Hindawi
International Journal of Intelligent Systems
Volume 2023, Article ID 1918409, 24 pages
https://doi.org/10.1155/2023/1918409

Research Article

Integrated Stochastic Investigation of Singularly Perturbed Delay Differential Equations for the Neuronal Variability Model

Iftikhar Ahmad,[1] **Syed Ibrar Hussain**,[2] **Hira Ilyas**,[1] **Layouni Zoubir**,[3] **Mariam Javed**,[1] **and Muhammad Asif Zahoor Raja**[4]

[1]*Department of Mathematics, University of Gujrat, Gujrat, Pakistan*
[2]*Dipartimento di Matematica e Informatica, Università degli Studi di Palermo, Via Archirafi 34, Palermo 90123, Italy*
[3]*Department of SEG, University of Badji Mokhtar, Annaba, Algeria*
[4]*Future Technology Research Center, National Yunlin University of Science and Technology, 123 University Road, Section 3, Douliou, Yunlin 64002, Taiwan*

Correspondence should be addressed to Syed Ibrar Hussain; syedibrar.hussain@unipa.it

Received 17 January 2023; Revised 5 May 2023; Accepted 19 May 2023; Published 26 May 2023

Academic Editor: Vasudevan Rajamohan

The proposed research utilizes a computational approach to attain a numerical solution for the singularly perturbed delay differential equation (SPDDE) problem arising in the neuronal variability model through artificial neural networks (ANNs) with different solvers. The log-sigmoid function is used to construct the fitness function. The implementation of ANN on SPDDE problems is formulated for different solvers and trained with different weights. The optimization solvers such as the genetic algorithm (GA), sequential quadratic programming (SQP), and pattern search (PS) are hybridized with the active set technique (AST) and the interior-point technique (IPT) and is used to check the accuracy and rapid convergence of the numerical results of the SPDDE model. The numerical outcomes demonstrate that the system is easy to handle and efficient to solve with boundary conditions. Moreover, we used the mean residual error for one hundred runs for each solver to validate the accuracy of the proposed scheme.

1. Introduction

The perturbed theory is a huge collection of algebraic methods that approximate the outcomes of problems that have no analytical solution in the closed form. This method reduces a hard problem to a comparatively simple problem of infinite sequence that may be solved logically. These problems are based on a small parameter. The perturbed theory is normally based on two forms [1, 2]. One is a regular perturbed theory whose series is a power series within ε, which has no vanished radius of convergent and defined as a singularly perturbed theory whose sequence either does not appear as a power series or if it appears, the power series has a vanished radius of convergence [3, 4].

A special feature of all regular perturbed theories is the accurate solution. However, a small nonzero $|\varepsilon|$ efficiently proceeds the unperturbed neither zeroth-order solution when $\varepsilon \longrightarrow 0$. Sometimes, there is no solution to unperturbed problems in singularly perturbed theory. The solution whose function ε may stop existing when $\varepsilon = 0$, but when a solution to the unperturbed problem does exist, its numerical features vary from the real result for arbitrarily little nonzero ε [5–7]. In these cases, the solution $\varepsilon = 0$ is essentially distinct in quality from the corresponding solutions attained in the boundary $\varepsilon \longrightarrow 0$. We should categorize the problem as the regular perturbed theory if no change exists in quality [8–10].

A singularly perturbed ordinary differential equation (ODE) whose highest derivative is multiplied with a small parameter is termed as a perturbed parameter. The solution of SPDDEs was initiated in the year 1968 [11]. In this era, different surveys were conducted by researchers [12–14].

Kadalbajoo and Reddy carried out asymptotic as well as numerical techniques for the solution of the singularly perturbed theory [15]. The solution of SPDDEs differs rapidly in the region which is known as layers that may be obvious in the solution or its gradient and frequently seem at the boundary region. Several problems in science as well as in engineering, elasticity, control theory, biosciences, and fluid mechanics are created by SPDDEs, such as those present in red cell models [16–20].

In the literature, the delay differential equations (DDEs) were considered since the 1940s, inspired by control problems; for instance, balancing the location of a ship with pushing water from a tank at one verge of the vessel to the tank on other verges, see also [21, 22]. In 1949, Myshkis described DDEs [23]. In early 1963, Cooke and Bellman described DDEs, which appeared outside SU [24], which represent the basic theory more than the dynamical system's standpoint about semigroups as well as semiflows, further parallel towards the ODE theory as long as possible.

The numerical solutions at the earlier times are used to determine the value of terms in SPDDEs. The delay in the process emerges because of the necessity of positive time to detect the guidance and respond to it [8, 25–28]. Recently, some researchers presented physical examples of delay differential equations, like the periodic oscillation of respiration frequency within constant conditions [29–33]. This delay is produced by cardiorespiratory inefficiency in the biological circuit commanding the CO_2 level in the blood [34, 35]. Furthermore, some applications of DDEs are in biological sciences. The DDE can be categorized into the retarded delay differential equation and the neutral differential equation. DDEs applications arise in the field of the control theory, explanation in human pupil reflex [36, 37], as well as on numerical modeling in biological sciences [38], HIV infection [39], and so on.

In early 1968, Kadalbajoo and Reddy approximated the solution of SPDDEs, and several surveys and reviews of various researchers have been presented. The study on different asymptotic and numerical methods for explaining singular perturbation queries is discussed [15].

Using a parametric cubic spline, the nonlinear singularly perturbed DDEs are changed into linear singularly perturbed DDEs by the quasilinearization method [40]. If the smaller order of the singular perturbation parameter in the delay is not sufficient, the approach of increasing the delay term in Taylor's series may lead to a bad approximation. We construct a special type of mesh in such a way that the term containing delay lies on the nodal points after discretization. The finite difference technique on Shishkin mesh is utilized to determine SPDDEs of convection diffusion kind with the integral border condition [41, 42]. This technique approximately belongs to first order convergent. The numerical solution for second-order SPDDEs is provided by the uniform finite difference method. The solution to the problem is utilized by a hybrid difference method that lies on a Shishkin-type mesh. The interior and boundary layer occurs in the exact solution because of the delay term [43, 44].

