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Research Article

Approximate Controllability and Ulam Stability for Second-Order Impulsive Integrodifferential Evolution Equations with State-Dependent Delay

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In this paper, we shall establish sufficient conditions for the existence, approximate controllability, and Ulam–Hyers–Rassias stability of solutions for impulsive integrodifferential equations of second order with state-dependent delay using the resolvent operator theory, the approximating technique, Picard operators, and the theory of fixed point with measures of noncompactness. An example is presented to illustrate the efficiency of the result obtained.

1. Introduction

In applied mathematics, control theory is crucial; it involves building and evaluating the control framework. Controllability analysis is used to solve a variety of real-world issues, such as issues with rocket launchers for satellite and aircraft control, issues with missiles and antimissile defense, and issues with managing the economy's inflation rate. Over the last twenty years, a lot of work has been done for controllability of evolution equations [1–13].

In addition, a key aspect of the field of mathematical analysis study is stability analysis. The concept of Ulam stability is applicable in various branches of mathematical analysis and is used in the cases where finding the exact solution is very difficult. A number of researchers have been working on the study of Ulam-type stabilities of differential and integrodifferential equations recently, and they have produced some remarkable findings, see [14–16], and the references therein.

During the past ten years, impulsive differential equations have attracted a lot of interest. Dynamic systems that contain jumps or discontinuities are represented using impulsive differential equations. In contrast, integrodifferential equations are found in many scientific fields where it is important to include aftereffect or delay (for example, in control theory, biology, ecology, and medicine). In fact, one always uses integrodifferential equations to describe a model that has heritable characteristics. As a result, these equations have attracted a lot of attention (see for instance, [17–23]). In [24], the authors studied some local and global existence and uniqueness results for abstract differential equations with state-dependent argument.

Second-order nonautonomous differential systems have received a lot of interest. There is no need to transform a second-order differential system into a first-order system in order to solve it. Various second-order nonautonomous differential systems existence results are presented in [5, 20, 25–29] and references therein.

In [30], Balachandran and Sakthivel considered the following integrodifferential system:

$$
\begin{aligned}
&\vartheta'(\varsigma) = \mathscr{Z}\vartheta(\varsigma) + (\mathrm{Bu})(\varsigma) + f\left(\varsigma, \vartheta(\varsigma), \int_0^{\varsigma} g(\varsigma, \varepsilon, \vartheta(\varepsilon))\mathrm{d}\varepsilon\right),\\
&\vartheta(0) = \vartheta_0, \quad \varsigma \in \Theta = [0, b],
\end{aligned} \tag{1}
$$

where $\vartheta(\cdot)$ takes values in a Banach space $\mathfrak{V}$ with the norm $\|\cdot\|$ and the control function $u(\cdot)$ is given in $L^2(\Theta, U)$, a Banach space of admissible control functions, with U being a Banach space. $g\colon \nabla \times \mathfrak{V} \longrightarrow \mathfrak{V}, f\colon \Theta \times \mathfrak{V} \times \mathfrak{V} \longrightarrow \mathfrak{V}$ are given functions, and B is a bounded linear operator from U into $\mathfrak{V}$. Here, $\nabla = \{(\varsigma, \varepsilon)\colon 0 \leqslant \varepsilon \leqslant \varsigma \leqslant b\}$.

In [8], the authors investigated the controllability of the functional differential equation with a random effect:

$$
\begin{cases}
\vartheta''(\varsigma, \delta) = \mathscr{Z}\vartheta(\varsigma, \delta) + \psi(\varsigma, \vartheta(\varrho, \delta), \delta) + B\mathfrak{f}(\varsigma, \delta), & \text{a.e. } \varsigma \in \mathbb{R}_+ := [0, \infty),\\
\vartheta(0, \delta) = \jmath_1(\delta), \quad \vartheta'(0, \delta) = \jmath_2(\delta),
\end{cases} \quad \delta \in \digamma, \tag{2}
$$

where $(\digamma, F, P)$ is a complete probability space with F being the event space and P being the probability function (see [31], for more information), $\psi\colon \mathbb{R}_+ \times \Xi \times \digamma \longrightarrow \Xi$ is a given function, $\jmath_1, \jmath_2\colon \digamma \longrightarrow \Xi$ are given measurable functions, and $(\Xi, |\cdot|)$ is a real Banach space. $\mathfrak{f}(\cdot, \delta)$ is the control function defined in $L^2(\mathbb{R}_+, \Lambda)$, a Banach space of admissible control functions with Λ being a Banach space, and B is a bounded linear operator from Λ into Ξ. The main result is based upon a generalization of the classical Darbo fixed-point theorem and the concept of measure of noncompactness combined with the family of cosine operators.

Arthi and Balachandran et al. [32] considered the following abstract control system:

$$
\begin{cases}
\vartheta''(\varsigma) = \mathscr{Z}\vartheta(\varsigma) + \mathrm{Bu}(\varsigma) + f\left(\varsigma, \vartheta_{\rho(\varsigma, \vartheta_\varsigma)}\right), & \varsigma \in I = [0, a], \varsigma \neq \varsigma_i,\\
\vartheta_0 = \varphi \in \mathscr{E}, & \vartheta'(0) = \eta \in \mathfrak{V},\\
\triangle\vartheta(\varsigma_i) = I_i\left(\vartheta_{\varsigma_i}\right), & i = 1, 2, \ldots, n,\\
\triangle\vartheta'(\varsigma_i) = \Theta_i\left(\vartheta_{\varsigma_i}\right), & i = 1, 2, \ldots, n,
\end{cases} \tag{3}
$$

where $L^2(I, U)$ a Banach space of admissible control functions with U being a Banach space and $B\colon U \longrightarrow \mathfrak{V}$ being a bounded linear operator; the function $\vartheta_\varsigma\colon (-\infty, 0] \longrightarrow \mathfrak{V}, \vartheta_\varsigma(\theta) = \vartheta(\varsigma + \theta)$, $\mathscr{E}$ is the phase space; $0 < \varsigma_1 < \cdots < \varsigma_n < a$ are prefixed numbers; $f\colon I \times \mathscr{E} \longrightarrow \mathfrak{V}$, $\rho\colon I \times \mathscr{E} \longrightarrow (-\infty, a], I_i(\cdot)\colon \mathscr{E} \longrightarrow \mathfrak{V}, \Theta_i(\cdot)\colon \mathscr{E} \longrightarrow \mathfrak{V}$ are appropriate functions.

Motivated by the abovementioned works, we derive some sufficient conditions for the existence, approximate controllability, and Ulam-type stability for impulsive integrodifferential equations of second order with state-dependent delay described in the form:

$$
\begin{cases}
\vartheta''(\varsigma) = \mathscr{Z}(\varsigma)\vartheta(\varsigma) + \Psi\left(\varsigma, \vartheta_{\rho(\varsigma, \vartheta_\varsigma)}, (\digamma\vartheta)(\varsigma)\right) + \int_0^{\varsigma} \Upsilon(\varsigma, \varepsilon)\vartheta(\varepsilon)d\varepsilon + \mathscr{P}u(\varsigma), & \text{if } \varsigma \in \widetilde{\Theta},\\
\vartheta(\varsigma_k^+) - \vartheta(\varsigma_k^-) = \nabla_k(\vartheta(\varsigma_k^-)), & k = 1, \ldots, m,\\
\vartheta'(\varsigma_k^+) - \vartheta'(\varsigma_k^-) = \widetilde{\nabla}_k(\vartheta(\varsigma_k^-)), & k = 1, \ldots, m,\\
\vartheta'(0) = \nu_0 \in \mathfrak{V}, \vartheta(\varsigma) = \Phi(\varsigma), & \text{if } \varsigma \in \mathbb{R}_-,
\end{cases} \tag{4}
$$

where $\Theta = [0, T], \widetilde{I} = (-\infty, T]$, $\widetilde{\Theta} = \Theta \backslash \hat{\Theta}_k$, and $\hat{\Theta}_k = \{\varsigma_1, \ldots, \varsigma_m\}$, with $0 = < \varsigma_1 < \varsigma_2 < \ldots < \varsigma_k < \varsigma_{m+1} = T$. $\mathscr{Z}(\varsigma)\colon D(\mathscr{Z}(\varsigma)) \subset \mathfrak{V} \longrightarrow \mathfrak{V}$, $\Upsilon(\varsigma, \varepsilon)$ are closed linear operators on $\mathfrak{V}$, with dense domain $D(\mathscr{Z}(\varsigma))$, which is independent of ς, and $D(\mathscr{Z}(\varepsilon)) \subset D(\Upsilon(\varsigma, \varepsilon))$; the operator $\digamma$ is defined by

$$
(\digamma\vartheta)(\varsigma) = \int_0^T \mathscr{K}(\varsigma, \varepsilon, \vartheta(\varepsilon))\mathrm{d}\varepsilon. \tag{5}
$$

The nonlinear term $\Psi\colon \Theta \times \mathscr{E} \times \mathfrak{V} \longrightarrow \mathfrak{V}, \mathfrak{J}\colon \mathbb{R}_- \longrightarrow \mathfrak{V}$, and $\rho\colon \Theta \times \mathscr{E} \longrightarrow \mathbb{R}$ are given functions. The jumps at the points $\varsigma_k \in (0, T)$ are given by $\vartheta(\varsigma_k^+) - \vartheta(\varsigma_k^-)$ and $\vartheta'(\varsigma_k^+) - \vartheta'(\varsigma_k^-)$, in the states ϑ and ϑ', respectively, where

$\vartheta(\varsigma_k^+), \vartheta(\varsigma_k^-)$ stand for left and right limits of ϑ at ς_k^-. Similarly, $\vartheta'(\varsigma_k^+), \vartheta'(\varsigma_k^-)$ stand for right and left limits of ϑ' at ς_k^-. The jumps at the points ς_k^- are determined by the nonlinear functions $\nabla_k, \widetilde{\nabla}_k\colon \mathfrak{B} \longrightarrow \mathfrak{B}$, where $k = 1, 2, 3, \ldots, m$. The control function u is a given function in the Banach space of admissible control $L^2(\Theta, U)$, where U is also a Banach space. $\mathscr{P}$ is a bounded linear operator from U into $\mathfrak{B}$, and $(\mathfrak{B}, \|\cdot\|)$ is a Banach space.

The work is organized as follows: In section two, we recall some definitions and facts about the resolvent operator, Picard operator, and measure of noncompactness. In section three, we give the existence of mild solutions to the problem (4). Section four is devoted to approximate controllability of mild solution and section five to the generalized Ulam–Hyers–Rassias (U-H-R) stability. In the last section, we present an example to illustrate our main result.

2. Preliminaries

Let $C(\Theta, \mathfrak{B})$ be the Banach space of all continuous functions ϑ mapping Θ into $\mathfrak{B}$. Let $\widetilde{\Theta}_0 = [0, \varsigma_1], \widetilde{\Theta}_k = (\varsigma_k, \varsigma_{k+1}], \bar{\Theta}_k = [\varsigma_k, \varsigma_{k+1}]$ for $k \in \{1, \ldots, m+1\}, u(\varsigma^+) = \lim_{\varsigma \longrightarrow \varsigma^+} u(\varsigma)$. We define the space of piecewise continuous functions:

$$\begin{aligned}\mathscr{W} = PC(\widetilde{I}, \mathfrak{B}) = \Big\{u : \widetilde{I} \longrightarrow \mathfrak{B} : u|_{\mathbb{R}^-} \in \mathscr{E}, u|_{\widetilde{\Theta}_k} \in C\big(\widetilde{\Theta}_k, \mathfrak{B}\big), \text{ such that } u(\varsigma_k^-) \text{ and} \\ u(\varsigma_k^+) \text{ exist and satisfy } u(\varsigma_k^-) = u(\varsigma_k), \quad \text{for } k = 1, \ldots, m\Big\},\end{aligned} \tag{6}$$

with the norm

$$\|\vartheta\|_{\mathscr{W}} = \sup_{\varsigma \in \widetilde{I}}\{\|\vartheta(\varsigma)\|\}. \tag{7}$$

Next, we consider the second-order integrodifferential system [26]:

$$\begin{aligned}&\gamma''(\varsigma) = \mathscr{Z}(\varsigma)\gamma(\varsigma) + \int_\varepsilon^\varsigma \Upsilon(\varsigma, \tau)\gamma(\tau)\mathrm{d}\tau, \quad \varepsilon \le \varsigma \le T \\ &\gamma(\varepsilon) = 0, \gamma'(\varepsilon) = x \in \mathfrak{B},\end{aligned} \tag{8}$$

for $0 \le \varepsilon \le T$. We denote $\nabla = \{(\varsigma, \varepsilon) : 0 \le \varepsilon \le \varsigma \le T\}$. Let:

(B1) For each $0 \le \varepsilon \le \varsigma \le T, \Upsilon(\varsigma, \varepsilon) : D(\mathscr{Z}) \longrightarrow \mathfrak{B}$ is a bounded linear operator, for every $\gamma \in D(\mathscr{Z}), \Upsilon(\cdot, \cdot)\gamma$ is continuous and

$$\|\Upsilon(\varsigma, \varepsilon)\gamma\| \le \iota\|\gamma\|_{[D(\mathscr{Z})]}, \tag{9}$$

for $\iota > 0$, $\varepsilon, \varsigma \in \nabla$.

(B2) There exists $L_\Upsilon > 0$ where

$$\|\Upsilon(\varsigma_2, \varepsilon)\gamma - \Upsilon(\varsigma_1, \varepsilon)\gamma\| \le L_\Upsilon|\varsigma_2 - \varsigma_1|\|\gamma\|_{[D(\mathscr{Z})]}, \tag{10}$$

for all $\gamma \in D(\mathscr{Z}), 0 \le \varepsilon \le \varsigma_1 \le \varsigma_2 \le T$.

