathscr{H}) \tag{27}
\end{equation*}
$$

Thus,

$$
\begin{equation*}
\left\langle I^{\prime}\left(u_{0}, v_{0}\right),\left(u_{0}, v_{0}\right)\right\rangle=\xi\left\langle\phi^{\prime}\left(u_{0}, v_{0}\right),\left(u_{0}, v_{0}\right)\right\rangle . \tag{28}
\end{equation*}
$$

However, $\left\langle\phi^{\prime}\left(u_{0}, v_{0}\right),\left(u_{0}, v_{0}\right)\right\rangle \neq 0$, since $\left(u_{0}, v_{0}\right) \notin \mathscr{M}^{0}$. Hence, $\xi=0$. This completes the proof.

Lemma 7. There exists a positive number $\Lambda_{0}$ such that, for all $\eta$ verifying

$$
\begin{equation*}
0<\eta<\Lambda_{0} \tag{29}
\end{equation*}
$$

we have $\mathscr{M}^{0}=\varnothing$.
Proof. Let us reason by contradiction.
Suppose $\mathscr{M}^{0} \neq \varnothing$ such that $0<\eta<\Lambda_{0}$. Then, by (24) and for $(u, v) \in \mathscr{M}^{0}$, we have

$$
\begin{align*}
\|(u, v)\|_{\mu, a}^{2} & =2_{*}\left(2_{*}-1\right) P(u, v)  \tag{30}\\
& =\left(\left(2_{*}-\eta\right) /\left(2_{*}-2\right)\right) Q(u, v) .
\end{align*}
$$

Moreover, by the Hölder inequality and the Sobolev embedding theorem, we obtain

$$
\begin{equation*}
\|(u, v)\|_{\mu, a} \geq[K(\alpha, \beta)]^{2_{*} / 2\left(2_{*}-2\right)}\left(S_{\mu}\right)^{2_{*} / 2\left(2_{*}-2\right)}\left[2_{*}\left(2_{*}-1\right)\left(\left|h^{+}\right|_{\infty}\right)^{-1}\right]^{-1 /\left(2_{*}-2\right)} \tag{31}
\end{equation*}
$$

and

$$
\begin{equation*}
\|(u, v)\|_{\mu, a} \leq\left[\left(\left(2_{*}-1\right) \eta\left(2_{*}-2\right)^{-1}\right) A\right] . \tag{32}
\end{equation*}
$$

From (31) and (32), we obtain $\eta \geq \Lambda_{0}$, which contradicts our hypothesis.

Thus, $\mathscr{M}=\mathscr{M}^{+} \cup \mathscr{M}^{-}$. Define

$$
\begin{equation*}
c:=\inf _{u \in \mathscr{M}} I(u, v), c^{+}:=\inf _{u \in \mathscr{M}^{+}} I(u, v) \text { and } c^{-}:=\inf _{u \in \mathscr{M}^{-}} I(u, v) . \tag{33}
\end{equation*}
$$

For the sequel, we need the following Lemma.

## Lemma 8

(i) For all $\eta$ such that $0<\eta<\Lambda_{0}$, one has $c \leq c^{+}<0$.
(ii) For all $\eta$ such that $0<\eta<(1 / 2) \Lambda_{0}$, one has

$$
\begin{equation*}
c^{-}>C_{1}=C_{1}\left(\eta, S_{\mu},\left|h^{+}\right|_{\infty}\right) \tag{34}
\end{equation*}
$$

where

$$
\begin{align*}
C_{1}:= & \frac{2_{*}-2}{2_{*} 2 A\left[2_{*}\left(2_{*}-1\right)\left|h^{+}\right|_{\infty}\right]^{2 /\left(2_{*}-2\right)}}[K(\alpha, \beta)]^{2_{*} /\left(2_{*}-2\right)}\left(S_{\mu}\right)^{2_{*} /\left(2_{*}-2\right)}+  \tag{35}\\
& -\left(\left(2_{*}-2\right) / 2_{*}\right) \eta .
\end{align*}
$$