Pramod Chakravarthy et al. utilized the finite difference technique with fitted operators by Numerov's method to approximate the solution of SPDDE [45]. The parameterized SPDDE system using a uniformly convergent numerical scheme is discussed in [46]. To approximate the numerical solution of SPDDE, Chakravarthy with his team utilized an exponentially fitted finite difference method to handle the large delay [47].

Shishkin mesh utilized a hybrid initial value technique to approximate the numerical solution of SPDDE for boundary value problems with a noncontinuous convection factor and a source term [48]. To approximate the numerical solution and its absolute error for the boundary value problem for the linear and nonlinear singular perturbed DDE, we utilize the fixed point method [49].

The singularly perturbed DDEs use the terminal boundary value method to obtain the solution such as exhibiting layer behaviour [50]. By presenting a terminal point, the unique problem is distributed into internal and external region problems. To solve both the internal and external region problems, the second-order finite difference method has been used. The technique is iterative to the terminal point [51, 52].

The SPDDE used reproducing kernel technique (RKM) and cannot obtain better approximate solutions. Now, the singularly perturbed DDE used a piecewise reproducing kernel technique to approximate the numerical solution [35]. The linear singular perturbed differential equation with delay in the convection term is changed into a linearized delay term by utilizing two-term Taylor series expansion. The SPDDE uses an asymptotic numerical hybrid technique to uniformly approximate the solution [53, 54]. Usually, the singularly perturbed nonlinear DDEs of the boundary value problems play an important role in clarifying different uses such as the theory of nonpremixed combustion [55] and chemical reactions [56]. Kadalbajoo and Sharma built a finite-difference technique to approximate the numerical solution of SP nonlinear DDEs [57, 58].

A finite difference technique and a B-spline collocation technique have been recommended for small delay queries, respectively [59, 60]. An initial value technique and a uniformly valid finite difference technique for convection diffusion problems with smooth data have been recommended, respectively [61], while the authors have recommended a singularly perturbed problem with nonsmooth data to be used as an initial value method. Boundary value problems containing DD calculations arose in reading the mathematical display of several practical occurrences, such as a microscale heat transfer, reaction-diffusion equations, stability, and control including control of chaotic systems [62, 63].

Due to their enormous importance in the control of ships, biological sciences, light absorption using stellar objects, chemistry, discrete mathematics, the medical industry, particle physics, object models, finance, engineering disciplines, community composition, medicine, electromagnetics, contagious diseases, telecommunications equipment, reaction mechanisms, and control models, delay differential mathematical systems have gained enormous significance for scientists/researchers [64]. Finding the solutions to singular systems, which have enormous significance and are thought to be difficult to replicate because of the solitary point at the origin, is never easy. Lane Emden, which has enormous relevance and a long history, is among the important singular types of the models. Many applications of SPDDEs can be found in astronomy, quantum mechanics, and gas cloud-based systems [11, 65].

This work presents a novel singular model that is rigid and more intricate due to the delay and perturbed factors. To solve the model, a stochastic numerical computational strategy is developed. The solvers for stochastic processing can be used to solve systems with numerical approximation, delayed, and fractional order. Some of the key features and highlights of the proposed study are listed as follows:

(1) A novel intelligent formulation is implemented to analyze the second-order SPDDE through ANN by using different solvers such as the GA, SQP, and PS, and hybridized with the AST and IPT.
(2) The accuracy, stability, and efficiency of the proposed approach are validated by presenting the detailed statistical analysis and mean residual error analysis.
(3) A comparative study is presented based on residual errors by comparing the results obtained from the GA, GA-IPT, and SQP for different problems in the form of tables and graphs.
(4) The hybrid optimum solvers have obtained the solution of SPDDE in less time and reduced the computational complexity and ensured the accuracy and rapid conversion of the obtained numerical solution.
(5) Some notable benefits of the technique include consistency, reliability, improved workflow, simplicity of comprehension, and encompassing pertinency, in addition to the accurate projections of the GA, SQP, and IPT framework.

The organization of the manuscript is designed as follows: Section 2 discusses the overall mathematical modeling, and it explains the modeling of neural networks, fitness function, learning techniques, and psuedocode for GA-IPT. Section 3 presents the results and discusses explaining the statistical analysis briefly. Section 4 explains the conclusions, remarks, and future recommendations.

2. Mathematical Designs of the Model

The SPDDE with its BVP is given as

$$\varepsilon \frac{d^2u}{dy^2} + m(y)\frac{du}{dy} + n(y)u(y-\delta) = l(y), \quad 0 < y < 2, \tag{1}$$

with the boundary conditions

$$u(y) = \varphi(y), \quad y \in [-\delta, 0] \text{ and } u(2) = B, \tag{2}$$

where $0 < \varepsilon \ll 1, \delta$ is the delay parameter and particular suitably flat functions on $[0, 2], \varphi(y)$ is a smooth function on $[-\delta, 0]$, and B is a given constant which is free of ε; the BVP (1) along with (2) shows a robust boundary level $y = 0$ [41].

If $m(y) < 0, m(y), n(y), and\, l(y)$ are particular suitably flat works on $[0, 2], \varphi(y)$ is flat work on $[-\delta, 0]$ and B is a constant which is free of ε, consequently the BVP (1) with (2) shows a robust border level at $y = 2$ [61].

2.1. Mathematical Modeling. Here, we have designed the neural networks in the form of linear second-order SPDDEs, along with their boundary conditions.