(B3) There exists $b_1 > 0$ such that

$$\left\|\int_\sigma^\varsigma S(\varsigma, \varepsilon)\Upsilon(\varepsilon, \sigma)\gamma \mathrm{d}\varepsilon\right\| \le b_1\|\gamma\|, \quad \text{for all } \gamma \in D(\mathscr{Z}). \tag{11}$$

Under these conditions, it has been established that there exists a resolvent operator $(\urcorner(\varsigma, \varepsilon))_{\varsigma \ge \varepsilon}$ associated with systems (2).

Definition 1 (see [26]). A family of bounded linear operators $(\urcorner(\varsigma, \varepsilon))_{\varsigma \ge \varepsilon}$ on $\mathfrak{B}$ is a resolvent operator for (2) if it verifies the following:

(a) The map $\urcorner\colon \nabla \longrightarrow \mathscr{L}(\mathfrak{B})$ is strongly continuous; $\urcorner(\varsigma, \cdot)\gamma$ is continuously differentiable for all $\gamma \in \mathfrak{B}, \urcorner(\varepsilon, \varepsilon) = 0, \partial/\partial\varsigma \urcorner(\varsigma, \varepsilon)|_{\varsigma=\varepsilon} = I$ and $\partial/\partial\varsigma \urcorner(\varsigma, \varepsilon)|_{\varepsilon=\varsigma} = -I$

(b) Assume $\bar{\vartheta} \in D(\mathscr{Z})$. The function $\urcorner(\cdot, \varepsilon)\bar{\vartheta}$ is a solution for systems (6) and (7). Thus,

$$\frac{\partial^2}{\partial\varsigma^2}\urcorner(\varsigma, \varepsilon)\bar{\vartheta} = \mathscr{Z}(\varsigma)\urcorner(\varsigma, \varepsilon)\bar{\vartheta} + \int_\varepsilon^\varsigma \Upsilon(\varsigma, \tau)\urcorner(\tau, \varepsilon)\bar{\vartheta}\mathrm{d}\tau, \tag{12}$$

for all $0 \le \varepsilon \le \varsigma \le T$.

By (a), there are $M_\urcorner > 0$ and $\widetilde{M_\urcorner} > 0$, such that

$$\|\urcorner(\varsigma, \varepsilon)\| \le M_\urcorner, \left\|\frac{\partial}{\partial\varepsilon}\urcorner(\varsigma, \varepsilon)\right\| \le \widetilde{M_\urcorner}, \quad (\varsigma, \varepsilon) \in \nabla. \tag{13}$$

Moreover,

$$G(\varsigma, \tau)\bar{\vartheta} = \int_\tau^\varsigma \Upsilon(\varsigma, \varepsilon)\urcorner(\varepsilon, \tau)\bar{\vartheta}\mathrm{d}\varepsilon, \bar{\vartheta} \in D(\mathscr{Z}), \quad 0 \le \tau \le \varsigma \le T \tag{14}$$

can be extended to $\mathfrak{B}$ where

$$\urcorner(\varsigma, \tau)\bar{\vartheta} = S(\varsigma, \tau) + \int_\tau^\varsigma S(\varsigma, \varepsilon)G(\varepsilon, \tau)\bar{\vartheta}\mathrm{d}\varepsilon, \quad \text{for all } \bar{\vartheta} \in \mathfrak{B}. \tag{15}$$

Then, there exists $L_\urcorner > 0$ where

$$\|\urcorner(\varsigma + h, \tau) - \urcorner(\varsigma, \tau)\| \le L_\urcorner|h|, \quad \text{for all } \varsigma, \varsigma + h, \tau \in [0, T]. \tag{16}$$

Let the state space $(\mathscr{E}, \|\cdot\|_{\mathscr{E}})$ be a seminorm linear space of functions mapping $(-\infty, 0]$ into $\mathbb{R}$, and verifying (see [33]):

(A_1) If $\vartheta \in C$ and $\vartheta_0 \in \mathscr{E}$, then for $\varsigma \in \Theta$:

(i) $\vartheta_\varsigma \in \mathscr{E}$
(ii) There exists $H > 0$ where $|\vartheta(\varsigma)| \leq H\|\vartheta_\varsigma\|_{\mathscr{E}}$
(iii) There exist $\Phi_1(\cdot)$ and $\Phi_2(\cdot)$: $\mathbb{R}_+ \longrightarrow \mathbb{R}_+$ with Φ_1 continuous and bounded and Φ_2 locally bounded where

$$\|\vartheta_\varsigma\|_{\mathscr{E}} \leq \Phi_1(\varsigma)\sup\{|\vartheta(\varepsilon)| : 0 \leq \varepsilon \leq \varsigma\} + \Phi_2(\varsigma)\|\vartheta_0\|_{\mathscr{E}}. \tag{17}$$

(A_2) For the function ϑ in $(A_1), \vartheta_\varsigma$ is a $\mathscr{E}$-valued continuous function on $\mathbb{R}^+$.

(A_3) The space $\mathscr{E}$ is complete. We denote $\Phi_1{}^* = \sup\{\Phi_1(\varsigma): \varsigma \in \Theta\}$, $\Phi_2{}^* = \sup\{\Phi_2(\varsigma): \varsigma \in \Theta\}$, $\aleph = \max\{\Phi_1{}^*, \Phi_2{}^*\}$.

For $I_{\theta_j} = \mathbb{R}\backslash\{\theta_j \in \mathbb{R}^-: \text{if } \varsigma \in \Theta, \text{then } (\theta_j + \varsigma) \in \hat{\Theta}_k\}$, we define the space

$$PC_\theta(\mathbb{R}^-, \mathfrak{B}) = \left\{\vartheta: \mathbb{R}^- \longrightarrow \mathfrak{B}: \vartheta|_{I_{\theta_j}} \text{ is continuous}, \vartheta(\theta_j^-), \quad \vartheta(\theta_j^+) \text{ exist with } \vartheta(\theta_j^-) = \vartheta(\theta_j)\right\}, \tag{18}$$

and the space

$$C_\theta := \left\{\daleth \in PC_\theta(\mathbb{R}^-, \mathfrak{B}): \lim_{\tau \longrightarrow -\infty} \daleth(\tau) \text{ exist in } \mathfrak{B}\right\}. \tag{19}$$

In the following, consider $\mathscr{E} = C_\theta$.

Lemma 2 (see [34]). *Let the following inequality holds:*

$$\daleth(\varsigma) \leq a(\varsigma) + \int_0^\varsigma b(\varepsilon)\daleth(\varepsilon)d\varepsilon + \sum_{0 \leq \varsigma_k < \varsigma} \beta_k \daleth(\varsigma_k^-), \quad \varsigma \geq 0, \tag{20}$$

where

$$\daleth, a, b \in PC(\mathbb{R}^+, \mathbb{R}^+) := \{\vartheta \in PC(\mathbb{R}^+, \mathbb{R}^+), \vartheta(\varsigma) \geq 0\}, \tag{21}$$

a is nondecreasing, $b(\varsigma) > 0, \beta_k > 0, k \in \mathbb{N}$. Then, for $\varsigma \in \mathbb{R}^+$, the following inequality is valid:

$$\daleth(\varsigma) \leq a(\varsigma) \prod_{0 < \varsigma_k < \varsigma} (1 + \beta_k)\exp\left(\int_0^\varsigma b(\varepsilon)\mathrm{d}\varepsilon\right), \quad \varsigma \in [\varsigma_k, \varsigma_{k+1}]. \tag{22}$$

Definition 3 (see [35]). Let $\widehat{\mathfrak{B}}$ be a metric space. $\mathfrak{Y}: \widehat{\mathfrak{B}} \longrightarrow \widehat{\mathfrak{B}}$ is a Picard operator if there exists $\vartheta^* \in \widehat{\mathfrak{B}}$, such that

(i) $\mathbb{F}_{\mathfrak{Y}} = \{\vartheta^*\}$ where $\mathbb{F}_{\mathfrak{Y}} = \{\vartheta \in \widehat{\mathfrak{B}}: \mathfrak{Y}(\vartheta) = \vartheta\}$ is the fixed point set of $\mathfrak{Y}$
(ii) $(\mathfrak{Y}^n(\vartheta_0))_{n\in\mathbb{N}}$ converges to ϑ^* for all $\vartheta_0 \in \widehat{\mathfrak{B}}$

Lemma 4 (see [35]). *Let $(\widehat{\mathfrak{B}}, d, \leq)$ be an ordered metric space and $\mathfrak{Y}$: $\widehat{\mathfrak{B}} \longrightarrow \widehat{\mathfrak{B}}$. We assume the following:*

(i) $\mathfrak{Y}$ is a Picard operator $(\mathbb{F}_{\mathfrak{Y}} = \{\vartheta^*_{\mathfrak{Y}}\})$
(ii) $\mathfrak{Y}$ is an increasing operator

Then, we have

(a) $\vartheta \in \widehat{\mathfrak{B}}, \vartheta \leq \mathfrak{Y}(\vartheta) \Rightarrow \vartheta \leq \vartheta^*_{\mathfrak{Y}}$
(b) $\vartheta \in \widehat{\mathfrak{B}}, \vartheta \geq \mathfrak{Y}(\vartheta) \Rightarrow \vartheta \geq \vartheta^*_{\mathfrak{Y}}$

Definition 5 (see [36]). Let $\widehat{\mathfrak{B}}$ be a Banach space and $\Delta_{\widehat{\mathfrak{B}}}$ be the bounded subsets of $\widehat{\mathfrak{B}}$. The Kuratowski measure of noncompactness is the map ζ: $\Delta_{\widehat{\mathfrak{B}}} \longrightarrow [0, \infty]$ given by

$$\zeta(\Omega) = \inf\{\epsilon > 0: \Omega \subseteq \cup_{i=1}^n \Omega_i \text{ and } \mathrm{diam}(\Omega_i) \leq \epsilon\}; \text{ here } \Omega \subset \Delta_{\widehat{\mathfrak{B}}}, \tag{23}$$

where

$$\mathrm{diam}(\Omega_i) = \sup\{\|u - v\|_X : u, v \in \Omega_i\}. \tag{24}$$

Lemma 6 ([37]). *If $\overline{\mathfrak{B}}$ is a bounded subset of a Banach space $\widehat{\mathfrak{B}}$, then for each $\epsilon > 0$, there is a sequence $\{\vartheta_k\}_{k=1}^\infty \subset \overline{\mathfrak{B}}$ such that*

$$\zeta(\overline{\mathfrak{B}}) \leq 2\zeta(\{\vartheta_k\}_{k=1}^\infty) + \epsilon. \tag{25}$$

Lemma 7 (see [38]). *If $\{\vartheta_k\}_{k=0}^\infty \subset L^1$ is uniformly integrable, then the function $\varsigma \longrightarrow \alpha(\{\vartheta_k(\varsigma)\}_{k=0}^\infty)$ is measurable and*

$$\zeta\left(\left\{\int_0^\varsigma \vartheta_k(\varepsilon)\mathrm{d}\varepsilon\right\}_{k=0}^\infty\right) \leq 2\int_0^\varsigma \zeta(\{\vartheta_k(\varepsilon)\}_{k=0}^\infty)\mathrm{d}\varepsilon. \tag{26}$$

Lemma 8 (see [36]).

(i) If $\overline{\mathfrak{B}} \subset PC(\Theta, \mathfrak{B})$ is bounded, then $\zeta(\overline{\mathfrak{B}}(\varsigma)) \leq \alpha_{PC}(\overline{\mathfrak{B}})$ for any $\varsigma \in \Theta$ where $\overline{\mathfrak{B}}(\varsigma) = \{u(\varsigma): u \in \overline{\mathfrak{B}}\} \subset \mathfrak{B}$.
(ii) If $\overline{\mathfrak{B}}$ is piecewise equicontinuous on Θ, then $\zeta(\overline{\mathfrak{B}}(\varsigma))$ is piecewise continuous for $\varsigma \in \Theta$, and

$$\alpha_{PC}(\overline{\mathfrak{B}}) = \sup\{\zeta(\bar{\mathfrak{B}}(\varsigma)), \varsigma \in \Theta\}. \tag{27}$$

(iii) If $\overline{\mathfrak{B}} \subset PC(\Theta, \mathfrak{B})$ *is bounded and piecewise equicontinuous, then* $\zeta(\overline{\mathfrak{B}}(\varsigma))$ *is piecewise continuous for* $\varsigma \in \Theta$ *and*

$$\zeta\left(\int_a^\varsigma \overline{\mathfrak{B}}(\varepsilon)\mathrm{d}\varepsilon\right) \leq 2\int_a^\varsigma \zeta(\overline{\mathfrak{B}}(\varepsilon))\mathrm{d}\varepsilon \quad \varsigma \in \Theta, \tag{28}$$

where α_{PC} *denotes the Kuratowski measure of noncompactness in the space* $PC(\Theta, \mathfrak{B})$.

Theorem 9 (see [39]). *Let* Δ *be a nonempty, bounded, closed, and convex subset of a Banach space* $\widehat{\mathfrak{B}}$ *and let* $\mathfrak{Y}: \Delta \longrightarrow \Delta$ *be a continuous mapping. Assume that there exists a constant* $k \in [0, 1)$, *such that*

$$\zeta(\mathfrak{Y}M) \leq k\zeta(M), \tag{29}$$

for any nonempty subset M *of* Δ. *Then,* $\mathfrak{Y}$ *has a fixed point in set* Δ.

