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Research Article

Comparison of Approximate Analytical and Numerical Solutions of the Allen Cahn Equation

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Allen Cahn (AC) equation is highly nonlinear due to the presence of cubic term and also very stiff; therefore, it is not easy to find its exact analytical solution in the closed form. In the present work, an approximate analytical solution of the AC equation has been investigated. Here, we used the variational iteration method (VIM) to find approximate analytical solution for AC equation. The obtained results are compared with the hyperbolic function solution and traveling wave solution. Results are also compared with the numerical solution obtained by using the finite difference method (FDM). Absolute error analysis tables are used to validate the series solution. A convergent series solution obtained by VIM is found to be in a good agreement with the analytical and numerical solutions.

1. Introduction

The AC equation is a nonlinear parabolic partial differential equation (PDE) which describes natural physical phenomenon in materials and multiphase systems. The model was firstly developed by Allen and Cahn to study the antiphase boundary motion in antiphase domain coarsening [1]. The AC model is mainly used to study various phase separation problems in different fields of sciences such as crystal growth, motion by mean curvature flows, image segmentation, and the mixture of two incompressible fluids [2–6]. It has become a rudimentary mathematical model to discuss the behavior of phase evolutions and interfacial dynamics in material sciences [7]. In order to study the phenomenon of phase transition, a capable and precise method for the solution of AC equation has practical significance. Physical importance of AC equation attracts many researchers to study the solution of this equation. Since the AC equation is highly stiff and nonlinear, therefore, it is not easy to find its exact analytical or numerical solution. Literature review reveals that many researchers use different numerical and analytical schemes and techniques to find the accurate solutions of AC equation. Tascan and Bekir [8] used the first integral method to find traveling wave solution of AC equation. Wazwaz [9] discussed solitons and kink solutions using the tanh-coth method. Tariq and Akram [10] investigated new traveling wave solution to the AC equation. Jeong et al. [11] performed a comparative study of various numerical techniques for solving the AC equation. Gui and Zhao [12] studied the traveling wave solutions of AC equation with a fractional Laplacian. Jeong and Kim [13] developed an explicit hybrid finite difference scheme for the numerical solution of AC equation. Shah et al. [14] proposed an algorithm based on the finite element method and implicit fractional-step θ−scheme to study the numerical solution of AC equation. Asif and Hasan [15] used FDM to obtain numerical solution of AC equation. Shin et al. [16] use hybrid finite element method for solving the AC equation.

Bulut [17] implemented the modified exponential function method to find analytical hyperbolic function solutions of AC equation.

In last few years, approximate analytical techniques attract attention of many researchers to find the approximate solutions of nonlinear PDE's due to their simplicity and easy approach. Series of methods have been established by mathematicians and researchers such as the homotopy perturbation method (HPM) [18, 19], homotopy analysis method (HAM) [20–22], variational iteration method (VIM) [23–25], Adomian decomposition method (ADM) [26–28], differential transform method (DTM) [29–31], tanh method (THM) and extended tanh method [32–34], use of trigamma function [35], Catalan-type numbers [36], and polynomials [37–39]. Traveling wave solutions are also prominent techniques in finding the closed form of solutions. In recent years, many researchers used these techniques to tackle various problems in physics and engineering [40–42]. Among the above stated methods and others, VIM is more dominant and power full due to its simplicity and fast convergence. Literature survey reveals that the AC equation has not been solved using this method. Therefore, in this research work, we have used VIM to obtain series solution of AC equation. The obtained results are compared with numerical solution of [15] and analytical hyperbolic function solution obtained in [17]. We also compare the VIM solution with traveling wave solution obtained in [8].

2. Mathematical Model

The AC equation is the mathematical model that describes phase transitions in material science,

$$\frac{\partial u}{\partial t} = \varepsilon^2 \Delta u - f(u), \quad (x,t) \in \Omega \times [0,T], \tag{1}$$

along with initial conditions (IC's) and boundary conditions (BC's)

$$u(x,0) = f(x), u(a_1,t) = u(a_n,t) = h(t). \tag{2}$$

In equation (1), ε is the thickness parameter which represents the width of transition region, and Ω is a bounded domain. The term $f(u) = F'(u)$ with $F(u) = 1/4(u^2-1)^2$ represents double well potential.