2.1.1. Neural Networks Modeling. The mathematical model of SPDDEs is presented by "feed-forward" ANNs through the following continual mapping founded in the form of a single "input, hidden, and output" layer by solution $\hat{u}(y)$ and its corresponding derivatives are given as follows:

$$\begin{cases} \hat{u}(y) = \sum_{j=1}^{p} \alpha_j f(\beta_j y + \gamma_j), \\ \frac{d\hat{u}}{dy} = \sum_{j=1}^{p} \alpha_j f'(\beta_j y + \gamma_j), \\ \frac{d^2\hat{u}}{dy^2} = \sum_{j=1}^{p} \alpha_j f''(\beta_j y + \gamma_j), \\ \cdots \\ \frac{d^k\hat{u}}{dy^k} = \sum_{j=1}^{p} \alpha_j f^k(\beta_j y + \gamma_j), \end{cases} \tag{3}$$

where p is the specified number of neurons, f is the activation function, and α, β, and γ represent real-valued component bounded weights and defined as follows:

$$\mathbf{w} = \left(\alpha_1, \alpha_2, \cdots, \alpha_p, \beta_1, \beta_2, \cdots, \beta_p, \gamma_1, \gamma_2, \cdots, \gamma_p\right),$$

$$\begin{cases} \widehat{u}(y) = \sum_{j=1}^{p} \dfrac{\alpha_j}{1 + e^{-\left(\beta_j y + \gamma_j\right)}}, \\ \dfrac{d\widehat{u}}{dy} = \sum_{j=1}^{p} \alpha_j \dfrac{\beta_j e^{-\left(\beta_j y + \gamma_j\right)}}{\left(1 + e^{-\left(\beta_j y + \gamma_j\right)}\right)^2}, \\ \dfrac{d^2\widehat{u}}{dy^2} = \sum_{j=1}^{p} \alpha_j \beta_j^{\,2} \left(\dfrac{2e^{-2\left(\beta_j y + \gamma_j\right)}}{\left(1 + e^{-\left(\beta_j y + \gamma_j\right)}\right)^3} - \dfrac{e^{-\left(\beta_j y + \gamma_j\right)}}{\left(1 + e^{-\left(\beta_j y + \gamma_j\right)}\right)^2} \right), \\ \vdots \\ \dfrac{d^k\widehat{u}}{dy^k} = \sum_{j=1}^{p} \alpha_j \beta_j^{\,k} \left(\dfrac{k e^{-k\left(\beta_j y + \gamma_j\right)}}{\left(1 + e^{-\left(\beta_j y + \gamma_j\right)}\right)^{k+1}} - \dfrac{e^{-(k-1)\left(\beta_j y + \gamma_j\right)}}{\left(1 + e^{-\left(\beta_j y + \gamma_j\right)}\right)^k} \right). \end{cases} \tag{4}$$

The networking in equation (3) utilized the log-sigmoid function $f(x) = 1/1 + e^{-x}$; furthermore, its certain derivatives for the activation function of the network displayed in (3) are written in equation (4). Figure 1 expresses the complete structure of the neural networks model for SPDDEs in the form of a structural diagram. It shows the complete features, layers, weights, derivatives, and functions involved in the construction of the input, hidden, and output layers, respectively.

2.1.2. Fitness Function Expression. The suited combination for the equations through a set of equation (4) is utilized to design the SPDDE for equations (1) and (2). A fitness function is an expression for (1) with boundary conditions to define an error which depends on the sum of two mean square error functions, given as

$$\widehat{E} = \widehat{E}_1 + \widehat{E}_2, \tag{5}$$

$$\widehat{E}_1 = \frac{1}{N}\sum_{i=1}^{N}\left[\varepsilon \frac{d^2\widehat{u}_i}{dy^2} + m_i \frac{d\widehat{u}_i}{dy} + n_i \widehat{u}_i(y-\delta) - l_i\right]^2, \tag{6}$$

$$\widehat{E}_2 = \frac{1}{2}\left[(\widehat{u}(y) - \varphi(y))^2 + (\widehat{u}(2) - B)^2\right], \tag{7}$$

where h denotes the step size for the whole domain.

$$\begin{aligned} &\text{Where } N = \frac{1}{h}, m_i = m(y_i), n_i = n(y_i), l_i = l(y_i), \\ &\widehat{u}_i = \widehat{u}(y_i), y_i = \text{ih}. \end{aligned} \tag{8}$$

2.2. Learning Techniques. Here, a detailed review of optimization solvers GA, SQP, IPT, and PS for ANN is expressed.

The IPT and SQPs are among a group of local search techniques that have been successfully applied to both unconstrained and constrained optimization problems. The proposed optimization problem is transformed into more manageable subproblems in the process of SQP algorithms, and methods mostly based on Karush–Kuhn–Tucker conditions are used for iterative revisions. By taking advantage of the viable interior region of the optimization problem, the IPT iteratively updates the weights. In numerous stiff and nonstiff optimization problems of practical interest, the SQP and IPT have been frequently employed [66].

The GA is inspired by Darwin's well-known theory of evolution. It was established properly in the early 1970s with great work by John Holland known as the pioneer of GA solver. Initially, it was designed for problem solving and analysis. The GA is working with a set of weights in the population sample. Solutions of single best fit are taken in the next step and utilized to create a new population that is inspired by the whole process, and a new population will be fitter as compared to the previous selection. In 1992, for the first time, John Koza (JK) developed a program for different tasks through the GA. He named his technique genetic programming. GAs are optimization techniques and stochastic search, encouraged by natural development.

"Pattern search" (also called direct search) is a sort of numeric optimization technique that does not need a gradient. "Pattern search" was invented by Hooke and Jeeves. Most of the literature about this topic is referred to as "Hooke and Jeeves" along with the theory published in their article. The title PS has been used as a collective term for all methods which search concerning the current point in a measured direction for best function value. When the search gets to the best point, that point is modified as a new base and this point is a restart point for searching. In case of an unsuccessful search, either the search route is changed or the search border is cut below by diminishing the step size. PS techniques tend to have very easy tactics and hence are easy to utilize as the initial optimization method. Moreover, the PS techniques are also flexile and reliable in their use.