Theorem 10 (see [40]). *Let* $(\widehat{\mathfrak{B}}, d)$ *be a nonempty complete metric space with a contraction mapping* $\mathfrak{Y}: \widehat{\mathfrak{B}} \longrightarrow \widehat{\mathfrak{B}}$. *Then,* $\mathfrak{Y}$ *admits a unique fixed point* x^* *in* $\widehat{\mathfrak{B}}$.

3. Existence of Mild Solutions

Definition 11. A function $\vartheta \in \widehat{\mathfrak{B}}$ is called a mild solution of problem (1) if it satisfies

$$\vartheta(\varsigma) = \begin{cases} -\dfrac{\partial \daleth(\varsigma, \varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=0} \mathfrak{I}(0) + \daleth(\varsigma, 0)\nu_0 \\ + \displaystyle\int_0^\varsigma \daleth(\varsigma, \varepsilon)\Big(\Psi\Big(\varepsilon, \vartheta_{\rho(\varepsilon,\vartheta_\varepsilon)}, (\digamma\vartheta)(\varepsilon)\Big) + \mathscr{P}u(\varepsilon)\Big)\mathrm{d}\varepsilon \\ - \displaystyle\sum_{0<\varsigma_k<\varsigma} \dfrac{\partial \daleth(\varsigma, \varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=\varsigma_k} \nabla_k(\vartheta(\varsigma_k^-)) + \displaystyle\sum_{0<\varsigma_k<\varsigma} \daleth(\varsigma, \varsigma_k)\widetilde{\nabla}_k(\vartheta(\varsigma_k^-)); & \text{if } \varsigma \in \Theta, \\ \mathfrak{I}(\varsigma); & \text{if } \varsigma \in \mathbb{R}_-. \end{cases} \tag{30}$$

The following assumption will be needed throughout the paper:

$(C1)$ $\Psi: \Theta \times \mathscr{E} \times \mathfrak{B} \longrightarrow \mathfrak{B}$ is a Carathéodory function, and there exist positive constants ξ_1, ξ_2 and continuous nondecreasing functions $\psi_\Psi^1, \psi_\Psi^2: \Theta \longrightarrow (0, +\infty)$ such that

$$\|\Psi(\varsigma, \vartheta_1, \vartheta_2) - \Psi(\varsigma, \vartheta_3, \vartheta_4)\| \leq \xi_1\psi_\Psi^1\left(\|\vartheta_1 - \vartheta_3\|_{\mathscr{E}}\right) + \xi_2\psi_\Psi^2\left(\|\vartheta_2 - \vartheta_4\|\right), \tag{31}$$

for $\vartheta_1, \vartheta_3 \in \mathscr{E}, \vartheta_2, \vartheta_4 \in \mathfrak{B}$. There exists a positive constant l_Ψ, such that for any bounded set $\Omega \subset \mathfrak{B}$ and $\Omega_\varsigma \in \mathscr{E}$, and each $\varsigma \in \mathbb{R}$, we have

$$\zeta\left(\Psi\left(\varsigma, \Omega_\varsigma, \digamma(\Omega(\varsigma))\right)\right) \leq l_\Psi\left(\zeta(\Omega(\varsigma)) + \sup_{\tau \in (-\infty, 0]} \zeta(\Omega(\tau + \varsigma))\right), \tag{32}$$

with

$$\Psi^* = \int_0^{\varsigma} \Psi(\varepsilon, 0, 0) d\varepsilon < \infty,\ \psi_{\Psi}^1(\varsigma) \le \varsigma,$$
$$\psi_{\Psi}^2(\varsigma) \le \varsigma. \quad (33)$$

($C2$) The function $\mathscr{K}\colon D_{\mathscr{K}} \times \mathfrak{B} \longrightarrow \mathfrak{B}$ is continuous, and there exists $\mathscr{K}_{c_1} > 0$, such that

$$\left\| \mathscr{K}(\varsigma, \varepsilon, \vartheta_1) - \mathscr{K}(\varsigma, \varepsilon, \vartheta_2) \right\| \le \mathscr{K}_{c_1} \left\| \vartheta_1 - \vartheta_2 \right\|, \text{ for each } (\varsigma, \varepsilon) \in D_{\mathscr{K}},$$
$$\vartheta_1, \vartheta_2 \in \mathfrak{B}. \quad (34)$$

Let

$$\sup_{D_{\mathscr{K}}} \{\|\mathscr{K}(\varsigma, \varepsilon, 0)\|\} = \mathscr{K}^* < \infty. \quad (35)$$

($C3$) Assume that $(B1) - (B3)$ hold, and there exist $M_{\urcorner}, \widetilde{M_{\urcorner}} \ge 1$, $\zeta \ge 0$, and $M_{\mathscr{P}} > 0$, such that

$$\|\urcorner(\varsigma, \varepsilon)\|_{\Upsilon(\mathfrak{B})} \le M_{\urcorner}, \left\| \frac{\partial \urcorner(\varsigma, \varepsilon)}{\partial \varepsilon} \right\|_{\Upsilon(\mathfrak{B})} \le \widetilde{M_{\urcorner}},$$
$$\|\mathscr{P}\| = M_{\mathscr{P}}. \quad (36)$$

($C4$) The functions Q_k^i, $\jmath\mathfrak{B} \longrightarrow \mathfrak{B}$ are continuous, and there exist positive constants $L_{Q_k^i}$; $k = 1, \ldots, m$, such that

$$\left\| Q_k^i(\vartheta_3) - Q_k^i(\vartheta_4) \right\| \le \frac{L_{Q_k^i}}{\tau} \left\| \vartheta_3 - \vartheta_4 \right\|, \text{ for all } \vartheta_3, \vartheta_4 \in \mathfrak{B}, \mathrm{k} = 1, \ldots, \mathrm{m}, \quad (37)$$

and

$$\sum_{1 \le k \le m+1} \left\| Q_k^i \right\| (0) = Q_i^* < +\infty, \quad (38)$$

where

$$Q_k^i = \begin{cases} \nabla_k & \text{if } i = 1, \\ \widetilde{\nabla}_k & \text{if } i = 2. \end{cases} \quad (39)$$

(C_H) Set $\urcorner(\rho^-) = \{\rho(\varepsilon, \varphi)\colon (\varepsilon, \varphi) \in \Theta \times \mathscr{E}, \rho(\varepsilon, \varphi) \le 0\}$. We assume that $\rho\colon \Theta \times \mathscr{E} \longrightarrow \mathbb{R}$ is continuous. Moreover, we assume the following assumption:

(i) ($H_{\mathfrak{J}}$) The function $\varsigma \longrightarrow \mathfrak{J}_{\varsigma}$ is continuous from $\urcorner(\rho^-)$ into $\mathscr{E}$, and there exists a continuous and bounded function $L^{\mathfrak{J}}\colon \urcorner(\rho^-) \longrightarrow (0, \infty)$ such that

$$\left\| \mathfrak{J}_{\varsigma} \right\|_{\mathscr{E}} \le L^{\mathfrak{J}}(\varsigma) \|\mathfrak{J}\|_{\mathscr{E}}, \text{ for every } \varsigma \in \urcorner(\rho^-). \quad (40)$$

Remark 12 (see [41]). The condition ($H_{\mathfrak{J}}$) is verified by functions continuous and bounded.

Lemma 13 (see [42]). *If* $\vartheta\colon (-\infty, +\infty) \longrightarrow \mathfrak{B}$ *is a function such that* $\vartheta_0 = \mathfrak{J}$, *then*

$$\left\| \vartheta_{\varepsilon} \right\|_{\mathscr{E}} \le \left({\Phi_2}^* + \mathscr{L}^{\mathfrak{J}} \right) \|\mathfrak{J}\|_{\mathscr{E}} + {\Phi_1}^* \sup\{|\vartheta(\theta)|; \theta \in [0, \max\{0, \varepsilon\}]\}, \quad \varepsilon \in \urcorner(\rho^-) \cup \Theta, \quad (41)$$

where $\mathscr{L}^{\mathfrak{J}} = \sup_{\varsigma \in \urcorner(\rho^-)} \mathscr{L}^{\mathfrak{J}}(\varsigma)$.

Now, we define a measure of noncompactness in the space $\mathscr{W}$. Let us fix a nonempty bounded subset S of the space $\mathscr{W}$ and $\times = \widetilde{\Theta}_k \cup \{I_{\theta_k} \cap [-T, 0]\}$. Then, for $\nu \in S$, $\epsilon > 0, \kappa_1, \kappa_2 \in \times$, such that $|\kappa_1 - \kappa_2| \le \epsilon$, we denote $\omega^T(\nu, \epsilon)$ the modulus of continuity of the function ν on $\times$, namely,

$$\omega^T(\nu, \epsilon) = \sup\left\{ \left\| e^{-\kappa_1} \nu(\kappa_1) - e^{-\kappa_2} \nu(\kappa_2) \right\|; \kappa_1, \kappa_2 \in \Theta \right\},$$
$$\omega^T(S, \epsilon) = \sup\left\{ \omega^T(\nu, \epsilon); \nu \in S \right\},$$
$$\omega_0(S) = \lim_{\epsilon \longrightarrow 0} \left\{ \omega^T(S, \epsilon) \right\}. \quad (42)$$

Consider the function χ_{PC} defined on the family of subset of $\mathscr{W}$ by

$$\chi_{\mathrm{PC}}(S) = \omega_0(S) + \sup\left\{e^{-\tau\Sigma(\varsigma)}\zeta(S(\varsigma))\right\}, \tag{43}$$

with $\tau > 1 + \sum_{k\geq 0}(\widetilde{M_\daleth}L_{Q_k^1} + M_\daleth L_{Q_k^2}), \Sigma(\varsigma) = 8M_\daleth l_\Psi \varsigma$ and $S(\varsigma) = \{v(\varsigma) \in \mathfrak{B}\,;\, v \in S\}$.

The function χ_{PC} is a sublinear measure of noncompactness on the space $\mathscr{W}$. For details on the definition and properties of the measure of noncompactness on the space of piecewise continuous functions PC, the reader is referred to [43].

Theorem 14. *Suppose that* $(C1)-(C4)$ *and* (C_H) *are verified. Then, (1) has at least one mild solution.*

Proof. We transform problem (1) into a fixed-point problem and define the operator $\Bbbk$: $\mathscr{W} \longrightarrow \mathscr{W}$ by

$$\Bbbk\vartheta(\varsigma) = \begin{cases} -\left.\dfrac{\partial\daleth(\varsigma,\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0}\mathfrak{I}(0) + \daleth(\varsigma,0)v_0 \\ + \displaystyle\int_0^\varsigma \daleth(\varsigma,\varepsilon)\Big(\Psi\Big(\varepsilon,\vartheta_{\rho(\varepsilon,\vartheta_\varepsilon)},(F\vartheta)(\varepsilon)\Big) + \mathscr{P}u(\varepsilon)\Big)\mathrm{d}\varepsilon \\ - \displaystyle\sum_{0<\varsigma_k<\varsigma}\left.\dfrac{\partial\daleth(\varsigma,\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=\varsigma_k}\nabla_k(\vartheta(\varsigma_k^-)) + \displaystyle\sum_{0<\varsigma_k<\varsigma}\daleth(\varsigma,\varsigma_k)\widetilde{\nabla}_k(\vartheta(\varsigma_k^-)); & \text{if } \varsigma\in\Theta, \\ \mathfrak{I}(\varsigma); & \text{if } \varsigma\in\mathbb{R}_-. \end{cases} \tag{44}$$

Let $x:(-\infty,T] \longrightarrow \mathfrak{B}$ be the function defined by

$$x(\varsigma) = \begin{cases} -\left.\dfrac{\partial\daleth(\varsigma,\varepsilon)\mathfrak{I}(0)}{\partial\varepsilon}\right|_{\varepsilon=0} + \daleth(\varsigma,0)v_0, & \text{if } \varsigma\in\Theta, \\ \mathfrak{I}(\varsigma), & \text{if } \varsigma\in\mathbb{R}_-. \end{cases} \tag{45}$$

Then, $x_0 = \mathfrak{I}$, and for each $w \in \mathscr{W}$, with $w(0) = 0$, we denote the function $\bar{w}$ by

$$\bar{w}(\varsigma) = \begin{cases} w(\varsigma), & \text{if } \varsigma\in\mathbb{R}^+, \\ 0, & \text{if } \varsigma\in\mathbb{R}_-. \end{cases} \tag{46}$$

If ϑ satisfies (3), we can decompose it as $\vartheta(\varsigma) = w(\varsigma) + x(\varsigma)$, which implies $\vartheta_\varsigma = w_\varsigma + x_\varsigma$, and the function $w(\cdot)$ satisfies

$$w(\varsigma) = \begin{cases} \displaystyle\int_0^\varsigma \daleth(\varsigma,\varepsilon)\Big(\Psi\Big(\varepsilon, w_{\rho(\varepsilon,w_\varepsilon+x_\varepsilon)} + x_{\rho(\varepsilon,w_\varepsilon+x_\varepsilon)}, F(w+x)(\varepsilon)\Big) + \mathscr{P}u(\varepsilon)\Big)\mathrm{d}\varepsilon \\ - \displaystyle\sum_{0<\varsigma_k<\varsigma}\left.\dfrac{\partial\daleth(\varsigma,\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=\varsigma_k}\nabla_k((w+x)(\varsigma_k^-)) \\ + \displaystyle\sum_{0<\varsigma_k<\varsigma}\daleth(\varsigma,\varsigma_k)\widetilde{\nabla}_k((w+x)(\varsigma_k^-)); & \text{if } \varsigma\in\Theta, \\ \mathfrak{I}(\varsigma); & \text{if } \varsigma\in\mathbb{R}_-. \end{cases} \tag{47}$$

Set

$$\Delta = \{w \in \mathscr{W}: w(0) = 0\}. \tag{48}$$

Let the operator $\widetilde{\Bbbk}: \Delta \longrightarrow \Delta$ be defined by

$$\widetilde{\Bbbk} w(\varsigma) = \begin{cases} \int_0^{\varsigma} \urcorner(\varsigma, \varepsilon)\Big(\Psi\Big(\varepsilon, w_{\rho(\varepsilon, w_\varepsilon + x_\varepsilon)} + x_{\rho(\varepsilon, w_\varepsilon + x_\varepsilon)}, F(w+x)(\varepsilon)\Big) + \mathscr{P}u(\varepsilon)\Big)d\varepsilon \\ - \sum_{0<\varsigma_k<\varsigma} \left.\dfrac{\partial \urcorner(\varsigma, \varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=\varsigma_k} \nabla_k\left((w+x)(\varsigma_k^-)\right) \\ + \sum_{0<\varsigma_k<\varsigma} \urcorner(\varsigma, \varsigma_k)\widetilde{\nabla}_k\left((w+x)(\varsigma_k^-)\right); & \text{if } \varsigma \in \Theta, \\ \mathfrak{J}(\varsigma); & \text{if } \varsigma \in \mathbb{R}_-. \end{cases} \tag{49}$$

The operator $\Bbbk$ has a fixed point which is equivalent to say that $\widetilde{\Bbbk}$ has one, so it turns to prove that $\widetilde{\Bbbk}$ has a fixed point. We shall check that the operator $\widetilde{\Bbbk}$ satisfies all conditions of Darbo's theorem.