3. Basic Idea of VIM

VIM is established by a Chinese scientist Ji-Huan He in 1999 using the concept of optimization and Lagrange multiplier [23]. In literature, this method is applied to solve large class of PDE's [23–25]. The key advantage of the suggested technique is that verity of the nonlinear problems can be solved without linearization or small perturbations. It is very straightforward in implementation and iterations can be performed using any symbolic mathematical software like Maple. Later on, many researchers make several modifications and use it to solve several mathematical models [43–51]. In order to elaborate the key notion of VIM, we take the differential equation of the form

$$L[u(x,t)] + N[u(x,t)] = f(x,t), \tag{3}$$

where "L" represents the linear operator, "N" represents the nonlinear operator, and $f(x,t)$ is the forcing function. By using VIM, the constructed correctional functional can be written as [23–25]

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda \left\{ L[u_n(x,\tau)] + N\left[\tilde{u}_n(x,\tau)\right] - f(x,\tau) \right\} d\tau. \quad n = 0,1,2,3\ldots, \tag{4}$$

where "λ" is the Lagrange multiplier, $\tilde{u}_n(x,\tau)$ is the restricted variation, and $u_n(x,\tau)$ shows the nth approximate solution. The Lagrange multiplier can be calculated using optimality conditions and integration by parts. There are some alternative ways to calculate the Lagrange multiplier discussed in [52]. The restricted variation gives $\delta\tilde{u}_n(x,\tau) = 0$. The approximate series solution can be obtained after successive iterations for $n \geq 0$. Finally, we can write the series solution as

$$u(x,t) = \lim_{n\longrightarrow\infty} [u_n(x,t)]. \tag{5}$$

4. Numerical Experiments

In this section, we have solved the AC equation considering various IC's with the help of VIM taking thickness parameter one, i.e., $\varepsilon = 1$. We compare the converging series solution with the existing numerical and analytical solutions through absolute error tables which validates the applicability of VIM. Furthermore, we also discuss the VIM solution of AC equation with the transition layer parameter. Effect of the transition parameter on concentration is also studied through plots. CPU time for various iterations in case of each example is also presented through tables.

4.1. Example 1. Consider the AC equation of the form [15, 17]

$$u_t(x,t) = u_{xx}(x,t) + u(x,t) - u^3(x,t), \quad x \in [0,1], t > 0, \tag{6}$$

with the IC

$$u(x,0) = -\frac{12(-1+\tanh[0.416667(0.3+0.848528x)])}{24+30(1+\tanh[0.416667(0.3+0.848528x)])}. \tag{7}$$

Taking

$$u(x,0) = u_0(x,t) = -\frac{12(-1+\tanh[0.416667(0.3+0.848528x)])}{24+30(1+\tanh[0.416667(0.3+0.848528x)])}. \tag{8}$$

The constructed functional for equation (6) using equation (4) is

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\tau)\left[\frac{\partial u_n(x,\tau)}{\partial t} - \frac{\partial^2 \tilde{u}_n(x,\tau)}{\partial x^2} + \left(\tilde{u}_n(x,\tau)\right)^3 - \tilde{u}_n(x,\tau)\right]d\tau, \tag{9}$$

where "$\lambda(\tau)$" represents the Lagrange multiplier which can be calculated via optimality conditions and restricted variation. Here, $\tilde{u}(x,\tau)$ shows the restricted variation, i.e., $\delta\tilde{u}(x,\tau) = 0$. Taking variation on both sides of equation (9), the equation becomes

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta\int_0^t \lambda(\tau)\left[\frac{\partial u_n(x,\tau)}{\partial t} - \frac{\partial^2 \tilde{u}_n(x,\tau)}{\partial x^2} + \left(\tilde{u}_n(x,\tau)\right)^3 - \tilde{u}_n(x,\tau)\right]d\tau, \tag{10}$$

or

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta\int_0^t \lambda(\tau)\left[\frac{\partial u_n(x,\tau)}{\partial t}\right]d\tau. \tag{11}$$

Now, using integration by parts, we have

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta\lambda(\tau)u_n(x,t) - \int_0^t \delta\lambda'(\tau)u_n(x,\tau)d\tau = 0. \tag{12}$$