Proof
(i) Let $(u, v) \in \mathscr{M}^{+}$. By (24), we have

$$
\begin{equation*}
\left[1 / 2_{*}\left(2_{*}-1\right)\right]\|(u, v)\|_{\mu, a}^{2}>P(u, v) \tag{36}
\end{equation*}
$$

and so

$$
\begin{align*}
I(u, v) & =(-1 / 2)\|(u, v)\|_{\mu, a}^{2}+\left(2_{*}-1\right) P(u, v)  \tag{37}\\
& <-\left(\left(2_{*}-\eta\right) / 2_{*} 2\right)\|(u, v)\|_{\mu, a}^{2} .
\end{align*}
$$

We conclude that $c \leq c^{+}<0$.
(ii) Let $(u, v) \in \mathscr{M}^{-}$. By (24), we get

$$
\begin{equation*}
\left[1 / 2_{*}\left(2_{*}-1\right)\right]\|(u, v)\|_{\mu, a}^{2}<P(u, v) . \tag{38}
\end{equation*}
$$

Moreover, by Sobolev embedding theorem, we have
$P(u, v) \leq[K(\alpha, \beta)]^{-2_{*} / 2}\left(S_{\mu}\right)^{-2_{*} / 2}\left|h^{+}\right|_{\infty}\|(u, v)\|_{\mu, a}^{2_{*}}$.
This implies

$$
\begin{equation*}
\|(u, v)\|_{\mu, a}>\left[2_{*}\left(2_{*}-1\right)\right]^{-1 /\left(2_{*}-2\right)} A[K(\alpha, \beta)]^{2_{*} / 2\left(2_{*}-2\right)}\left(S_{\mu}\right)^{2_{*} / 2\left(2_{*}-2\right)}, \quad \text { for all } u \in \mathscr{M}^{-} \tag{40}
\end{equation*}
$$

By (20), we get
$I(u, v) \geq\left(\frac{\left(2_{*}-2\right)}{2_{*} 2}\right)\|(u, v)\|_{\mu, a}^{2}+-\left(1-\left(1 / 2_{*}\right)\right) \eta\|(u, v)\|_{\mu, a}$.

Thus, for all $\eta$ such that $0<\eta<(1 / 2) \Lambda_{0}$, we have $I(u, v) \geq C_{1}$.

For each $(u, v) \in \mathscr{H}$, we write

$$
\begin{equation*}
t_{m}:=t_{\max }(u, v)=\left[\frac{\|(u, v)\|_{\mu, a}}{2_{*}\left(2_{*}-1\right) \int_{\Omega}|u|^{\alpha+1}|v|^{\beta+1} h|x|^{-2_{*} b} \mathrm{~d} x}\right]^{1 /\left(2_{*}-2\right)}>0 \tag{42}
\end{equation*}
$$

Lemma 9. Let $\eta$ satisfy $0<\eta<\Lambda_{0}$. For each $(u, v) \in \mathscr{H}$ with $\int_{\Omega}|u|^{\alpha+1}|v|^{\beta+1} h|x|^{-2, b} \mathrm{~d} x>0$, one has the following:
(i) If $\mathrm{Q}(u, v) \leq 0$, then there exists a unique $t^{-}>t_{m}$ such that $\left(t^{-} u, t^{-} v\right) \in \mathscr{M}^{-}$and

$$
\begin{equation*}
I\left(t^{-} u, t^{-} v\right)=\sup _{t \geq 0}(t u, t v) . \tag{43}
\end{equation*}
$$

(ii) If $Q(u, v)>0$, then there exists unique $t^{+}$and $t^{-}$such that $0<t^{+}<t_{m}<t^{-},\left(t^{+} u, t^{+} v\right) \in \mathscr{M}^{+}$, and $\left(t^{-} u, t^{-} v\right)$ $\in \mathscr{M}^{-}$

$$
\begin{equation*}
I\left(t^{+} u, t^{+} v\right)=\inf _{0 \leq t \leq t_{m}} I(t u, t v) \text { and } I\left(t^{-} u, t^{-} v\right)=\sup _{t \geq 0} I(t u, t v) . \tag{44}
\end{equation*}
$$

Proof. With minor modifications, we refer to [18].
Taking the idea of the work of Brown-Zhang [18], we prove the following result.