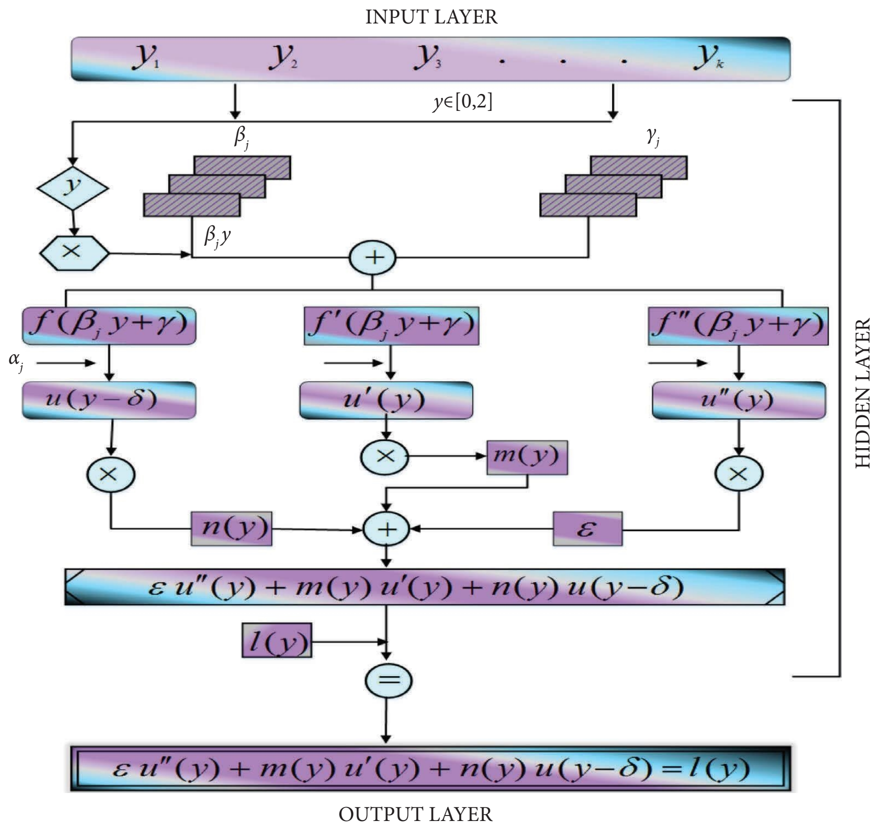

Figure 1: Structural diagram of the neural network model for SPDDEs.

The "interior-point technique" (also known as "barrier techniques") is a group of algorithms to solve nonlinear as well as linear convexity optimization models. "John von Neumann" [67] recommended an IPT for linear programming that was neither an efficient nor a "polynomial-time technique" in practice. It revolved around being slower than the commonly utilized simplex technique [68]. "Narendra Karmarkar" established a system for linear programming named Karmarkar's computation, in 1984, that runs likely in polynomial time which is also efficient for practice. It allowed solutions for linear programming methods that are behind the talents of the simplex technique.

Rather, the simplex technique attains the best result by passing through the interior of the feasible area. Eventually, modern IPTs have been instilled virtually in all fields for continuous optimization and have forced excellent improvements into the earlier procedures. In [69–72], the authors have presented the applications of higher-order differential equations in financial and business predictions/forecasts based on the nonlinear and linear types of models. Figure 2 represents the complete structure of the hybrid solver for the singularly perturbed delay model.

3. Results and Discussion

In this section, the proposed technique is implemented on three different problems for singularly perturbed delay differential equations and the results and discussions are presented. The numerical results of the SPDDE equation are obtained, analyzed, and presented by using "SQP," "GA," "PS," and hybrid techniques "GA-AST," "PS-AST," "GA-IPT," and "PS-IPT." The outcome of these three SPDDEs optimized the results by local and global techniques in neural network designs which are given. Algorithm 1 elaborates a thorough pseudocode for GA-IPT to find weights based on ANN for solving the SPDDE, which plays an essential role in this study.

3.1. Problem 1. Consider the following BVP for an SPDDE as

$$\begin{cases} \varepsilon \dfrac{d^2u}{dy^2} + 0.25\,u(y-\delta) - u(y) = 1, \\ u(0) = 1 \text{ and } u(1) = 0, \end{cases} \tag{9}$$

with boundary conditions at $\varepsilon = 2^{-4}$ and $\delta = 0.03$.

$$\begin{cases} \widehat{E} = \dfrac{1}{10}\displaystyle\sum_{i=1}^{10}\left[\varepsilon \dfrac{d^2\widehat{u}_i}{dy^2} + 0.25\,\widehat{u}_i(y-\delta) - \widehat{u}_i(y) - 1\right]^2, \\ + \dfrac{1}{2}\left[(\widehat{u}(0)-1)^2 + (\widehat{u}(1)-0)^2\right]. \end{cases} \tag{10}$$

We need to obtain the solution to the problem by utilizing the proposed methodology, as it depends upon the fitness function $\widehat{E}$ along with step size $h = 0.1$. Here, we determine the fitness function $\widehat{E}$ taking $N = 10$.

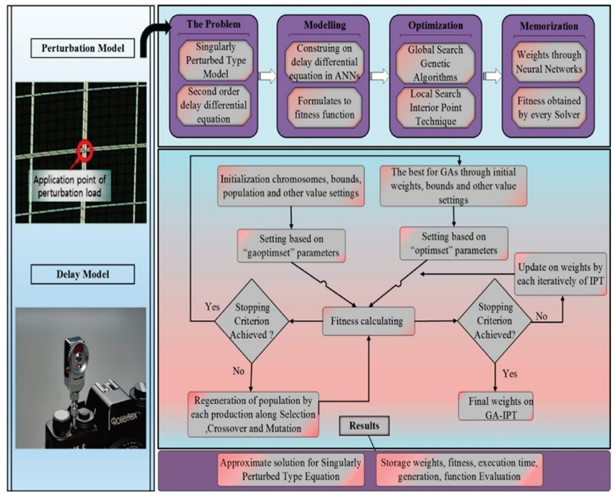

Figure 2: Flowchart of hybrid solver for the singularly perturbed delay model.

GA-IPT technique Start.
STEP: Initialization

Initially, the population is selected arbitrarily through the entries on the real line (number) toward express weights along with the equivalent elements identical toward the unknown weights of ANN paradigms.

Initialize routine for GAs as:

Population: 300, bounds: (–10, 10).

STEP: (Fitness calculation)

To estimate the fitness value of each member of the population in equation (5) utilizing the values for the network specified set on equations (3) and (4).

STEP: Ranking

In order of minimum value for fitness functions to the models, rank each individual concerning the populations. Members who performed well often had lower fitness values and vice versa.