Let $\Pi_{\theta'} = \{w \in \Delta: \|w\|_\Delta \leq \theta'\}$, with

$$\begin{aligned} &M_{\urcorner}\Big(\Big(\xi_1 \psi_\Psi^1(\eta_{\theta'}^*) + \xi_2 \psi_\Psi^2\big(\bar{\eta}^*\big) + \Psi^*\Big)T + M_{\mathscr{P}} T^{1/2}\|u\|_{L^2} \\ &+ \sum_{k\geq 0}\Big(L_{Q_k^i} + L_{Q_k^2}\Big) + Q_1^* + Q_2^*\Big) \leq \theta', \end{aligned} \tag{50}$$

such that $\eta_{\theta'}^*, \bar{\eta}^*$ are constants, they will be specific later.

The set $\Pi_{\theta'}$ is bounded, closed, and convex.

Step 1: $\widetilde{\Bbbk}(\Pi_{\theta'}) \subset \Pi_{\theta'}$.

For $w \in \Pi_{\theta'}$, $\varsigma \in \Theta$ and by $(C1)-(C3)$, we have

$$\begin{aligned} &\left\|w_{\rho(\varepsilon, w_\varepsilon + x_\varepsilon)} + x_{\rho(\varepsilon, w_\varepsilon + x_\varepsilon)}\right\|_{\mathscr{C}} \\ &\leq \left\|w_{\rho(\varepsilon, w_\varepsilon + x_\varepsilon)}\right\|_{\mathscr{C}} + \left\|x_{\rho(\varepsilon, w_\varepsilon + x_\varepsilon)}\right\|_{\mathscr{C}} \\ &\leq \Phi_1(\varsigma)\sup_{[0,\varepsilon]}|w(\varsigma)| + \Big(\Phi_2(\varsigma) + \mathscr{L}^{\mathfrak{J}}\Big)\|\mathfrak{J}\|_{\mathscr{C}} + \Phi_1(\varsigma)\sup_{[0,\varepsilon]}\|x(\theta)\| \\ &\leq \Phi_1^*\theta' + \Big(\Phi_2^* + \mathscr{L}^{\mathfrak{J}}\Big)\left\|\mathfrak{J}_{\mathscr{C}} + \Phi_1^*\Big(\widetilde{M_{\urcorner}}\|\mathfrak{J}_0\| + M_{\urcorner}\|\nu_0\|\Big)H\right\|\mathfrak{J}_{\mathscr{C}} \\ &\leq \Phi_1^*\theta' + \Big[\Phi_2^* + \mathscr{L}^{\mathfrak{J}} + \Phi_1^*\Big(\widetilde{M_{\urcorner}}\|\mathfrak{J}_0\| + M_{\urcorner}\|\nu_0\|\Big)H\Big]\|\mathfrak{J}\|_{\mathscr{C}} \\ &= \eta_{\theta'}^*, \end{aligned} \tag{51}$$

and

$$\|F(w+x)(\varepsilon)\| \leq a\mathscr{K}_{c_1}\Big(\theta' + \widetilde{M_{\urcorner}}\|\mathfrak{J}_0\| + M_{\urcorner}\|\nu_0\|\Big) + a\mathscr{K}^* = \bar{\eta}^*. \tag{52}$$

Then,

$$\left\|\widetilde{\mathbb{k}}w(\varsigma)\right\| \leq M_{\neg}\Big(\big(\xi_1\psi_\Psi^1(\eta_{\theta'}^*) + \xi_2\psi_\Psi^2(\overline{\eta}^*)\big)T + \Psi^* + M_{\mathscr{P}}T^{1/2}\|u\|_{L^2} + \sum_{k\geq 0}\Big(L_{Q_k^i} + L_{Q_k^2}\Big) + Q_1^* + Q_2^*\Big). \tag{53}$$

Thus,

$$\left\|\widetilde{\mathbb{k}}w\right\|_\Delta \leq \theta'. \tag{54}$$

Therefore, $\widetilde{\mathbb{k}}(\Pi_{\theta'}) \subset \Pi_{\theta'}$, which implies that $\widetilde{\mathbb{k}}(\Pi_{\theta'})$ is bounded.

Step 2: $\widetilde{\mathbb{k}}$ is continuous.

Let $\{w_m\}_{m\in\mathbb{N}}$ be a sequence such that $w_m \longrightarrow \beth^*$ in $\Pi_{\theta'}$. At the first, we study the convergence of the sequences $(w^m_{\rho(\varepsilon,w_\varepsilon^m)})_{m\in\mathbb{N}}, \varepsilon \in \Theta$. If $\varepsilon \in \Theta$ is such that $\rho(\varepsilon, w_\varepsilon) > 0$, then we have

$$\begin{aligned}\left\|w^m_{\rho(\varepsilon,w_\varepsilon^m)} - \beth^*_{\rho(\varepsilon,\beth_\varepsilon^*)}\right\|_{\mathscr{C}} &\leq \left\|w^m_{\rho(\varepsilon,w_\varepsilon^n)} - \beth^*_{\rho(\varepsilon,w_\varepsilon^m)}\right\|_{\mathscr{C}} + \left\|\beth^*_{\rho(\varepsilon,w_\varepsilon^m)} - \beth^*_{\rho(\varepsilon,\beth_\varepsilon^*)}\right\|_{\mathscr{C}}\\ &\leq \Phi_1{}^*\left\|w_m - \beth^*\right\| + \left\|\beth^*_{\rho(\varepsilon,w_\varepsilon^n)} - \beth^*_{\rho(\varepsilon,\beth_\varepsilon^*)}\right\|_{\mathscr{C}},\end{aligned} \tag{55}$$

which proves that $w^m_{\rho(\varepsilon,w_\varepsilon^m)} \longrightarrow \beth^*_{\rho(\varepsilon,w_\varepsilon)}$ in $\mathscr{C}$, as $m \longrightarrow \infty$, for $\varepsilon \in \Theta$ where $\rho(\varepsilon, w_\varepsilon) > 0$. If $\rho(\varepsilon, w_\varepsilon) < 0$, we get

$$\left\|w^m_{\rho(\varepsilon,w_\varepsilon^m)} - \beth^*_{\rho(\varepsilon,w_\varepsilon)}\right\|_{\mathscr{C}} = \left\|\mathfrak{I}^m_{\rho(\varepsilon,w_\varepsilon^m)} - \mathfrak{I}_{\rho(\varepsilon,\beth_\varepsilon^*)}\right\|_{\mathscr{C}} = 0, \tag{56}$$

which also shows that $w^m_{\rho(\varepsilon,w_\varepsilon^m)} \longrightarrow \beth^*_{\rho(\varepsilon,w_\varepsilon)}$ in $\mathscr{C}$, as $m \longrightarrow \infty$, for every $\varepsilon \in \Theta$ such that $\rho(\varepsilon, w_\varepsilon) < 0$. Then, for $\varsigma \in \Theta$, we have

$$\begin{aligned}\left\|\left(\widetilde{\mathbb{k}}w^m\right)(\varsigma) - \left(\widetilde{\mathbb{k}}w^*\right)(\varsigma)\right\| \leq M_{\neg}\int_0^\varsigma &\| \Psi\Big(\varepsilon, w^m_{\rho(\varepsilon,w_\varepsilon^m)} + x_{\rho(\varepsilon,w_\varepsilon^m+x_\varepsilon)}, H(w^m + x)(\varepsilon)\Big)\\ &- \Psi\Big(\varepsilon, \Big(w^*_{\rho(\varepsilon,w_\varepsilon^*)} + x_{\rho(\varepsilon,w_\varepsilon^*+x_\varepsilon)}\Big), F(\beth^* + x)(\varepsilon)\Big)\Big\|d\varepsilon\\ &+ \sum_{0<\varsigma_k<\varsigma}\left\|\nabla_k(w_n(\varsigma)) - \nabla_k(w^*(\varsigma))\right\|\\ &+ \sum_{0<\varsigma_k<\varsigma}\left\|\widetilde{\nabla}_k(w_n(\varsigma)) - \widetilde{\nabla}_k(w^*(\varsigma))\right\|.\end{aligned} \tag{57}$$

Since $\mathscr{K}$ and Ψ are continuous, we obtain

$$\mathscr{K}(\varsigma, \varepsilon, (w^m + x)(\varepsilon)) \longrightarrow \mathscr{K}(\varsigma, \varepsilon, (\beth^* + x)(\varepsilon)), \quad \text{as } m \longrightarrow +\infty, \tag{58}$$

and

$$\left\|\mathscr{K}(\varsigma, \varepsilon, (w^m + x)(\varepsilon)) - \mathscr{K}(\varsigma, \varepsilon, (\beth^* + x)(\varepsilon))\right\| \leq \mathscr{K}^*_{c_1}\left\|w^m(\varepsilon) - \beth^*(\varepsilon)\right\|. \tag{59}$$

By the Lebesgue-dominated convergence theorem,

$$\int_0^{\varsigma} \mathscr{K}(\varsigma,\varepsilon,(w^m+x)(\varepsilon))\,d\varepsilon \underset{[m\longrightarrow+\infty]}{\longrightarrow} \int_0^{\varsigma} \mathscr{K}(\varsigma,\varepsilon,(ב^*+x)(\varepsilon))\mathrm{d}\varepsilon. \tag{60}$$

Then, by $(C1)$, we get

$$\begin{aligned}&\Psi\Big(\varepsilon, w^m_{\rho(\varepsilon,w^m_\varepsilon)} + x_{\rho(\varepsilon,w^m_\varepsilon+x_\varepsilon)}, F(w^m+x)(\varepsilon)\Big)\\&\underset{[m\longrightarrow+\infty]}{\longrightarrow} \Psi\Big(\varepsilon,\Big(w^*_{\rho(\varepsilon,w^*_\varepsilon)} + x_{\rho(\varepsilon,w^*_\varepsilon+x_\varepsilon)}\Big), F(ב^*+x)(\varepsilon)\Big).\end{aligned} \tag{61}$$

Since $\nabla_k,\widetilde{\nabla}_k$ are continuous, by the Lebesgue-dominated convergence theorem, we obtain

$$\left\|\left(\widetilde{\mathbb{k}}w^m\right)(\varsigma)-\left(\widetilde{\mathbb{k}}w^*\right)(\varsigma)\right\| \longrightarrow 0,\quad \text{as } m\longrightarrow +\infty. \tag{62}$$

Thus, $\widetilde{\mathbb{k}}$ is continuous.

Step 3: $\widetilde{\mathbb{k}}$ is ζ_C-contraction.