Hence, we have the following stationary conditions:

$$\begin{aligned} \lambda'(\tau) &= 0|_{\tau=t}, \\ 1+\lambda(\tau) &= 0|_{\tau=t}, \end{aligned} \tag{13}$$

which gives

$$\lambda(t) = -1. \tag{14}$$

Equation (9) becomes

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left[\frac{\partial u_n(x,\tau)}{\partial t} - \frac{\partial^2 u_n(x,\tau)}{\partial x^2} + \left(u_n(x,\tau)\right)^3 - u_n(x,\tau)\right]d\tau, \quad n = 0,1,2\ldots \tag{15}$$

For $n = 0, 1$, equation (15) can be written as

$$u_1(x,t) = u_0(x,t) - \int_0^t \left[\frac{\partial u_0(x,\tau)}{\partial t} - \frac{\partial^2 \tilde{u}_0(x,\tau)}{\partial x^2} + \left(\tilde{u}_0(x,\tau)\right)^3 - \tilde{u}_0(x,\tau)\right]d\tau, \tag{16}$$

$$u_2(x,t) = u_1(x,t) - \int_0^t \left[\frac{\partial u_1(x,\tau)}{\partial t} - \frac{\partial^2 \tilde{u}_1(x,\tau)}{\partial x^2} + \left(\tilde{u}_1(x,\tau)\right)^3 - \tilde{u}_1(x,\tau)\right]d\tau. \tag{17}$$

Using $u_0(x,t)$ from equation (8) to equation (16) and after integration, we get

$$u_1(x,t) = \frac{-12(-1+\tanh(0.1250001+0.3535536162x))}{(54+30\tanh(0.1250001+0.3535536162x))} + \left\{ \begin{array}{c} \left(\frac{8.485286789\tanh(0.1250001+0.3535536162x)}{54+30\tanh(0.1250001+0.3535536162x)}\right) \\ \left(\frac{(0.3535536162-0.3535536162\tanh(0.1250001+0.3535536162x)^2)}{54+30\tanh(0.1250001+0.3535536162x)}\right)+\ldots+ \\ \frac{-12(-1+\tanh(0.1250001+0.3535536162x))}{(54+30\tanh(0.1250001+0.3535536162x))} \end{array} \right\} t. \tag{18}$$

Equation (17) gives

$$u_2(x,t) = u_1(x,t) - \int_0^t \left[\frac{\partial u_1(x,\tau)}{\partial t} - \frac{\partial^2 \tilde{u}_1(x,\tau)}{\partial x^2} + \left(\tilde{u}_1(x,\tau)\right)^3 - \tilde{u}_1(x,\tau)\right] d\tau. \tag{19}$$

Using equation (18) in equation (19), we get

$$u_2(x,t) = u_1(x,t) - \left[-\frac{1728(-1+\tanh(0.1250001+0.3535536162x))^3}{(54.+30\tanh(0.1250001+0.3535536162x))^3}\right] t - 0.5 \left\{ \begin{array}{c} \left(\frac{8.485286789\tanh(0.1250001+0.3535536162x)}{54.+30.*\tanh(.1250001+0.3535536162x)}\right) \\ \left(\frac{0.3535536162-0.3535536162\tanh(0.1250001+0.3535536162x)^2}{54+30\tanh(.1250001+0.3535536162x)}\right) \\ +\ldots+\left(\frac{\left(1.099693168\times 10^5(-1+\tanh(0.1250001+0.3535536162x))^3\right)}{(54+30\tanh(0.1250001+0.3535536162x))^4}\right) \\ \left(\frac{(0.3535536162-0.3535536162\tanh(0.1250001+0.3535536162x)^2)}{(54+30\tanh(0.1250001+0.3535536162x))^4}\right) \\ \left(\frac{\tanh(0.1250001+0.3535536162x)}{(54+30\tanh(0.1250001+0.3535536162x))^4}\right) \end{array} \right\} t^2 + \ldots. \tag{20}$$

Similarly, for $n = 2, 3, 4, \ldots$, we can get $u_3(x,t), u_4(x,t), u_5(x,t), \ldots$.

4.1.1. Absolute Error Analysis. Table 1 shows absolute error analysis between the solutions obtained by the modified exponential function method [17], finite difference method [15], and variational iteration method. Absolute error is calculated for fourth and fifth iterations of the variational iteration method. It is clear from Table 1 that error decreases with the increase of the number of iterations. The desired and accurate result can be obtained by performing more iterations.