STEP: Terminate criteria

Termination of the algorithm:

(ii) Obtained the objective level
(ii) Required generations achieved
(iii) Obtained tolerance and generations

Function tolerance: $1e^{-13}$, fitness limit: $1e^{-13}$

Other settings: By default. If the terminate criterion fits, then the hybridization level is.

Reproductive:

Create the following population by each generation.

Selection: Calls stochastic uniform selection as a function.

Crossover: Call for constraint dependent as function.

Mutation: Call for constraint dependent as function.

Others: By default.

Hybridization:

The interior point technique is incorporated for fine tuning of parameters by taking the best chromosome of GAs as a starting point. Setting up the parameters for IPT:

Finite difference: Forward differences.

X-tolerance: $1e^{-13}$

Function-tolerance: $1e^{-16}$; constraint tolerance: By default.

STEP: Recurrence

Repeat the whole hybrid procedure for hidden layers.

GA-IPT procedure end.

Algorithm 1: Pseudocode for GA-IPT to find weights based on ANN for solving the SPDDE.

$$\hat{u}_{PS-IPT}(y) = \frac{(3.14689)}{1+e^{-(4.32612\,y+1.73076)}} + \frac{(0.22124)}{1+e^{-(1.83084\,y+1.53809)}} + \cdots + \frac{(0.27836)}{1+e^{-(1.80760\,y+0.79412)}}, \tag{11}$$

$$\hat{u}(y) = \frac{\alpha_1}{1+e^{-(\beta_1 y+\gamma_1)}} + \frac{\alpha_2}{1+e^{-(\beta_2 y+\gamma_2)}} + \cdots + \frac{\alpha_{10}}{1+e^{-(\beta_{10} y+\gamma_{10})}},$$

$$\hat{u}_{PS}(y) = \frac{(0.91647)}{1+e^{-(8.19245\,y+1.40890)}} + \frac{(6.18157)}{1+e^{-((-5.18712)y+7.46590)}} + \cdots + \frac{(-8.56837)}{1+e^{-(8.98990\,y+2.59702)}}, \tag{12}$$

$$\hat{u}_{GA}(y) = \frac{(0.77375)}{1+e^{-(0.55587\,y+(-1.60025))}} + \frac{(3.25069)}{1+e^{-(2.03191\,y+1.06558)}} + \cdots + \frac{(-2.23298)}{1+e^{-(0.69839\,y+1.07222)}}, \tag{13}$$

$$\hat{u}_{GA-IPT}(y) = \frac{(-1.33463)}{1+e^{-(0.64838\,y+1.87804)}} + \frac{(1.48967)}{1+e^{-((-1.9882)y+(-1.1972))}} + \cdots + \frac{(1.35239)}{1+e^{-(1.89166\,y+0.84600)}}. \tag{14}$$

To obtain the solution for equation (9) with boundary conditions, we have utilized the GA, PS, and hybrid technique of GA-IPT and PS-IPT by MATLAB built-in functions, given in Table 1. Group of prepared weights with individual fitness for PS, GA, PS-IPT, and GA-IPT algorithms are displayed in (Figures3 and 4) as fitness functions expressed in equation (10). Tables 2 and 3 present the values of "number of weights for PS, GA, PS-IPT and GA-IPT" and residual errors of the proposed technique. The suggested solution of (given in equation (4)) the proposed model is designed by utilizing optimal weights and equations. (10)–(13) shows the optimal solutions obtained from PS-IPT, PS, GA, and GA-IPT optimization solvers.

The proposed technique of sorted, scattered numbers of runs and residual error of four individual solvers for problem 1 is shown in Figure 5, and the y-axes show that y lies in the selected interval from 0 to 1, and the residual error's graphical representation shows that the behaviour of the PS technique lies from 10^{-5} to 10^{-2} and the hybrid PS technique with IPT lies from 10^{-9} to 10^{-7}, GA technique lies from 10^{-5} to 10^{-3}, and hybrid GA technique with IPT lies from 10^{-9} to 10^{-7}. The 3-D display of best weights α, β, γ with the GA, GA-IPT, PS, and PS-IPT for problem 1 is displayed in Figure 6, and the learning curves of the proposed techniques PS, PS-IPT, and GA, GA-IPT for problem 1 are shown in Figures 3 and 4.

$$\begin{cases} \varepsilon \dfrac{d^2u}{dy^2} + 128\dfrac{du}{dy} + 0.25\,u\,(y-1) = 0.25\,(y-1), \\ 0 < y < 1.5, \\ u\,(y) = y, -1 \le y < 0 \text{ and } u\,(1.5) = 2. \end{cases} \tag{15}$$

3.2. Problem 2. We consider the following BVP for a SPDDE:

With boundary conditions at $\varepsilon = 10^{-2}$, we need to obtain the solution to the problem by utilizing the proposed methodology, which depends upon fitness function $\hat{E}$ along with the step size $h = 0.1$. Here, we have determined the fitness function $\hat{E}$, taking $N = 10$.

$$\hat{E} = \frac{1}{10}\sum_{i=1}^{10}\left[\varepsilon\frac{d^2\hat{u}_i}{dy^2} + 128\frac{d\hat{u}_i}{dy} + 0.25\,\hat{u}_i\,(y-1) - 0.25\,(y-1)\right]^2 + \frac{1}{2}\left[(\hat{u}\,(y)-y)^2 + (\hat{u}\,(1.5)-2)^2\right]. \tag{16}$$

$$\hat{u}_{\text{GA-IPT}}(y) = \frac{(-0.59867)}{1+e^{-(9.00000y+8.62809)}} + \frac{(8.11615)}{1+e^{-(4.64693\,y+7.91614)}} + \cdots + \frac{(8.22812)}{1+e^{-(0.06273\,y+2.86697)}}, \tag{17}$$

$$\hat{u}_{\text{SQP}}(y) = \frac{(0.02071)}{1+e^{-((-0.37235)y+1.13109)}} + \frac{(1.47340)}{1+e^{-(2.52201\,y+1.57181)}} + \cdots + \frac{(-1.28929)}{1+e^{-(2.55527\,y+1.54266)}}. \tag{18}$$

To find the results for equation (15) with the given conditions, we applied the GA and SQP for refinement, and we have hybridized it with IPT, i.e., GA-IPT by utilizing MATLAB with different parameter settings, shown in Table 1. Obtained weights with individual fitness for SQP, GA, and GA-IPT algorithms are shown in Figure 7 as fitness functions given in equation (16). Tables 4 and 5 present values of "number of weights for GA, GA-IPT, and SQP" and residual errors of the proposed technique. The suggested solution $\hat{u}(y)$ is given in equation (4); the proposed model is designed by utilizing optimal weights, and equations (17)–(19) shows the optimal solutions obtained from the GA, GA-IPT, and SQP optimization solvers.