Let Π be a bounded equicontinuous subset of $\Pi_{\theta'}$, $w\in\Pi$, and $\kappa_1,\kappa_2\in\Theta$, with $\kappa_2>\kappa_1$, we have

$$\begin{aligned}
&\left\|\widetilde{\mathbb{k}}w(\kappa_1)-\widetilde{\mathbb{k}}w(\kappa_2)\right\|\\
&\le \int_{\kappa_1}^{\kappa_2}\|\urcorner(\kappa_2,\varepsilon)\|\Big(\Big\|\Psi\Big(\varepsilon,w_{\rho(\varepsilon,w_\varepsilon+x_\varepsilon)}+x_{\rho(\varepsilon,w_\varepsilon+x_\varepsilon)},F(w+x)(\varepsilon)\Big)\Big\|+\|\mathscr{P}u(\varepsilon)\|\Big)\mathrm{d}\varepsilon\\
&\quad+\int_0^{\kappa_1}\|\urcorner(\kappa_2,\varepsilon)-\urcorner(\kappa_1,\varepsilon)\|\Big(\Big\|\Psi\Big(\varepsilon,w_{\rho(\varepsilon,w_\varepsilon+x_\varepsilon)}+x_{\rho(\varepsilon,w_\varepsilon+x_\varepsilon)},F(w+x)(\varepsilon)\Big)\Big\|\\
&\quad+\|\mathscr{P}u(\varepsilon)\|\Big)\mathrm{d}\varepsilon+\sum_{0<\varsigma_k<\kappa_1}\left\|\frac{\partial\urcorner(\kappa_1,\varsigma_k)}{\partial\varepsilon}-\frac{\partial\urcorner(\kappa_2,\varsigma_k)}{\partial\varepsilon}\right\|\left\|\nabla_k(w+x)(\varsigma_k^-)\right\|\\
&\quad+\sum_{0<\varsigma_k<\kappa_1}\|\urcorner(\kappa_1,\varsigma_k)-\urcorner(\kappa_2,\varsigma_k)\|\left\|\widetilde{\nabla}_k(w+x)(\varsigma_k^-)\right\|\\
&\quad+\widetilde{M}_\urcorner\sum_{\kappa_1<\varsigma_k<\kappa_2}\left\|\nabla_k(w+x)(\varsigma_k^-)\right\|+M_\urcorner\sum_{\kappa_1<\varsigma_k<\kappa_2}\left\|\widetilde{\nabla}_k(w+x)(\varsigma_k^-)\right\|\\
&\le\left[\psi^1_\Psi(\eta^*_{\theta'})\xi_1+\psi^2_\Psi(\overline{\eta}^*)\xi_2\right]\Big(M_\urcorner|\kappa_2-\kappa_1|+\int_0^{\kappa_1}\|\urcorner(\kappa_2,\varepsilon)-\urcorner(\kappa_1,\varepsilon)\|\mathrm{d}\varepsilon\Big)\\
&\quad+\int_{\kappa_1}^{\kappa_2}\|\mathscr{K}(\varepsilon,0,0)\|\mathrm{d}\varepsilon+M_{\mathscr{P}}\Big(\int_0^{\varsigma}\|\urcorner(\kappa_1,\varepsilon)-\urcorner(\kappa_2,\varepsilon)\|^2\Big)^{1/2}\|u\|_{L^2}\\
&\quad+M_\urcorner M_{\mathscr{P}}(\kappa_2-\kappa_1)^{1/2}\|u\|_{L^2}+\sum_{0<\varsigma_k<\kappa_1}\left\|\frac{\partial\urcorner(\kappa_1,\varsigma_k)}{\partial\varepsilon}-\frac{\partial\urcorner(\kappa_2,\varsigma_k)}{\partial\varepsilon}\right\|\left\|\nabla_k(w+x)(\varsigma_k^-)\right\|\\
&\quad+\sum_{0<\varsigma_k<\kappa_1}\|\urcorner(\kappa_1,\varsigma_k)-\urcorner(\kappa_2,\varsigma_k)\|\left\|\widetilde{\nabla}_k(w+x)(\varsigma_k^-)\right\|\\
&\quad+\widetilde{M}_\urcorner\sum_{\kappa_1<\varsigma_k<\kappa_2}\left\|\nabla_k(w+x)(\varsigma_k^-)\right\|+M_\urcorner\sum_{\kappa_1<\varsigma_k<\kappa_2}\left\|\widetilde{\nabla}_k(w+x)(\varsigma_k^-)\right\|.
\end{aligned}\tag{63}$$

By the strong continuity of $\urcorner$, we get

$$\left\|\widetilde{\mathbb{k}}w(\kappa_1)-\widetilde{\mathbb{k}}w(\kappa_2)\right\|\longrightarrow 0, \text{as } \kappa_1\longrightarrow\kappa_2. \tag{64}$$

Thus, $\widetilde{\mathbb{k}}(\Pi)$ is equicontinuous; then, $\omega_0(\widetilde{\mathbb{k}}(\Pi))=0$.

Now, for $w\in\Pi$, and for any $\varrho>0$, there exists a sequence $\{w^k\}_{k=0}^{\infty}\subset\Pi$ such that for $\varsigma\in\Theta$, we have

$$\begin{aligned}
\zeta(\widetilde{\mathbb{k}}(\Pi)(\varsigma)) &\leq \zeta\left(\left\{\int_0^\varsigma \urcorner(\varsigma,\varepsilon)\Psi\left(\varepsilon, w_{\rho(\varepsilon,w_\varepsilon)}+x_{\rho(\varepsilon,w_\varepsilon+x_\varepsilon)}, F(w+x)(\varepsilon)\right)\mathrm{d}\varepsilon;\ w\in\Pi\right\}\right)\\
&\quad+\zeta\left(\left\{\sum_{0<\varsigma_k<\varsigma}\left(-\frac{\partial\urcorner(\varsigma,\varsigma_k)}{\partial\varepsilon}\nabla_k((w+x)(\varsigma_k^-))\right.\right.\right.\\
&\quad\left.\left.\left.+\urcorner(\varsigma,\varsigma_k)\widetilde{\nabla}_k((w+x)(\varsigma_k^-))\right);\ w\in\Pi\right\}\right)\\
&\leq 2\zeta\left(\left\{\int_0^\varsigma \urcorner(\varsigma,\varepsilon)\Psi\left(\varepsilon, w^k_{\rho(\varepsilon,w^k_\varepsilon)}+x_{\rho(\varepsilon,w^k_\varepsilon+x_\varepsilon)}, F\left(w^k+x\right)(\varepsilon)\right)\mathrm{d}\varepsilon;\ w\in\Pi\right\}\right)\\
&\quad+\sum_{k\geq 0}\left(\widetilde{M}_\urcorner L_{Q_k^1}+M_\urcorner L_{Q_k^2}\right)\zeta(\Pi(\varsigma))+\varrho\\
&\leq \int_0^\varsigma 4M_\urcorner l_\Psi\left(\zeta(\Pi(\theta))+\sup_{\tau\in(-\infty,0]}\zeta(\Pi(\tau+\varepsilon))\right)\mathrm{d}\varepsilon\\
&\quad+\frac{1}{\tau}\sum_{k\geq 0}\left(\widetilde{M}_\urcorner L_{Q_k^1}+M_\urcorner L_{Q_k^2}\right)\zeta(\Pi(\varsigma))+\varrho\\
&\leq \int_0^\varsigma e^{8\tau M_\urcorner l_\Psi\varepsilon}e^{-8\tau M_\urcorner l_\Psi\varepsilon}8M_\urcorner l_\Psi\zeta(\Pi(\varepsilon))\mathrm{d}\varepsilon\\
&\quad+\frac{1}{\tau}\sum_{k\geq 0}\left(\widetilde{M}_\urcorner L_{Q_k^1}+M_\urcorner L_{Q_k^2}\right)\zeta(\Pi(\varsigma))+\varrho\\
&\leq \int_0^\varsigma 8M_\urcorner l_\Psi e^{8\tau M_\urcorner l_\Psi\varepsilon}\sup_{\varepsilon\in[0,\varsigma]}e^{-8\tau M_\urcorner l_\Psi\varepsilon}\zeta(\Pi(\varepsilon))\mathrm{d}\varepsilon\\
&\quad+\frac{1}{\tau}\sum_{k\geq 0}\left(\widetilde{M}_\urcorner L_{Q_k^1}+M_\urcorner L_{Q_k^2}\right)\zeta(\Pi(\varsigma))+\varrho\\
&\leq \zeta_{PC}(\Pi)\int_0^\varsigma\left(\frac{e^{8\tau M_\urcorner l_\Psi\varepsilon}}{\tau}\right)'d\varepsilon+\frac{1}{\tau}\sum_{k\geq 0}\left(\widetilde{M}_\urcorner L_{Q_k^1}+M_\urcorner L_{Q_k^2}\right)\zeta(\Pi(\varsigma))+\varrho\\
&\leq\frac{e^{8\tau M_\urcorner l_\Psi\varsigma}}{\tau}\left(1+\sum_{k\geq 0}\left(\widetilde{M}_\urcorner L_{Q_k^1}+M_\urcorner L_{Q_k^2}\right)\right)\zeta_{PC}(\Pi)+\varrho.
\end{aligned}\tag{65}$$

Since ϱ is arbitrary, we get

$$\zeta(\widetilde{\mathbb{k}}(\Pi)(\varsigma))\leq\frac{e^{8\tau M_\urcorner l_\Psi\varsigma}}{\tau}\left(1+\sum_{k\geq 0}\left(\widetilde{M_\urcorner}L_{Q_k^1}+M_\urcorner L_{Q_k^2}\right)\right)\zeta_{PC}(\Pi). \tag{66}$$

Thus,

$$\zeta_{PC}(\widetilde{\mathbb{k}}(\Pi))\leq\frac{1}{\tau}\left(1+\sum_{k\geq 0}\left(\widetilde{M_\urcorner}L_{Q_k^1}+M_\urcorner L_{Q_k^2}\right)\right)\zeta_{PC}(\Pi). \tag{67}$$

By Theorem 9, it follows that there exists at least one fixed point $\beth^*$ within $\widetilde{\Bbbk}$. Consequently, the point $\beth^* + x$ is a fixed point for the operator $\Bbbk$, which is a mild solution to (1). ☐

4. Controllability Results

Definition 15. The reachable set of system (1) is given by

$$\mathcal{S}_T(\Psi) = \{\vartheta(T) \in \mathfrak{B} : (\vartheta) \text{ represents a mild solution of system (1)}\}. \quad (68)$$

In case $\Psi \equiv 0$, system (1) reduces to the corresponding linear system. The reachable set in this case is denoted by $\mathcal{S}_c(0)$.

Definition 16. If $\overline{\mathcal{S}_T(\Psi)} = \mathfrak{B}$, then the semilinear control system is approximately controllable on $[0, T]$. Here, $\overline{\mathcal{S}_T(\Psi)}$ represents the closure of $\mathcal{S}_T(\Psi)$. Clearly, if $\overline{\mathcal{S}_T(0)} = \mathfrak{B}$, then the linear system is approximately controllable.

We define the operator $\mathcal{N}: \mathfrak{B} = L^2(\Theta, \mathfrak{B}) \longrightarrow \mathfrak{B}$ as follows:

$$\mathcal{N}\vartheta(\varsigma) = \Psi\left(\varsigma, \vartheta_{\rho(\varsigma,\vartheta_\varsigma)}, (F\vartheta)(\varsigma)\right); \quad 0 < \varsigma \leq T. \quad (69)$$

It is demonstrated that the approximate controllability of the linear system extends from the semilinear system, given certain conditions on the nonlinear component. Let us now consider the ensuing linear system:

$$\begin{cases} \jmath''(\varsigma) = \mathcal{Z}(\varsigma)\jmath(\varsigma) + \int_0^\varsigma \Upsilon(\varsigma,\varepsilon)\jmath(\varepsilon)d\varepsilon + \mathcal{P}v(\varsigma), & \text{if } \varsigma \in \widetilde{\Theta}, \\ \jmath(\varsigma_k^+) - \jmath(\varsigma_k^-) = \nabla_k(\jmath(\varsigma_k^-)), & k = 1, \ldots, m, \\ \jmath'(\varsigma_k^+) - \jmath'(\varsigma_k^-) = \widetilde{\nabla}_k(\jmath(\varsigma_k^-)), & k = 1, \ldots, m, \\ \jmath'(0) = v_0 \in \mathfrak{B}, \jmath(\varsigma) = \mathfrak{J}(\varsigma), & \text{if } \varsigma \in \mathbb{R}_-, \end{cases} \quad (70)$$

and the semilinear system

$$\begin{cases} \vartheta''(\varsigma) = \mathcal{Z}(\varsigma)\vartheta(\varsigma) + \Psi\left(\varsigma, \vartheta_{\rho(\varsigma,\vartheta_\varsigma)}, (F\vartheta)(\varsigma)\right) + \int_0^\varsigma \Upsilon(\varsigma,\varepsilon)\vartheta(\varepsilon)d\varepsilon + \mathcal{P}u(\varsigma), & \text{if } \varsigma \in \widetilde{\Theta}, \\ \vartheta(\varsigma_k^+) - \vartheta(\varsigma_k^-) = \nabla_k(\vartheta(\varsigma_k^-)), & k = 1, \ldots, m, \\ \vartheta'(\varsigma_k^+) - \vartheta'(\varsigma_k^-) = \widetilde{\nabla}_k(\vartheta(\varsigma_k^-)), & k = 1, \ldots, m, \\ \vartheta'(0) = v_0 \in \mathfrak{B}, \vartheta(\varsigma) = \mathfrak{J}(\varsigma), & \text{if } \varsigma \in \mathbb{R}_-. \end{cases} \quad (71)$$

The following hypotheses must be introduced in order to demonstrate the main aim of this section, that is, the approximate controllability of system (5):

(1) [(C5)] Linear system (4) is approximately controllable
(2) [(C6)] Range of the operator $\aleph$ is a subset of the closure of range of $\mathcal{P}$, i.e.,

$$\text{Range}(\mathcal{N}) \subseteq \overline{\text{Range}(\mathcal{P})}. \quad (72)$$

Theorem 17. *If hypotheses* $(C1)-(C6)$ *are verified, then system (1) is approximately controllable.*