4.2. Example 2. In this section, we consider the AC equation (8).

$$u_t(x,t) = u_{xx}(x,t) + u(x,t) - u^3(x,t), \quad x \in [0,1], t > 0, \tag{21}$$

Table 1: Absolute error between numerical results obtained by FDM, analytical solution obtained by using the tanh function method, and series solution obtained by VIM at $t = 0.01$.

x_i	Numerical solution (FDM)	Tan hyperbolic functions solution (u_{THFS})	Solution by VIM after four and five iterations		Absolute error in case of FDM $\lvert u_{\text{THFS}} - u_{\text{FDM}}\rvert$	Absolute error in case of VIM after four iterations $\lvert u_{\text{THFS}} - u_4\rvert$	Absolute error in case of VIM after five iterations $\lvert u_{\text{THFS}} - u_5\rvert$
			u_4	u_5			
0.00	0.184247	0.184258	0.184163	0.184218	1.06626×10^{-5}	9.3825×10^{-5}	4.0028×10^{-5}
0.01	0.183187	0.183197	0.183105	0.183158	1.06500×10^{-5}	9.2078×10^{-5}	3.9320×10^{-5}
0.02	0.182131	0.182142	0.182051	0.182103	1.06370×10^{-5}	9.0363×10^{-5}	3.8625×10^{-5}
0.03	0.181080	0.181091	0.181002	0.181052	1.06236×10^{-5}	8.8678×10^{-5}	3.7940×10^{-5}
0.04	0.180034	0.180044	0.179957	0.180007	1.06098×10^{-5}	8.7025×10^{-5}	3.7268×10^{-5}
0.05	0.178992	0.179003	0.178917	0.178966	1.05956×10^{-5}	8.5401×10^{-5}	3.6607×10^{-5}
0.06	0.177955	0.177966	0.177882	0.177929	1.05810×10^{-5}	8.3807×10^{-5}	3.5957×10^{-5}

with the IC

$$u(x,0) = -\frac{1}{1+e^{-(\sqrt{2}/2)x+C}}, \tag{22}$$

where C_0 is the integrating constant. Taking

$$u(x,0) = u_0(x,t) = -\frac{1}{1+e^{-(\sqrt{2}/2)x+C}}. \tag{23}$$

Using equation (23) in equation (21) and after integration, we get

$$u_1(x,t) = u_0(x,t) - \frac{3}{2}\left(\frac{e^{-(\sqrt{2}/2)x+C}}{\left(1+e^{-(\sqrt{2}/2)x+C}\right)^2}\right)t. \tag{24}$$

Similarly, using equation (24) in equation (19), we get

$$u_2(x,t) = u_1(x,t) - \frac{9}{32}\left[\frac{t^2 e^{-(\sqrt{2}/2)x+C}\left(-8e^{-(\sqrt{2}/2)x+C} + 3t^2 e^{-\sqrt{2}x+2C} - 4e^{-2\sqrt{2}x+4C} - 8e^{-\sqrt{2}x+2C} - 8e^{-(3\sqrt{2}/2)x+3C} - 4\right)}{\left(1+e^{-3\sqrt{2}/2x+C}\right)^6}\right]. \tag{25}$$

Similarly, for $n = 2, 3, 4, \ldots$ we can get $u_3(x,t), u_4(x,t), u_5(x,t), \ldots$.

4.2.1. Absolute Error Analysis. Table 2 represents the absolute error between traveling wave solution [8] and the series solution obtained by using VIM. Absolute error is estimated for fourth and fifth iterations of VIM solution. The decreasing observation of error indicates the convergence of series solution. Higher accuracy can be attained by performing more iterations.

4.3. Example 3. Here, we consider ε-version of AC equation of the form [51]

$$u_t(x,t) = \varepsilon^2 u_{xx}(x,t) + u(x,t) - u^3(x,t), \quad x \in [0,1], t > 0, \tag{26}$$

with initial profile containing the transition layer thickness parameter.

$$u(x,0) = \frac{1}{2}\left[1 - \tanh\left(\frac{x}{2\sqrt{2}\,\varepsilon}\right)\right]. \tag{27}$$