Table 1: Setting of parameters for optimization solvers.

Techniques	Parameters	Settings
IPT	Solver	"Fmincon"
	Algorithm	"Interior-point"
	Start point	Randn (1, 30)
	Derivative	Approximated by solver
	Max iteration	By default
	Max- function evaluations	200000
	X-tolerance	$1e^{-15}$
	Function tolerance	$1e^{-18}$
	Constraint tolerance	$1e^{-15}$
	Others	By default
PS	Solver	Pattern search (PS)
	Start point	Randn (1, 30)
	Max iterations	3000
	X-tolerance	$1e^{-13}$
	Max-function evaluation	200000
	Function tolerance	10^{-16}
	Constraint tolerance	By default
	Others	By default
GA	Solver	Genetic algorithm (GA)
	Variables	30
	Generations	3000
	Selection	"Stochastic uniform"
	Initial scores	20
	Mutation/Crossover	Constraint
	Population size	300
	Fitness limit	$1e^{-13}$
	Function tolerance	$1e^{-13}$
	Constraint tolerance	By default

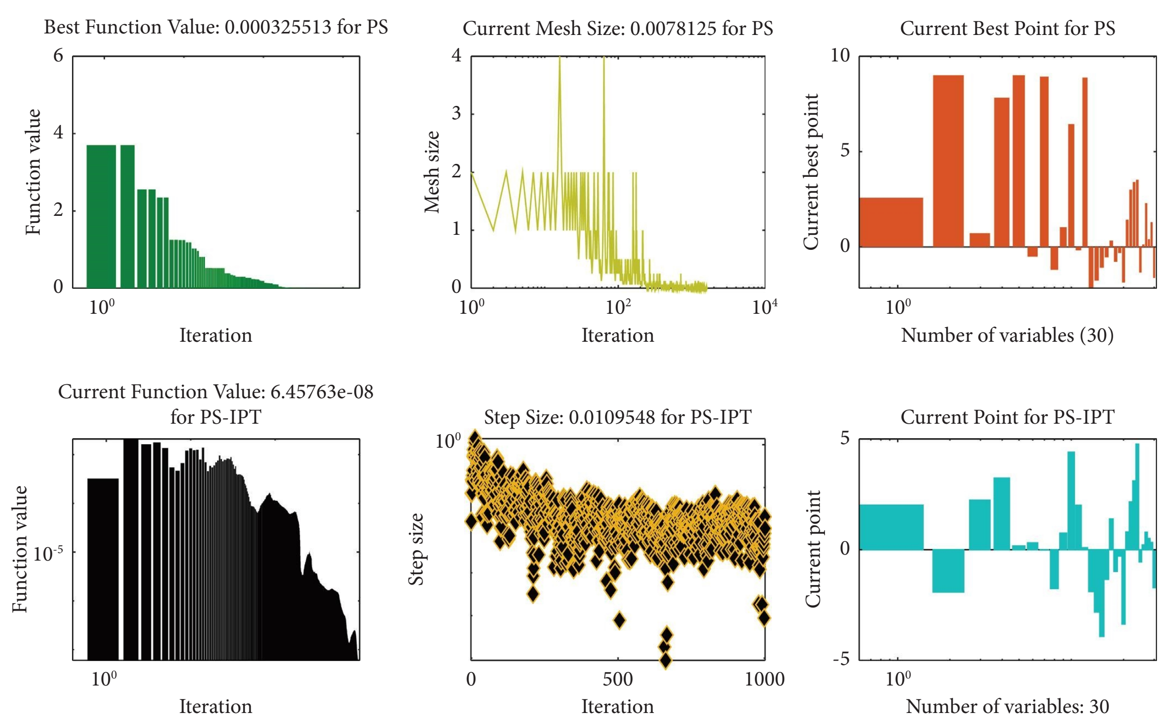

Figure 3: Learning curve of the proposed techniques PS and PS-IPT for problem 1.