## Proposition 10

(i) For all $\eta$ such that $0<\eta<\Lambda_{0}$, there exists a $(P S)_{c^{+}}$ sequence in $\mathscr{M}^{+}$
(ii) For all $\eta$ such that $0<\eta<(1 / 2) \Lambda_{0}$, there exists a $(P S)_{c^{-}}$sequence in $\mathscr{M}^{-}$

## 4. Conclusion

In our work, we have searched the critical points as the minimizers of the energy functional associated with the problem on the constraint defined by the Nehari manifold $\mathscr{M}$, which are solutions to our problem. Under some sufficient conditions on coefficients of the proposed problem (2) such that $-\infty<a<(N-2) / 2, a \leq b<a+1$, $0 \leq \gamma<N(\alpha+\beta+1+\theta) /(\alpha+\beta+2), \quad 0<\theta<1, \quad 2_{*}=2 N /$ $(N-2+2 \quad(b-a)), \quad-\infty<\mu<\bar{\mu}_{a}:=((N-2(a+1)) / 2)^{2}$, $\alpha+\beta=2_{*}-2,0<\beta<1$, and $0<\eta<(1 / 2) \Lambda_{0}$, we split $\mathscr{M}$ into two disjoint subsets $\mathscr{M}^{+}$and $\mathscr{M}^{-}$; thus, we obtain that (2) has two nontrivial solutions $\left(u_{0}^{+}, v_{0}^{+}\right) \in \mathscr{M}^{+}$and $\left(u_{0}^{-}, v_{0}^{-}\right) \in \mathscr{M}^{-}$. Since $\mathscr{M}^{+} \cap \mathscr{M}^{-}=\varnothing$, this implies that $\left(u_{0}^{+}, v_{0}^{+}\right)$and ( $u_{0}^{-}, v_{0}^{-}$) are distinct.

## Appendix

## A. Proof of Theorem 2

Drawing on the works of $[18,19]$, we establish the existence of a local minimum for $I$ on $\mathscr{M}^{+}$.

Proposition 11. For all $\eta$ such that $0<\eta<\Lambda_{0}$, the functional I has a minimizer $\left(u_{0}^{+}, v_{0}^{+}\right) \in \mathscr{M}^{+}$and it satisfies
(i) $M\left(u_{0}^{+}, v_{0}^{+}\right)=c=c^{+}$
(ii) $\left(u_{0}^{+}, v_{0}^{+}\right)$is a nontrivial solution of (2)

Proof. If $0<\eta<\Lambda_{0}$, then by Proposition 10 (i), there exists a $\left(u_{n}, v_{n}\right)_{n}(P S)_{c^{+}}$sequence in $\mathscr{M}^{+}$; thus, it is bounded by Lemma 5. Then, there exists $\left(u_{0}^{+}, v_{0}^{+}\right) \in \mathscr{H}$, and we can extract a subsequence which will be denoted by $\left(u_{n}, v_{n}\right)_{n}$ such that