Proof. The mild solution of system (4) corresponding to the control v is given by

$$\jmath(\varsigma) = \begin{cases} -\dfrac{\partial \daleth(\varsigma,\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=0}\mathfrak{J}(0) + \daleth(\varsigma,0)v_0 + \int_0^\varsigma \daleth(\varsigma,\varepsilon)\mathcal{P}v(\varepsilon)d\varepsilon \\ \quad - \sum_{0<\varsigma_k<\varsigma} \dfrac{\partial \daleth(\varsigma,\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=\varsigma_k}\nabla_k(\jmath(\varsigma_k^-)) + \sum_{0<\varsigma_k<\varsigma} \daleth(\varsigma,\varsigma_k)\widetilde{\nabla}_k(\jmath(\varsigma_k^-)); & \text{if } \varsigma \in \Theta, \\ \mathfrak{J}(\varsigma); & \text{if } \varsigma \in \mathbb{R}_-. \end{cases} \quad (73)$$

Assume the following system:

$$\begin{cases} \vartheta''(\varsigma) = \mathscr{X}(\varsigma)\vartheta(\varsigma) + \Psi\Big(\varsigma, \vartheta_{\rho(\varsigma,\vartheta_\varsigma)}, (F\vartheta)(\varsigma)\Big) + \int_0^\varsigma \Upsilon(\varsigma,\varepsilon)\vartheta(\varepsilon)\mathrm{d}\varepsilon & \\ +\mathscr{P}v(\varsigma) - \Psi\Big(\varsigma, \jmath_{\rho(\varsigma,\jmath_\varsigma)}, (F\jmath)(\varsigma)\Big), & \text{if } \varsigma \in \widetilde{\Theta}, \\ \vartheta(\varsigma_k^+) - \vartheta(\varsigma_k^-) = \nabla_k(\vartheta(\varsigma_k^-)), & k = 1,\ldots,m, \\ \vartheta'(\varsigma_k^+) - \vartheta'(\varsigma_k^-) = \widetilde{\nabla}_k(\vartheta(\varsigma_k^-)), & k = 1,\ldots,m, \\ \vartheta'(0) = \nu_0 \in \mathfrak{B}, \vartheta(\varsigma) = \mathfrak{I}(\varsigma), & \text{if } \varsigma \in \mathbb{R}_-. \end{cases} \tag{74}$$

Since $\mathscr{N}\vartheta \in \overline{\mathrm{Range}(\mathscr{P})}$, there exists a control function $u \in L^2(\Theta, U)$ such that

$$\|\mathscr{N}\vartheta - \mathscr{P}u\| \le \epsilon, \text{for given } \epsilon > 0. \tag{75}$$

Now, assume that ϑ is the mild solution of system (1) corresponding to $(v-u)$ given by

$$\vartheta(\varsigma) = \begin{cases} -\dfrac{\partial \urcorner(\varsigma,\varepsilon)}{\partial\varepsilon}\Big|_{\varepsilon=0}\mathfrak{I}(0) + \urcorner(\varsigma,0)\nu_0 + \int_0^\varsigma \urcorner(\varsigma,\varepsilon)(\mathscr{N}\vartheta + \mathscr{P}(v-u))(\varepsilon)\mathrm{d}\varepsilon & \\ -\sum_{0<\varsigma_k<\varsigma} \dfrac{\partial \urcorner(\varsigma,\varepsilon)}{\partial\varepsilon}\Big|_{\varepsilon=\varsigma_k}\nabla_k(\vartheta(\varsigma_k^-)) + \sum_{0<\varsigma_k<\varsigma}\urcorner(\varsigma,\varsigma_k)\widetilde{\nabla}_k(\vartheta(\varsigma_k^-)); & \text{if } \varsigma \in \Theta, \\ \mathfrak{I}(\varsigma); & \text{if } \varsigma \in \mathbb{R}_-. \end{cases} \tag{76}$$

Then, if $\varsigma \in \mathbb{R}_-$, we get $\|\jmath(\varsigma) - \vartheta(\varsigma)\| = 0$.

And, for $\varsigma \in \Theta$, we have

$$\begin{aligned} \|\jmath(\varsigma) - \vartheta(\varsigma)\| \le{}& \int_0^\varsigma \|\urcorner(\varsigma,\varepsilon)\|\|(\mathscr{P}u(\varepsilon) - \mathscr{N}\vartheta(\varepsilon))\|\mathrm{d}\varepsilon + \sum_{0<\varsigma_k<\varsigma}\left\|\frac{\partial\urcorner(\varsigma,\varepsilon)}{\partial\varepsilon}|_{\varepsilon=\varsigma_k}\right\| \\ &\times \|\nabla_k(\jmath(\varsigma_k^-)) - \nabla_k(\vartheta(\varsigma_k^-))\| \\ &+ \sum_{0<\varsigma_k<\varsigma}\|\urcorner(\varsigma,\varsigma_k)\|\|\widetilde{\nabla}_k(\jmath(\varsigma_k^-)) - \widetilde{\nabla}_k(\vartheta(\varsigma_k^-))\| \\ \le{}& \int_0^\varsigma \|\urcorner(\varsigma,\varepsilon)\|\|(\mathscr{P}u(\varepsilon) - \mathscr{N}\jmath(\varepsilon))\|\mathrm{d}\varepsilon \\ &+ \int_0^\varsigma \|\urcorner(\varsigma,\varepsilon)\|\|(\mathscr{N}\jmath(\varepsilon) - \mathscr{N}\vartheta(\varepsilon))\|\mathrm{d}\varepsilon \\ &+ \widetilde{M_\urcorner}\sum_{0<\varsigma_k<\varsigma} L_{Q_k^2}\|\jmath(\varsigma_k^-) - \vartheta(\varsigma_k^-)\| \\ &+ M_\urcorner \sum_{0<\varsigma_k<\varsigma} L_{Q_k^1}\|\jmath(\varsigma_k^-) - \vartheta(\varsigma_k^-)\|. \end{aligned} \tag{77}$$

Now, for any $\varsigma \in \widetilde{\Theta}$, we define the function $F(\varsigma) = \sup_{\varepsilon \in [0,\varsigma]} \| \jmath(\varepsilon) - \vartheta(\varepsilon) \|$, and from the definition of the function ρ and Lemma 13, we obtain

$$\left\| \jmath_{\rho(\varepsilon, \jmath_\varepsilon)} - \vartheta_{\rho(\varepsilon, \vartheta_\varepsilon)} \right\| \leq {\Phi_1}^* \sup_{\theta \in (0,\varepsilon)} \| \jmath(\theta) - \vartheta(\theta) \| \leq {\Phi_1}^* F(\varepsilon). \tag{78}$$

Then,

$$F(\varsigma) \leq M_\neg T^{1/2} \epsilon + M_\neg \left(\xi_1 {\Phi_1}^* + \xi_2 \mathscr{K}_{c_1} T \right) \int_0^\varsigma F(\varepsilon) \mathrm{d}\varepsilon + \sum_{0<\varsigma_k<\varsigma} \left(\widetilde{M_{\mathscr{Q}^*}} L_{Q_k^2} + M_\neg L_{Q_k^1} \right) F(\varsigma_k^-). \tag{79}$$

Therefore, according to Lemma 2, we get

$$F(\varsigma) \leq \epsilon M_\neg T^{1/2} e^{M_\neg \left(\xi_2 \mathscr{K}_{c_1} T + \xi_1 {\Phi_1}^* \right) T} \prod_{0<\varsigma_k<\varsigma} \left(1 + \widetilde{M_{\mathscr{Q}^*}} L_{Q_k^2} + M_\neg L_{Q_k^1} \right). \tag{80}$$

By taking suitable control function u, we make $\| \jmath(\varsigma) - \vartheta(\varsigma) \|$ arbitrary small. Therefore, the reachable set of (4) is dense in the reachable set of (70), which is dense in $\mathfrak{B}$ due to $(C5)$. Hence, the approximate controllability of (70) implies that of the semilinear control system (4). ☐

5. Ulam–Hyers–Rassias Stability Results

Let $\nabla_1, \nabla_2, \nabla_3 \geq 0$ and $\nu \in C(\Theta, \mathbb{R}_+)$ be nondecreasing and consider the following inequalities:

$$\begin{cases} \left\| ℶ''(\varsigma) - \mathscr{Z}(\varsigma) ℶ(\varsigma) - \Psi\left(\varsigma, ℶ_{\rho(\varsigma, ℶ_\varsigma)}, (H ℶ)(\varsigma) \right) - \int_0^\varsigma \Upsilon(\varsigma, \varepsilon)(ℶ(\varepsilon)) \mathrm{d}\varepsilon \right\| \leq \nu(\varsigma), & \varsigma \in \widetilde{\Theta}, \\ \| ℶ(\varsigma_k^+) - ℶ(\varsigma_k^-) - \nabla_k(ℶ(\varsigma_k^-)) \| \leq \nabla_1, & k = 1, \ldots, m, \\ \| ℶ'(\varsigma_k^+) - ℶ'(\varsigma_k^-) - \widetilde{\nabla}_k(ℶ(\varsigma_k^-)) \| \leq \nabla_2, & k = 1, \ldots, m, \\ \| ℶ(\varsigma) - \mathfrak{I}(\varsigma) \| \leq \nabla_3, & \text{if } \varsigma \in \mathbb{R}_-. \end{cases} \tag{81}$$

Let the space be

$$\widetilde{X} = \left\{ u \in PC^2(\widetilde{I}, \mathfrak{B}) \colon u(\varsigma)|_{\widetilde{\Theta}} \in D(\mathscr{Z}) \right\}. \tag{82}$$

The following concepts are inspired by papers [14, 15] and references therein.

Definition 18. Equation (1) is generalized U-H-R stable with respect to $(\nu, \nabla_1, \nabla_2, \nabla_3)$, if there exists $\theta_{\Psi,\nabla,\nu} > 0$, such that for each solution $\jmath \in \widetilde{X}$ of inequality (6), there exists a mild solution $\widehat{\jmath} \in \mathscr{W}$ of equation (1), with

$$\left\| \jmath(\varsigma) - \widehat{\jmath}(\varsigma) \right\| \leq \theta_{\Psi,\nabla,\nu} (\nabla_1 + \nabla_2 + \nabla_3 + \nu(\varsigma)), \ \varsigma \in \widetilde{I}. \tag{83}$$

Remark 19. A function $\jmath \in \widetilde{X}$ is a solution of inequality (6) if and only if there exist $\wp_1, \wp_2 \in PC(\widetilde{I}, \mathfrak{B})$ and $\lambda_k, \widehat{\lambda}_k \in \mathbb{R}$, such that

(a_1) $\| \wp_1(\varsigma) \| \leq \nu(\varsigma)$; $\varsigma \in \widetilde{\Theta}$, $\lambda_k \leq \nabla_1, \widehat{\lambda}_k \leq \nabla_2$ and $\wp_2(\varsigma) \leq \nabla_3$; $\varsigma \in \mathbb{R}_-$,

(a_2) $ℶ'' - \mathscr{Z}(\varsigma) ℶ(\varsigma) = \Psi(\varsigma, ℶ_{\rho(\varsigma, ℶ_\varsigma)}, (H ℶ)(\varsigma)) + \int_0^\varsigma \Upsilon(\varsigma, \varepsilon) ℶ(\varepsilon) \mathrm{d}\varepsilon + \wp_1(\varsigma)$, if $\varsigma \in \widetilde{\Theta}$,

(a_3) $ℶ(\varsigma_k^+) - ℶ(\varsigma_k^-) = \nabla_k(ℶ(\varsigma_k^-)) + \lambda_k$, $\quad k = 1, \ldots, m$,

(a_4) $ℶ'(\varsigma_k^+) - ℶ'(\varsigma_k^-) = \widetilde{\nabla}_k(ℶ(\varsigma_k^-)) + \widehat{\lambda}_k$, $\quad k = 1, \ldots, m$,

(a_5) $ℶ(\varsigma) = \mathfrak{I}(\varsigma) + \wp_2(\varsigma)$, $\quad$ if $\varsigma \in \mathbb{R}_-$.

Remark 20. If $\daleth \in \widetilde{X}$ is a solution of inequality (6) then $\daleth$ is a solution of the following integral inequality:

$$\begin{cases} \left\| \daleth(\varsigma) + \dfrac{\partial \urcorner(\varsigma,\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=0} \mathfrak{I}(0) - \urcorner(\varsigma,0)\nu_0 - \int_0^\varsigma \urcorner(\varsigma,\varepsilon)\Big(\Psi\Big(\varepsilon, \daleth_{\rho(\varepsilon,\daleth_\varepsilon)}, (F\daleth)(\varepsilon)\Big) + \mathscr{P}u(\varepsilon)\Big)d\varepsilon \right. \\ \left. + \sum_{0<\varsigma_k<\varsigma} \left(\dfrac{\partial \urcorner(\varsigma,\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=\varsigma_k} \nabla_k(\daleth(\varsigma_k^-)) - \urcorner(\varsigma,\varsigma_k)\widetilde{\nabla}_k(\daleth(\varsigma_k^-)) \right) \right\| \\ \leq M_\urcorner \int_0^\varsigma \nu(\varepsilon)\mathrm{d}\varepsilon + M_\urcorner \nabla_1 + \widetilde{M_\urcorner}\nabla_2, \quad \text{if } \varsigma \in \widetilde{\Theta}, \\ \|\daleth(\varsigma) - \mathfrak{I}(\varsigma)\| \leq \nabla_3, \quad \text{if } \varsigma \in \mathbb{R}_-. \end{cases} \tag{84}$$

We also need the following additional assumption to discuss about stability: (C_ν). We assume that for a non-decreasing function $\nu \in C(\widetilde{\Theta}, \mathfrak{B})$, there exists $c_\nu > 0$, such that

$$\int_0^\varsigma \nu(\varepsilon)\mathrm{d}\varepsilon \leq c_\nu \nu(\varsigma). \tag{85}$$

Theorem 21. *If* $(C1)-(C4)$, (C_H), *and* (C_ν) *are satisfied, with*

$$M_\urcorner\Big(\xi_1\Phi_1{}^* + \xi_2\mathscr{K}_{c_1}T\Big)T + \sum_{0<\varsigma_k<\varsigma}\Big(\widetilde{M_\urcorner}L_{Q_k^2} + M_\urcorner L_{Q_k^1}\Big) < 1, \tag{86}$$

then, equation (1) is generalized U-H-R stable with respect to $(\nu, \nabla_1, \nabla_2, \nabla_3)$.