In order to implement VIM, taking

$$u(x,0) = u_0(x,t) = \frac{1}{2}\left[1 - \tanh\left(\frac{x}{2\sqrt{2}\,\varepsilon}\right)\right], \tag{28}$$

the correctional functional for equation (26) will take the form

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left[\frac{\partial u_n(x,\tau)}{\partial t} - \varepsilon^2 \frac{\partial^2 \tilde{u}_n(x,\tau)}{\partial x^2} + \left(\tilde{u}_n(x,\tau)\right)^3 - \tilde{u}_n(x,\tau)\right] d\tau. \quad n = 0, 1, 2 \ldots. \tag{29}$$

For $n = 0, 1$, equation (29) gives

Table 2: Absolute error between the VIM solution and traveling wave solution at $C = 4.0$ and $t = 0.01$.

x_i	Traveling wave solution (u_{TWS})	Solution by VIM after four and five iterations		Absolute error in case of VIM after four iterations $\lvert u_{\mathrm{TWS}} - u_4 \rvert$	Absolute error in case of VIM after five iterations $\lvert u_{\mathrm{TWS}} - u_5 \rvert$
		u_4	u_5		
−5	−0.0005415497601	−0.0005415497570	−0.0005415497601	3.1×10^{-12}	0.00000
−4	−0.001097714003	−0.001097713996	−0.001097713993	7.00×10^{-12}	1.00×10^{-12}
−3	−0.002223780506	−0.002223780417	−0.002223780437	8.90×10^{-11}	6.910×10^{-11}
−2	−0.004499794675	−0.004499793394	−0.004499794109	1.281×10^{-9}	5.66×10^{-10}
−1	−0.009084075273	−0.009084065692	−0.009084070631	9.581×10^{-9}	4.642×10^{-9}
0	−0.01825307504	−0.01825299748	−0.01825303736	7.756×10^{-8}	3.768×10^{-8}
1	−0.03633741749	−0.03633679823	−0.03633712022	6.1926×10^{-7}	2.9727×10^{-7}
2	−0.07104238533	−0.07103762964	−0.07104015939	4.75569×10^{-6}	2.22594×10^{-6}
3	−0.1342747341	−0.1342406215	−0.1342595369	3.4116×10^{-5}	1.51972×10^{-5}
4	−0.2392905691	−0.2390661759	−0.2391995376	2.24393×10^{-4}	9.10315×10^{-5}
5	−0.3894877140	−0.3880553670	−0.3889625354	1.43235×10^{-3}	5.251786×10^{-4}

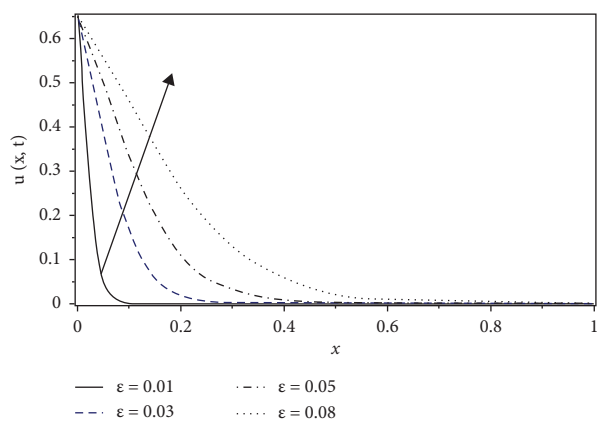

Figure 1: Effect of the transition parameter ε on concentration $u(x,t)$ for $t = 0.5$.

$$u_1(x,t) = \frac{1}{8\cosh^2(x\sqrt{2}/4\varepsilon)}\left[4\cosh^2\left(\frac{x\sqrt{2}}{4\varepsilon}\right) - 4\sinh\left(\frac{x\sqrt{2}}{4\varepsilon}\right)\cosh\left(\frac{x\sqrt{2}}{4\varepsilon}\right) + 3t\right],$$