$$
\begin{align*}
\left(u_{n}, v_{n}\right) & -\left(u_{0}^{+}, v_{0}^{+}\right) \text {weakly in } \mathscr{H} \\
\left(u_{n}, v_{n}\right) & \longrightarrow\left(u_{0}^{+}, v_{0}^{+}\right) \text {weakly in }\left(L^{2_{*}}\left(\Omega,|x|^{-2_{*} b}\right)\right)^{2}, \\
u_{n} & \longrightarrow u_{0}^{+} \text {a.e in } \Omega  \tag{45}\\
v_{n} & \longrightarrow v_{0}^{+} \text {a.e in } \Omega \\
u_{n} & \longrightarrow u_{0}^{+} \text {strongly in } L^{1-\beta}\left(\Omega,|x|^{-\gamma}\right), \\
v_{n} & \longrightarrow v_{0}^{+} \text {strongly in } L^{1-\beta}\left(\Omega,|x|^{-\gamma}\right)
\end{align*}
$$

and we have

$$
\begin{align*}
& \lim _{n \longrightarrow \infty} \int_{\Omega} \frac{\left|u_{n}+\omega\right|^{1-\beta}}{|x|^{\alpha}} \mathrm{d} x=\int_{\Omega} \frac{\left|u_{0}^{+}+\omega\right|^{1-\beta}}{|x|^{\alpha}} \mathrm{d} x+o(1) \\
& \lim _{n \longrightarrow \infty} \int_{\Omega} \frac{\left|v_{n}+\omega\right|^{1-\beta}}{|x|^{\gamma}} \mathrm{d} x=\int_{\Omega} \frac{\left|v_{0}^{+}+\omega\right|^{1-\beta}}{|x|^{\gamma}} \mathrm{d} x+o(1) \tag{46}
\end{align*}
$$

Thus, by (45), $\left(u_{0}^{+}, v_{0}^{+}\right)$is a weak nontrivial solution of (2). Now, we show that $\left(u_{n}, v_{n}\right)$ converges to $\left(u_{0}^{+}, v_{0}^{+}\right)$strongly in
$\mathscr{H}$. . Suppose this is not true, then by the lower semicontinuity of the norm, either $\left\|u_{0}^{+}\right\|_{\mu, a}<\lim \inf _{n \rightarrow \infty}\left\|u_{n}\right\|_{\mu, a}$ or $\left\|v_{0}^{+}\right\|_{\mu, a}<\lim \inf _{n \rightarrow \infty}\left\|v_{n}\right\|_{\mu, a}$, we obtain

$$
\begin{align*}
c & \leq I\left(u_{0}^{+}, v_{0}^{+}\right)=\left(\frac{\left(2_{*}-2\right)}{2_{*} 2}\right)\left\|\left(u_{0}^{+}, v_{0}^{+}\right)\right\|_{\mu, a}^{2}-\left(1-\left(\frac{1}{2_{*}}\right)\right) \eta Q\left(u_{0}^{+}, v_{0}^{+}\right)  \tag{47}\\
& <\lim _{n \longrightarrow \infty} \inf I\left(u_{n}, v_{n}\right)=c .
\end{align*}
$$

We get a contradiction. Therefore, $\left(u_{n}, v_{n}\right)$ converges to ( $u_{0}^{+}, v_{0}^{+}$) strongly in $\mathscr{H}$. Moreover, we have $\left(u_{0}^{+}, v_{0}^{+}\right) \in \mathscr{M}^{+}$. If not, then by Lemma 9, there are two numbers $t_{0}^{+}$and $t_{0}^{-}$, uniquely defined so that $\left(t_{0}^{+} u_{0}^{+}, t_{0}^{+} v_{0}^{+}\right) \in \mathscr{M}^{+}$and $\left(t^{-} u_{0}^{+}, t^{-} v_{0}^{+}\right)$ $\in \mathscr{M}^{-}$. In particular, we have $t_{0}^{+}<t_{0}^{-}=1$. Since

$$
\begin{equation*}
\frac{d}{\mathrm{~d} t} I\left(t u_{0}^{+}, t v_{0}^{+}\right)_{J t=t_{0}^{+}}=0 \text { and } \frac{d^{2}}{\mathrm{~d} t^{2}} I\left(t u_{0}^{+}, t v_{0}^{+}\right)_{J t=t_{0}^{+}}>0 \tag{48}
\end{equation*}
$$