Proof. Let $\jmath$ be a solution of (6) and $\hat{\jmath} \in \mathscr{W}$ be the mild solution of (1) with $\hat{\jmath}(0) = \jmath(0) = \mathfrak{I}(0)$ and $\jmath'(0) = \jmath'(0) = \nu_0$.

Then, we get

$$\hat{\jmath}(\varsigma) = \begin{cases} -\dfrac{\partial \urcorner(\varsigma,\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=0}\mathfrak{I}(0) + \urcorner(\varsigma,0)\nu_0 \\ + \int_0^\varsigma \urcorner(\varsigma,\varepsilon)\Big(\Psi\Big(\varepsilon, \hat{\jmath}_{\rho(\varepsilon,\hat{\jmath}_\varepsilon)}, (F\hat{\jmath})(\varepsilon)\Big) + \mathscr{P}u(\varepsilon)\Big)\mathrm{d}\varepsilon \\ - \sum_{0<\varsigma_k<\varsigma} \dfrac{\partial \urcorner(\varsigma,\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=\varsigma_k}\nabla_k\big(\hat{\jmath}(\varsigma_k^-)\big) + \sum_{0<\varsigma_k<\varsigma}\urcorner(\varsigma,\varsigma_k)\widetilde{\nabla}_k\big(\hat{\jmath}(\varsigma_k^-)\big); \quad \text{if } \varsigma \in \widetilde{\Theta}, \\ \mathfrak{I}(\varsigma); \quad \text{if } \varsigma \in \mathbb{R}_-. \end{cases} \tag{87}$$

On the other hand, we get

$$\begin{cases} \Big\| \jmath(\varsigma) + \dfrac{\partial \daleth(\varsigma,\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=0} \mathfrak{J}(0) - \daleth(\varsigma,0)\nu_0 \\ - \displaystyle\int_0^{\varsigma} \daleth(\varsigma,\varepsilon)\Big(\Psi\Big(\varepsilon, \jmath_{\rho(\varepsilon,\jmath_\varepsilon)}, (F\jmath)(\varepsilon)\Big) + \mathscr{P}u(\varepsilon)\Big) d\varepsilon \\ + \displaystyle\sum_{0<\varsigma_k<\varsigma} \left(\dfrac{\partial \daleth(\varsigma,\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=\varsigma_k} \nabla_k(\jmath(\varsigma_k^-)) - \daleth(\varsigma,\varsigma_k)\widetilde{\nabla}_k(\jmath(\varsigma_k^-))\right)\Big\| \\ \leq M_\neg c_\nu \nu(\varsigma) + M_\neg \nabla_1 + \widetilde{M_\neg}\nabla_2, ; \quad \text{if } \varsigma \in \widetilde{\Theta}, \\ \|\jmath(\varsigma) - \mathfrak{J}(\varsigma)\| \leq \nabla_3, \quad \text{if } \varsigma \in \mathbb{R}_-. \end{cases} \tag{88}$$

Hence, for $\varsigma \in \widetilde{\Theta}$, we have

$$\begin{aligned} \|\jmath(\varsigma) - \hat{\jmath}(\varsigma)\| = & \Big\| \jmath(\varsigma) + \frac{\partial \daleth(\varsigma,\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=0} \mathfrak{J}(0) - \daleth(\varsigma,0)\nu_0 \\ & - \int_0^{\varsigma} \daleth(\varsigma,\varepsilon)\Big(\Psi\Big(\varepsilon, \hat{\jmath}_{\rho(\varepsilon,\hat{\jmath}_\varepsilon)}, (F\hat{\jmath})(\varepsilon)\Big) + \mathscr{P}u(\varepsilon)\Big) d\varepsilon \\ & + \sum_{0<\varsigma_k<\varsigma} \frac{\partial \daleth(\varsigma,\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=\varsigma_k} \nabla_k\big(\hat{\jmath}(\varsigma_k^-)\big) - \sum_{0<\varsigma_k<\varsigma} \daleth(\varsigma,\varsigma_k)\widetilde{\nabla}_k\big(\hat{\jmath}(\varsigma_k^-)\big) \\ & - \int_0^{\varsigma} \daleth(\varsigma,\varepsilon)\Big(\Psi\Big(\varepsilon, \jmath_{\rho(\varepsilon,\jmath_\varepsilon)}, (F\jmath)(\varepsilon)\Big) + \mathscr{P}u(\varepsilon)\Big) d\varepsilon \\ & + \int_0^{\varsigma} \daleth(\varsigma,\varepsilon)\Big(\Psi\Big(\varepsilon, \jmath_{\rho(\varepsilon,\jmath_\varepsilon)}, (F\jmath)(\varepsilon)\Big) + \mathscr{P}u(\varepsilon)\Big) d\varepsilon \\ & + \sum_{0<\varsigma_k<\varsigma} \left(\frac{\partial \daleth(\varsigma,\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=\varsigma_k} \nabla_k(\jmath(\varsigma_k^-)) - \daleth(\varsigma,\varsigma_k)\widetilde{\nabla}_k(\jmath(\varsigma_k^-))\right) \\ & - \sum_{0<\varsigma_k<\varsigma} \left(\frac{\partial \daleth(\varsigma,\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=\varsigma_k} \nabla_k(\jmath(\varsigma_k^-)) - \daleth(\varsigma,\varsigma_k)\widetilde{\nabla}_k(\jmath(\varsigma_k^-))\right)\Big\| \\ \leq & M_\neg c_\nu \nu(\varsigma) + M_\neg \nabla_1 + \widetilde{M}_\neg \nabla_2 + M_\neg \int_0^{\varsigma} \xi_1 \Big\| \jmath_{\rho(\varepsilon,\widehat{\jmath_\varepsilon})} - \hat{\jmath}_{\rho(\varepsilon,\hat{\jmath}_\varepsilon)} \Big\|_{\mathscr{C}} \\ & + \xi_2 \mathscr{K}_{c_1} T \big\| \jmath(\varepsilon) - \hat{\jmath}(\varepsilon) \big\| d\varepsilon \\ & + \widetilde{M}_\neg \sum_{0<\varsigma_k<\varsigma} L_{Q_k^2} \big\| \jmath(\varsigma_k^-) - \hat{\jmath}(\varsigma_k^-) \big\| \end{aligned} \tag{89}$$

Let $\widetilde{\Delta}_\varsigma = [\varsigma_k, \varsigma_{k+1}]$, and

$$\begin{aligned} PC_{\beth}(\mathbb{R},\mathbb{R}^+) = \Big\{ & \beth : \widetilde{I} \longrightarrow \mathbb{R}^+ : w|_{\mathbb{R}^-} \in PC_\theta, w|_{\widetilde{\Delta}_\varsigma} \text{ is continuous and} \\ & \beth(\varsigma_k^-), \beth(\varsigma_k^+) \text{ exist with } \beth(\varsigma_k^-) = \beth(\varsigma_k) \Big\}. \end{aligned} \tag{90}$$

For $\beth \in PC_{\beth}(\widetilde{I}, \mathbb{R}^{+})$, let

$$\mathscr{W}\beth(\varsigma) = \begin{cases} M_{\neg}c_{\gamma}\nu(\varsigma) + M_{\neg}\nabla_1 + \widetilde{M_{\neg}}\nabla_2 \\ + M_{\neg}\int_0^{\varsigma} \xi_1 (\beth(\rho(\varepsilon, \beth(\varepsilon+\theta)) + \theta)) + \xi_2 \mathscr{K}_{c_1} T\beth(\varepsilon) d\varepsilon \\ + \sum_{0<\varsigma_k<\varsigma} \left(\widetilde{M_{\neg}}L_{Q_k^2} + M_{\neg}L_{Q_k^1}\right)\beth(\varsigma_k^-), & \text{if } \varsigma \in \widetilde{\Theta}, \text{and } \theta \in \mathbb{R}^-, \\ 0, & \text{otherwise.} \end{cases} \tag{91}$$

Now, we will prove that $\mathscr{W}$ is a Picard operator. For that, let $\varsigma \in \widetilde{I}$ and $\beth_1, \beth_2 \in PC(\widetilde{I}, \mathbb{R}^{+})$, if $\varsigma \notin \widetilde{\Theta}$, we get $|\mathscr{W}\beth_1(\varsigma) - \mathscr{W}\beth_2(\varsigma)| = 0$, and if $\varsigma \in \widetilde{\Theta}$, we have

$$\left|\mathscr{W}\beth_1(\varsigma) - \mathscr{W}\beth_2(\varsigma)\right| \leq \left(M_{\neg}\left(\xi_1 \Phi_1{}^{*} + \xi_2 \mathscr{K}_{c_1} T\right)T + \sum_{0<\varsigma_k<\varsigma}\left(\widetilde{M_{\neg}}L_{Q_k^2} + M_{\neg}L_{Q_k^1}\right)\right)\left|\beth_1 - \beth_2\right|. \tag{92}$$

Therefore, $\mathscr{W}$ is a contraction; hence, from Theorem 10, there exists a unique $\beth^*$ in $F_{\mathscr{W}}$, and from Definition 3, we deduce that $\mathscr{W}$ is a Picard operator.

Furthermore, we have

$$\begin{aligned} \beth^*(\varsigma) = {} & M_{\neg}c_{\gamma}\nu(\varsigma) + M_{\neg}\nabla_1 + \widetilde{M_{\neg}}\nabla_2 \\ & + M_{\neg}\int_0^{\varsigma} \xi_1 (\beth^*(\rho(\varepsilon, w(\varepsilon+\theta)) + \theta)) + \xi_2 \mathscr{K}_{c_1} T\beth^*(\varepsilon) \mathrm{d}\varepsilon \\ & + \sum_{0<\varsigma_k<\varsigma} \left(\widetilde{M_{\neg}}L_{Q_k^2} + M_{\neg}L_{Q_k^1}\right)\beth^*(\varsigma_k^-). \end{aligned} \tag{93}$$

We can see that $\beth^*$ is an increasing function and $(\beth^*)'$ is nonnegative.

So, for $\varsigma \in \widetilde{I}$ and $\theta \in \mathbb{R}^-$, we have

$$\beth^*(\rho(\varsigma, \beth^*(\varsigma+\theta)) + \theta) \leq \beth^*(\varsigma+\theta) \leq \beth^*(\varsigma). \tag{94}$$

Then,

$$\begin{aligned} \beth^*(\varsigma) \leq {} & M_{\neg}c_{\gamma}\nu(\varsigma) + M_{\neg}\nabla_1 + \widetilde{M_{\neg}}\nabla_2 + M_{\neg}\int_0^{\varsigma}\left(\xi_1 + \xi_2 \mathscr{K}_{c_1} T\right)\beth^*(\varepsilon)\mathrm{d}\varepsilon \\ & + \sum_{0<\varsigma_k<\varsigma} \left(\widetilde{M_{\neg}}L_{Q_k^2} + M_{\neg}L_{Q_k^1}\right)\beth^*(\varsigma_k^-). \end{aligned} \tag{95}$$

From Lemma 2, we get

$$\beth^*(\varsigma) \leq \left(M_{\neg}c_{\gamma}\nu(\varsigma) + M_{\neg}\nabla_1 + \widetilde{M_{\neg}}\nabla_2\right)e^{M_{\neg}\left(\xi_1+\xi_2\mathscr{K}_{c_1}T\right)\varsigma} \prod_{0<\varsigma_k<\varsigma}\left(1 + \widetilde{M_{\neg}}L_{Q_k^2} + M_{\neg}L_{Q_k^1}\right). \tag{96}$$

In particular, if $\daleth(\varsigma) = \left\| \jmath(\varsigma) - \hat{\jmath}(\varsigma) \right\|$, then we have $\daleth(\varsigma) \leq \mathscr{W} \daleth(\varsigma)$, and applying the abstract Gronwall lemma, we obtain $\daleth(\varsigma) \leq \daleth^*(\varsigma)$. It follows that

$$\begin{aligned} \left\| \jmath(\varsigma) - \hat{\jmath}(\varsigma) \right\| \leq & \left(M_{\neg} c_{\nu} \nu(\varsigma) + M_{\neg} \nabla_1 + \widetilde{M_{\neg}} \nabla_2 \right) e^{M_{\neg}\left(\xi_1 + \xi_2 \mathscr{K}_{c_1} T\right)\varsigma} \\ & \times \prod_{0<\varsigma_k<\varsigma} \left(1 + \widetilde{M_{\neg}} L_{Q_k^2} + M_{\neg} L_{Q_k^1} \right). \end{aligned} \tag{97}$$

Now, if $\varsigma \in \mathbb{R}^-$, we get

$$\begin{aligned} \left\| \jmath(\varsigma) - \hat{\jmath}(\varsigma) \right\| &= \left\| \jmath(\varsigma) - \mathfrak{F}(\varsigma) \right\| \\ &\leq \nabla_3. \end{aligned} \tag{98}$$