$$u_2(x,t) = \frac{1}{6144(\sinh(x\sqrt{2}/4\varepsilon) + \cosh(x\sqrt{2}/4\varepsilon))\cosh^6(x\sqrt{2}/4\varepsilon)}\left[\begin{array}{c} -1024\cosh^7\left(\frac{x\sqrt{2}}{4\varepsilon}\right) + 1024\sinh\left(\frac{x\sqrt{2}}{4\varepsilon}\right)\cosh^6\left(\frac{x\sqrt{2}}{4\varepsilon}\right) \\ +\left(1728t^2 + 2304t + 3840\right)\cosh^5\left(\frac{x\sqrt{2}}{4\varepsilon}\right) \\ +1728\sinh\left(\frac{x\sqrt{2}}{4\varepsilon}\right)\left(t^2 + \frac{4}{3}t - \frac{4}{27}\right)\cosh^4\left(\frac{x\sqrt{2}}{4\varepsilon}\right) \\ -1728\cosh^3\left(\frac{x\sqrt{2}}{4\varepsilon}\right)t^2 + \left(-81t^4 - 432t^3\right)\cosh\left(\frac{x\sqrt{2}}{4\varepsilon}\right) \\ -81\sinh\left(\frac{x\sqrt{2}}{4\varepsilon}\right)t^4 \end{array}\right]. \quad (30)$$

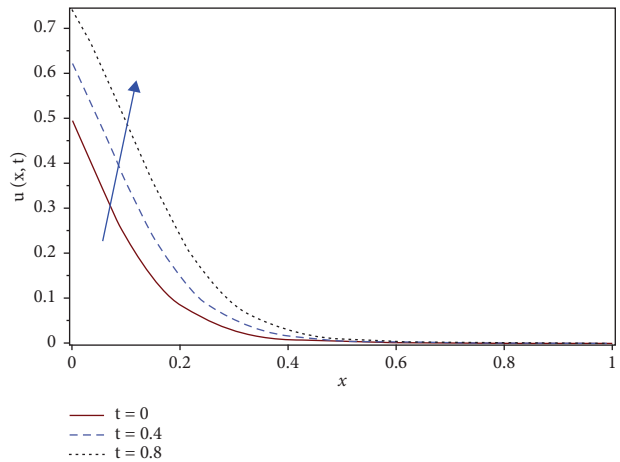

Figure 2: Variation of concentration $u(x,t)$ for the fixed value of transition parameter $\varepsilon = 0.06$ at various time scales.

Table 3: CPU timing for different iterations of Example 1.

Iteration nos.	CPU time in seconds
1st	0.53
2nd	0.57
3rd	0.82
4th	25.85
5th	725.35

Table 4: CPU timing for different iterations of Example 2.

Iteration nos.	CPU time in seconds
1st	0.54
2nd	0.87
3rd	2.43
4th	452.73
5th	1374.76

Table 5: CPU timing for different iterations of Example 3.

Iteration nos.	CPU time in seconds
1st	0.51
2nd	0.54
3rd	0.62
4th	1.45
5th	326.23

Further iterations can be performed for $n = 2, 3, \cdots$ up to required accuracy.

4.3.1. Relation between the Transition Parameter and Concentration. Study of relation between the transition parameter and concentration plays an important role in engineering processes, especially formation of alloys and multiphase systems. Figures 1 and 2 depict the effect of the transition parameter ε on concentration of binary fluids in binary alloys. Figure 1 explores that concentration increases as we increase the values of the transition parameter. Physically larger estimations of transition variable upsurges the rate of diffusion and, therefore, concentration increases. Figure 2 indicates the increasing behavior of concentration with the passage of time for the fixed value of transition parameter. In fact, at fixed diffusion rate, the concentration of molecules in the lower concentrated region increases with the time when diffusion takes place from the higher concentrated region to the lower concentrated region and thus concentration enhances.

5. Conclusion

In the present work, the VIM is effectively applied to obtain an approximate analytical solution of the AC equation. The AC model is solved for different kinds of initial conditions. A series solution obtained using VIM is compared with the numerical results obtained by using FDM and analytical solutions obtained by the traveling wave solution method and the tanh function method. Computed results are illustrated via absolute error tables, and a good agreement of the results is observed. Decreasing behavior of error in Tables 1 and 2 validates the application of VIM for solving highly nonlinear PDE's. Results are compared after fourth and fifth iterations of VIM. The error tables show the congregating behavior of series solution to the analytical solution and numerical solution. CUP time for different iterations in case of every example is tabulated through Tables 3–5. A mathematical software Maple is used to perform iterations and absolute error calculations.

Data Availability

Data sharing is not applicable to this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Safdar Hussain and Fazal Haq contributed to formal Analysis, methodology, and writing the original draft. Abdullah Shah supervised over all activities. Dilsora

Abduvalieva contributed to formal analysis and code validation. Ali Shokri analyzed and interpreted the data and contributed to funding acquisition.

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