there exists $t_{0}^{+}<t^{-} \leq t_{0}^{-}$such that $I\left(t_{0}^{+} u_{0}^{+}, t_{0}^{+} v_{0}^{+}\right)<I\left(t^{-} u_{0}^{+}\right.$, $t^{-} v_{0}^{+}$). By Lemma 9, we get

$$
\begin{equation*}
I\left(t_{0}^{+} u_{0}^{+}, t_{0}^{+} v_{0}^{+}\right)<I\left(t^{-} u_{0}^{+}, t^{-} v_{0}^{+}\right)<I\left(t_{0}^{-} u_{0}^{+}, t_{0}^{-} v_{0}^{+}\right)=I\left(u_{0}^{+}, v_{0}^{+}\right), \tag{49}
\end{equation*}
$$

which is a contradiction.

## B. Proof of Theorem 3

Next, we establish the existence of a local minimum for $I$ on $\mathscr{M}^{-}$. For this, we require the following lemma.

Lemma 12. For all $\eta$ such that $0<\eta<(1 / 2) \Lambda_{0}$, the functional I has a minimizer $\left(u_{0}^{-}, v_{0}^{-}\right)$in $\mathscr{M}^{-}$, and it satisfies
(i) $I\left(u_{0}^{-}, v_{0}^{-}\right)=c^{-}>0$
(ii) $\left(u_{0}^{-}, v_{0}^{-}\right)$is a nontrivial solution of (2) in $\mathscr{H}$

Proof. If $0<\eta<(1 / 2) \Lambda_{0}$, then by Proposition 10 (ii), there exists a $\left(u_{n}, v_{n}\right)_{n},(P S)_{c^{-}}$sequence in $\mathscr{M}^{-}$; thus, it is bounded by Lemma 5 . Then, there exists $\left(u_{0}^{-}, v_{0}^{-}\right) \in \mathscr{H}$, and we can extract a subsequence which will be denoted by $\left(u_{n}, v_{n}\right)_{n}$ such that

$$
\begin{align*}
\left(u_{n}, v_{n}\right) & \rightharpoonup\left(u_{0}^{-}, v_{0}^{-}\right) \text {weakly in } \mathscr{H}, \\
\left(u_{n}, v_{n}\right) & \rightharpoonup\left(u_{0}^{-}, v_{0}^{-}\right) \text {weakly in }\left(L^{2_{*}}\left(\Omega,|x|^{-2_{*} b}\right)\right)^{2}, \\
u_{n} & \longrightarrow u_{0}^{-} \text {a.e in } \Omega,  \tag{50}\\
v_{n} & \longrightarrow v_{0}^{-} \text {a.e in } \Omega, \\
u_{n} & \longrightarrow u_{0}^{+} \text {strongly in } L^{1-\beta}\left(\Omega,|x|^{-\gamma}\right), \\
v_{n} & \longrightarrow v_{0}^{+} \text {strongly in } L^{1-\beta}\left(\Omega,|x|^{-\gamma}\right) .
\end{align*}
$$

This implies

$$
\begin{equation*}
P\left(u_{n}, v_{n}\right) \longrightarrow P\left(u_{0}^{-}, v_{0}^{-}\right), \text {as } n \longrightarrow \infty . \tag{51}
\end{equation*}
$$

Moreover, by (24), we obtain

$$
\begin{equation*}
P\left(u_{n}, v_{n}\right)>\left[2_{*}\left(2_{*}-1\right)\right]^{-1}\left\|u_{n}, v_{n}\right\|_{\mu, a}^{2} \tag{52}
\end{equation*}
$$

thus, by (31) and (52), there exists a positive number

$$
\begin{equation*}
C_{2}:=\left[2_{*}\left(2_{*}-1\right) K(\alpha, \beta)\right]^{2_{*} /\left(2_{*}-2\right)}\left(S_{\mu}\right)^{2_{*} /\left(2_{*}-2\right)} \tag{53}
\end{equation*}
$$

such that

$$
\begin{equation*}
P\left(u_{n}, v_{n}\right)>C_{2} . \tag{54}
\end{equation*}
$$