Then, if we put

$$\theta_{\Psi,\nabla,\nu} = \begin{cases} \left(M_{\neg} c_{\nu} + M_{\neg} + \widetilde{M_{\neg}} + 1 \right) e^{M_{\neg} L_f \left(\xi_1 + \xi_2 \mathscr{K}_{c_1} T \right)\varsigma} & \\ \times \prod_{0<\varsigma_k<\varsigma} \left(1 + \widetilde{M_{\mathscr{Q}^*}} L_{Q_k^2} + M_{\neg} L_{Q_k^1} \right), & \text{if } \varsigma \in \widetilde{\Theta}, \\ 1, & \text{if } \varsigma \in \mathbb{R}_-. \end{cases} \tag{99}$$

Thus, we have for all $\varsigma \in \widetilde{I}$

$$\left\| \jmath(\varsigma) - \widehat{\jmath}(\varsigma) \right\| \leq \theta_{\Psi,\nabla,\nu} \left(\nabla_1 + \nabla_2 + \nabla_3 + \nu(\varsigma) \right), \tag{100}$$

which implies that (4) is generalized U-H-R stable with respect to $(\nu, \nabla_1, \nabla_2, \nabla_3)$. □

6. An Example

Consider the following class of partial integrodifferential system:

$$\begin{cases} \dfrac{\partial^2 \nu(\varsigma, x)}{\partial^2 \varsigma} = \dfrac{\partial^2 \nu(\varsigma, x)}{\partial^2 x} + \displaystyle\int_0^{\varsigma} \Gamma(\varsigma - \varepsilon) \dfrac{\partial^2 \nu(\varepsilon, x)}{\partial^2 x} \mathrm{d}\varepsilon & \\ + \displaystyle\int_{-\infty}^{-\varsigma} \dfrac{e^{-8\tau} \| \nu(\varsigma + \sigma(\varsigma, \nu(\varsigma + \tau, x)), x) \|_{L^2}}{129\left((\varsigma + \tau)^2 + 2\varsigma + 1 \right)} \mathrm{d}\tau & \\ + \displaystyle\int_0^1 \dfrac{\cosh(\varsigma) \ln\left(\pi + e^{-\varsigma^2} \right) (1 + \nu(\varepsilon, x))}{222\left(1 + 2\varsigma^2 + \varepsilon^2 \right) e^{11\varsigma}} \mathrm{d}\varepsilon + \widetilde{\sigma}(\varsigma) \nu(\varsigma, x) & \\ + \mathscr{L}(\varsigma, x), & \text{if } \varsigma \in I \backslash \{1, 2, \ldots, 8\} \text{ and } x \in (0, \pi), \\ \nabla \nu(\varsigma_k, x) = \alpha_k \displaystyle\int_0^{\varsigma_k} \nu(\varepsilon, x) \mathrm{d}\varepsilon;\ k = \overline{1:8}, \text{and } x \in (0, \pi), & \\ \widetilde{\nabla}\left(\left. \dfrac{\partial \nu(\varsigma, x)}{\partial \varsigma} \right|_{\varsigma=\varsigma_k} \right) = \beta_k \displaystyle\int_0^{\varsigma_k} \nu(\varepsilon, x) \mathrm{d}\varepsilon; & k = \overline{1:8}, \text{and } x \in (0, \pi), \\ \nu(\varsigma, 0) = \nu(\varsigma, 1) = 0, & \text{for } \varsigma \in I, \\ \left. \dfrac{\partial \nu(\varsigma, x)}{\partial \varsigma} \right|_{\varsigma=0} = \nu_1(x), \nu(\varsigma, x) = \mathfrak{F}(\varsigma, x), & \text{if } \varsigma \in \mathbb{R}_- \text{ and } x \in (0, \pi), \end{cases} \tag{101}$$

where $I = [0, 1], \sigma\colon \Theta \times \mathbb{R} \longrightarrow \mathbb{R}$, $\mathscr{L}\colon [0, 1] \times [0, \pi] \longrightarrow [0, \pi]$, $\alpha_k, \beta_k \in (0, e^{-\pi}/17)$.

Let

$$\mathscr{H} := L^2(0, \pi) = \left\{u\colon (0, \pi) \longrightarrow \mathbb{R} : \int_0^\pi |u(x)|^2 \mathrm{d}x < \infty\right\}, \tag{102}$$

be the Hilbert space with the scalar product $\langle u, v\rangle = \int_0^\pi u(x)v(x)\mathrm{d}x$, and the norm

$$\|u\|_2 = \left(\int_0^\pi |u(x)|^2 \mathrm{d}x\right)^{1/2}, \tag{103}$$

and the phase space $\mathscr{E}$ be $\mathrm{BUC}(\mathbb{R}^-, \mathscr{H})$, the space of bounded uniformly continuous functions endowed with the following norm: $\|\psi_{\mathscr{E}} = \sup_{-\infty<\tau\le 0}\|\psi(\tau)_{L^2}, \psi \in \mathscr{E}$. It is well known that $\mathscr{E}$ satisfies the axioms (A_1) and (A_2) with $K = 1$ and $\Phi_1(\varsigma) = \Phi_2(\varsigma) = 1$ (see [41]). We define $\mathscr{Z}$ induced on $\mathscr{H}$ as

$$\widehat{\mathscr{Z}}\gamma = \gamma'', \text{and}\, D(\mathscr{Z}) = \left\{\gamma \in H^2(0, \pi) : \gamma(0) = \gamma(\pi) = 0\right\}. \tag{104}$$

Then, $\widehat{\mathscr{Z}}$ is the infinitesimal generator of a cosine function of operators $(C_0(\varsigma))_{\varsigma\in\mathbb{R}}$ on H associated with sine function $(S_0(\varsigma))_{\varsigma\in\mathbb{R}}$. In addition, $\widehat{\mathscr{Z}}$ has discrete spectrum which consists of eigenvalues $-n^2$ for $n \in \mathbb{N}$, with corresponding eigenvectors $w_n(x) = 1/\sqrt{2\pi}e^{\mathrm{inx}}$. The set $\{w_n\colon n \in \mathbb{N}\}$ is an orthonormal basis of $\mathscr{H}$. Applying this idea, we can write

$$\widehat{\mathscr{Z}}\gamma = \sum_{n=1}^{\infty} -n^2\langle \gamma, w_n\rangle w_n, \quad \gamma \in D(\mathscr{Z}). \tag{105}$$

The cosine family associated with $\widehat{\mathscr{Z}}$ is given by $(C_0(\varsigma))_{\varsigma\in\mathbb{R}}$:

$$C_0(\varsigma)\gamma = \sum_{n=1}^{\infty} \cos(n\varsigma)\langle \gamma, w_n\rangle w_n, \quad \varsigma \in \mathbb{R}, \tag{106}$$

and the sine function is given by

$$S_0(\varsigma)\gamma = \sum_{n=1}^{\infty} \frac{\sin(n\varsigma)}{n}\langle \gamma, w_n\rangle w_n, \quad \varsigma \in \mathbb{R}. \tag{107}$$

Thus, $\|C_0(\varsigma)\| \le 1$ and $S_0(\varsigma)$ is compact for all $\varsigma \in \mathbb{R}$. We define $\mathscr{Z}(\varsigma)\gamma = \widehat{\mathscr{Z}}\gamma + \widetilde{\sigma}(\varsigma)\gamma$ on $D(\mathscr{Z})$. Clearly, $\mathscr{Z}(\varsigma)$ is a closed linear operator. Therefore, $\mathscr{Z}(\varsigma)$ generates $(S(\varsigma, \varepsilon))_{(\varsigma,\varepsilon)\in\nabla}$ such that $S(\varsigma, \varepsilon)$ is compact and self-adjoint for all $(\varsigma, \varepsilon) \in \nabla = \{(\varsigma, \varepsilon)\colon 0 \le \varepsilon \le \varsigma \le 1\}$ (see [26]).

We define the operators $\Lambda(\varsigma, \varepsilon)\colon D(\mathscr{Z}) \subset \mathscr{H} \mapsto \mathscr{H}$ as follows:

$$\Lambda(\varsigma, \varepsilon)\gamma = \Gamma(\varsigma - \varepsilon)\widehat{\mathscr{Z}}\gamma, \quad \text{for}\, 0 \le \varepsilon \le \varsigma \le 1, \gamma \in D(\mathscr{Z}). \tag{108}$$

The assumption $(C4)$ holds under more suitable conditions on the operator Γ. Furthermore, $(B1) - (B3)$ are fulfilled. Then, there exists a resolvent compact operator [26, 44].

Now, let $\mathscr{P}\colon U \longrightarrow \mathscr{H}$ be defined by $\mathscr{P}u(\varsigma)(x) = \mathscr{L}(\varsigma, x), x \in [0, \pi], u \in U$, where $\mathscr{L}\colon [0, 1] \times [0, \pi] \longrightarrow \mathscr{H}$ is linearly continuous, and for $\mathfrak{J} \in BUC(\mathbb{R}^-, H)$, we put $\rho(\varsigma, \mathfrak{J})(\nu) = \sigma(\varsigma, \nu(\varsigma + \tau, x))$, such that $(C_{\mathfrak{J}})$ holds, and let $\varsigma \longrightarrow \mathfrak{J}_\varsigma$ be continuous on $\urcorner(\rho^-)$.

We put $\nu(\varsigma)(x) = \nu(\varsigma, x)$, for $\varsigma \in [0, 1]$, and define

$$\begin{aligned}
\Psi(\varsigma, \vartheta_1, \vartheta_2)(x) &= \int_{-\infty}^{-\varsigma} \frac{e^{-8\tau}\|\vartheta_1(\varsigma + \sigma(\varsigma, \nu(\varsigma + \tau, x)), x)\|_{L^2}}{129\left((\varsigma + \tau)^2 + 2\varsigma + 1\right)}\mathrm{d}\tau + \frac{\cosh(\varsigma)\vartheta_2(\varsigma)(x)}{e^{11\varsigma}},\\
\vartheta_2(\varsigma)(x) &= F(\vartheta_1)(x) = \int_0^1 \frac{\ln\left(\pi + e^{-\varsigma^2}\right)(1 + \vartheta_1(\varepsilon, x))}{222\left(1 + 2\varsigma^2 + \varepsilon^2\right)}\mathrm{d}\varepsilon,\\
\nabla_k(\vartheta_1(\varsigma_k^-))(x) &= \alpha_k \int_0^{\varsigma_k} \vartheta_1(\varepsilon, x)\mathrm{d}\varepsilon,\\
\widetilde{\nabla}_k(\vartheta_1(\varsigma_k^-))(x) &= \beta_k \int_0^{\varsigma_k} \vartheta_1(\varepsilon, x)\mathrm{d}\varepsilon.
\end{aligned} \tag{109}$$

These definitions allow us to depict system (7) in the abstract form (4).

Now, for $\varsigma \in [0, 1]$, we have

$$\left|\Psi\left(\varsigma, \varkappa_{1(\varsigma)}, \varkappa_2(\varsigma)\right)\right| \le \frac{1 - e^{-16\pi}}{258(\varsigma + 1)^2}\left(\frac{1}{2}\|\varkappa_1\|_{\mathscr{E}}\right) + \cosh(\varsigma)e^{-11\varsigma}\left(|\varkappa_2(\varsigma)|\right). \tag{110}$$

So, $\psi_{i+1}(\varsigma) = \varsigma/1+i;\ i=0,1$ are continuous non-decreasing functions, and we have

$$\xi_1 = \frac{\left(1-e^{-16\pi}\right)\left(1-(1+\pi)^{-3}\right)}{258\sqrt{3}},$$
$$\xi_2 = \frac{1}{4}\sqrt{\frac{1}{330}\left(241 - \frac{55+120e^2+66e^4}{e^{24}}\right)}. \tag{111}$$

And for any bounded set $\Pi \subset \mathscr{H}$, and $\Pi_\varsigma \in \mathscr{E}$, we get

$$\zeta\Big(\Psi\big(\varsigma, \Pi_\varsigma, \digamma(\Pi(\varsigma))\big)\Big) \le (\xi_1+\xi_2)\zeta(\Pi(\varsigma)). \tag{112}$$

Now, about $\mathscr{K}, \nabla_k, \widetilde{\nabla}_k$, we obtain

$$\left\|\mathscr{K}(\varsigma,\varepsilon,\varkappa_1) - \mathscr{K}(\varsigma,\varepsilon,\varkappa_2)\right\|_2 \le \frac{\ln(\pi+1)}{222}\left\|\varkappa_1-\varkappa_2\right\|_2,$$
$$\left\|\nabla_k(\varkappa_1) - \nabla_k(\varkappa_2)\right\|_2 \le \alpha_k\left\|\varkappa_1-\varkappa_2\right\|_2, \tag{113}$$
$$\left\|\widetilde{\nabla}_k(\varkappa_1) - \widetilde{\nabla}_k(\varkappa_2)\right\|_2 \le \beta_k\left\|\varkappa_1-\varkappa_2\right\|_2.$$

Now, similar reasoning as in [28], if the corresponding linear system is approximately controllable, then system (7) is approximately controllable.

Furthermore, we have

$$M_\neg\left(\xi_2\mathscr{K}_{c_1}T + \xi_1\Phi_1^{\ *}\right)T + \sum_{0<\varsigma_k<\varsigma}\left(\widetilde{M_\neg} L_{Q_k^2} + M_\neg L_{Q_k^1}\right) \le 0,0239. \tag{114}$$

Thus, all the assumptions of Theorem 21 are fulfilled. Consequently, the mild solution of problem (101) is generalized U-H-R stable.

Data Availability

Data sharing is not applicable to this paper as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The study was carried out in collaboration of all authors. All the authors read and approved the final manuscript.

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