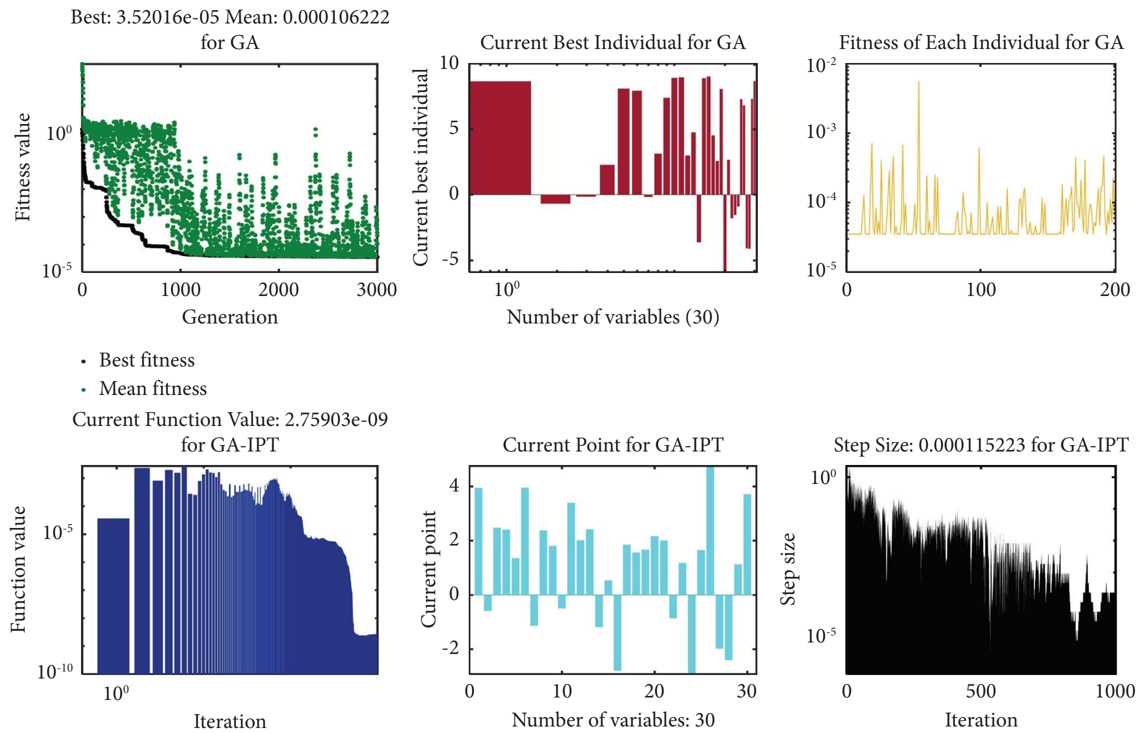

Figure 4: Learning curve of the proposed techniques GA and GA-IPT for problem 1.

Table 2: Optimized weights towards ANN through the designed technique in problem 1.

Indices	Optimized parameters with ANN models				
	j	PS	GA	PS-IPT	GA-IPT
α_j	1	0.9164750	0.7737568	3.1468994	−1.3346375
	2	6.1815714	3.2506946	0.2212471	1.4896793
	3	0.8292910	−0.7627727	1.2065874	5.1029969
	4	0.9478915	1.9019516	1.3411143	−0.1489729
	5	0.8707260	8.9998875	−1.0288080	−1.5649892
	6	−0.6630079	8.9995217	−2.6301087	1.8198148
	7	1.7726430	0.5511131	5.0873706	2.7695111
	8	−0.4553805	1.0497176	2.6232159	2.4811050
	9	−1.3061185	1.9138552	0.6367644	−0.8498850
	10	−8.5683795	−2.2329875	0.2783610	1.3523990
β_j	1	8.1924536	0.5558713	4.3261220	0.6483823
	2	−5.1871205	2.0319120	1.8308430	−1.9882203
	3	6.9167275	1.5949496	3.5293137	−2.2880579
	4	2.4095697	1.4065516	−1.3911130	0.7626022
	5	8.9866846	−1.7199822	−1.8664458	1.2618407
	6	1.7570514	2.8712440	−1.3487284	−2.2251951
	7	−8.9702150	5.3995972	2.5253342	−0.8397408
	8	2.0824703	1.3348065	−2.0723136	−3.5820637
	9	6.7255338	1.7258145	1.7239297	0.0462442
	10	8.9899067	0.6983951	1.8076067	1.8916643
γ_j	1	1.4089070	−1.6002584	1.7307678	1.8780401
	2	7.4659014	1.0655847	1.5380952	−1.1972149
	3	2.1152584	−1.1868336	0.8564890	4.0942542
	4	−0.0512475	1.2500164	1.9968474	2.7371949
	5	−0.2437504	3.4325258	1.6953031	−1.1186656
	6	−0.3976007	2.5739728	−0.4572963	−1.2461224
	7	−0.7923369	1.7820793	−4.3782226	1.6118485
	8	−1.1717247	1.3343562	−0.3893486	−1.8269782
	9	0.3539055	1.5395085	2.1204743	1.1849277
	10	2.5970263	1.0722228	0.7941233	0.8460052

Table 3: Residual error for the proposed techniques in problem 1.

Y	Residual errors			
	PS	GA	PS-IPT	GA-IPT
0.0	1.67245$E-05$	9.31838$E-06$	3.30328$E-09$	4.13518$E-09$
0.1	1.53841$E-03$	5.43679$E-04$	5.47624$E-08$	4.19201$E-08$
0.2	2.43828$E-03$	1.38167$E-03$	3.26438$E-07$	3.08938$E-07$
0.3	4.11607$E-03$	5.03819$E-03$	4.36329$E-07$	6.44600$E-07$
0.4	3.96567$E-03$	1.16996$E-02$	4.91945$E-08$	4.66148$E-07$
0.5	4.30404$E-03$	1.65080$E-02$	2.87183$E-07$	1.20356$E-06$
0.6	1.85032$E-03$	1.87952$E-02$	1.74559$E-07$	1.58017$E-06$
0.7	2.02445$E-03$	1.66318$E-02$	1.85185$E-07$	1.05983$E-06$
0.8	5.79775$E-03$	1.03803$E-02$	6.10314$E-07$	1.22827$E-06$
0.9	6.46407$E-03$	2.14090$E-03$	2.28990$E-07$	4.75857$E-07$
1	4.65838$E-03$	1.39188$E-04$	1.47434$E-08$	3.02143$E-08$