This implies that

$$
\begin{equation*}
P\left(u_{0}^{-}, v_{0}^{-}\right) \geq C_{2} . \tag{55}
\end{equation*}
$$

Now, we prove that $\left(u_{n}, v_{n}\right)_{n}$ converges to $\left(u_{0}^{-}, v_{0}^{-}\right)$ strongly in $\mathscr{H}$. Suppose this is not true, then either $\left\|u_{0}^{-}\right\|_{\mu, a}<$ $\lim \inf _{n \longrightarrow \infty}\left\|u_{n}\right\|_{\mu, a}$ or $\left\|v_{0}^{-}\right\|_{\mu, a}<\liminf _{n \longrightarrow \infty}\left\|v_{n}\right\|_{\mu, a}$. By Lemma 9 , there is a unique $t_{0}^{-}$such that $\left(t_{0}^{-} u_{0}^{-}, t_{0}^{-} v_{0}^{-}\right) \in \mathcal{N}^{-}$. Since

$$
\begin{equation*}
\left(u_{n}, v_{n}\right) \in \mathscr{M}^{-}, I\left(u_{n}, v_{n}\right) \geq I\left(t u_{n}, t v_{n}\right), \text { for all } t \geq 0 \tag{56}
\end{equation*}
$$

we have

$$
\begin{equation*}
I\left(t_{0}^{-} u_{0}^{-}, t_{0}^{-} v_{0}^{-}\right)<\lim _{n \longrightarrow \infty} I\left(t_{0}^{-} u_{n}, t_{0}^{-} v_{n}\right) \leq \lim _{n \longrightarrow \infty} I\left(u_{n}, v_{n}\right)=c^{-}, \tag{57}
\end{equation*}
$$

and this is a contradiction. Hence,

$$
\begin{equation*}
\left(u_{n}, v_{n}\right)_{n} \longrightarrow\left(u_{0}^{-}, v_{0}^{-}\right) \text {strongly in } \mathscr{H} . \tag{58}
\end{equation*}
$$

Thus,

$$
\begin{equation*}
I\left(u_{n}, v_{n}\right) \text { converges to } I\left(u_{0}^{-}, v_{0}^{-}\right)=c^{-} \text {as } n \text { tends to }+\infty . \tag{59}
\end{equation*}
$$

By (54) and Lemma 6, we may assume that ( $u_{0}^{-}, v_{0}^{-}$) is a nontrivial solution of (2).

Now, we complete the proof of Theorem 3. By Proposition 11 and Lemma 12, we obtain that (2) has two nontrivial solutions $\left(u_{0}^{+}, v_{0}^{+}\right) \in \mathscr{M}^{+}$and $\left(u_{0}^{-}, v_{0}^{-}\right) \in \mathscr{M}^{-}$. Since $\mathscr{M}^{+} \cap \mathscr{M}^{-}=\varnothing$, this implies that $\left(u_{0}^{+}, v_{0}^{+}\right)$and $\left(u_{0}^{-}, v_{0}^{-}\right)$are distinct.

Finally, for every $\omega \in(0,1)$, problem (2) has a solution $u_{\omega} \in \mathscr{H}$ such that $I\left(u_{\omega}, v_{\omega}\right)=0$. Thus, there exist $\left\{\omega_{n}\right\} \subset$ $(0,1)$ with $\omega_{n} \longrightarrow 0$ as $n \longrightarrow \infty$. Then, we get $(u, v)=$ $\lim _{n \longrightarrow \infty}\left(u_{\omega}, v_{\omega}\right)$.

## Data Availability

The functional analysis data used to support the findings of this study are included within the article.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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