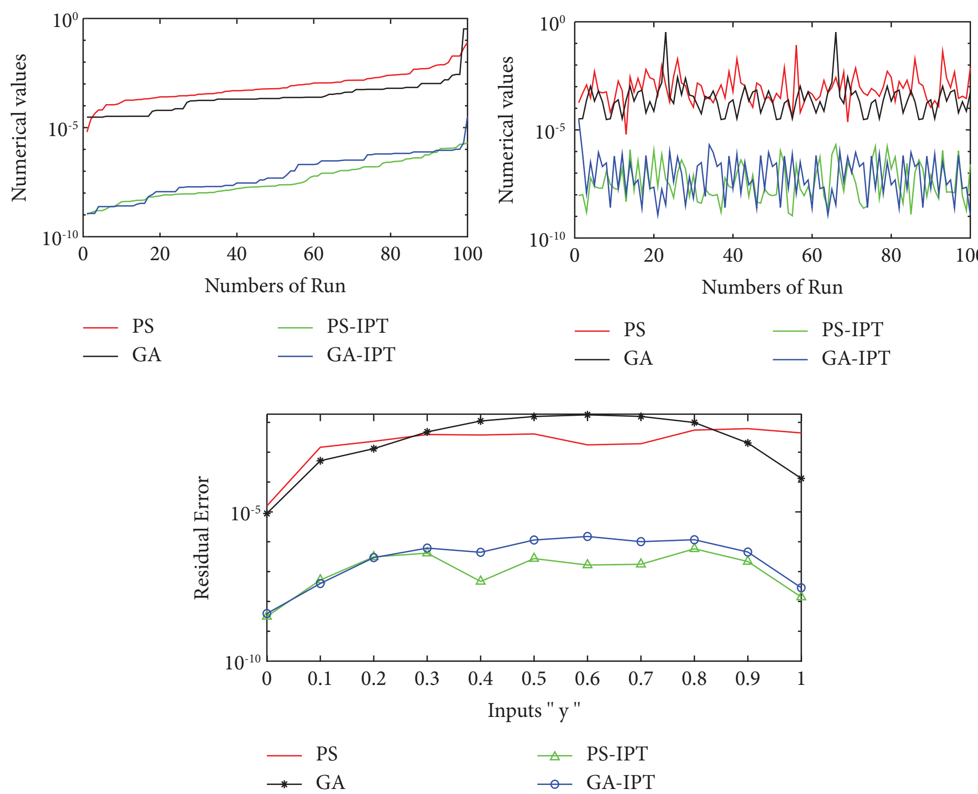

Figure 5: Sorted and scattered numbers of runs and the residual error for problem 1.

$$\hat{u}_{\mathrm{GA}}(y) = \frac{(-0.30664)}{1+e^{-((-7.73043)y+(-2.96601))}} + \frac{(-1.20102)}{1+e^{-((-2.35878)y+(-2.67451))}} + \cdots + \frac{(-8.07646)}{1+e^{-(5.16724\,y+7.08939)}}. \tag{19}$$

The proposed technique of sorted, scattered numbers of runs and residual error of four individual solvers for problem 2 are shown in Figure 8, and the y-axes display that y lies in the selected interval from 0 to 1, and the residual error of obtained graphical representation shows that the behaviour of the SQP technique lies from 10^{-5} to 10^{-4}, GA technique lies from 10^{-4} to 10^{-2}, and hybrid GA technique with IPT lies from 10^{-5} to 10^{-4}. The 3-D display of best weights α, β, γ with GA, GA-IPT, and SQP for problem 2 is displayed in Figure 9, and the learning curve of the proposed techniques GA, GA-IPT, and SQP for problem 2 is shown in Figure 7.

$$\begin{cases} \varepsilon \dfrac{d^2u}{dy^2} - 3\dfrac{du}{dy} + u\,(y-1) = 0, 0 < y < 2, \\ u(y) = 1, -1 \le y < 0 \text{ and } u\,(2) = 2. \end{cases} \tag{20}$$

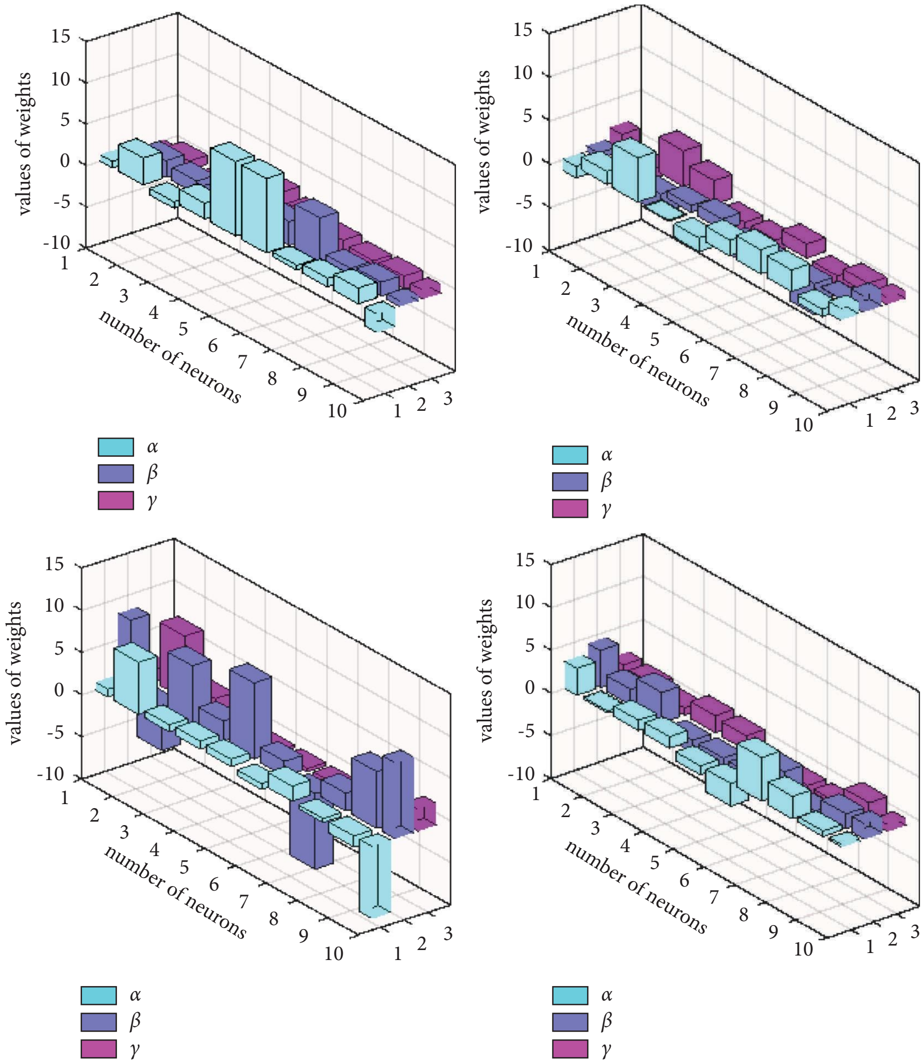

Figure 6: 3-D display of best weights of the GA, GA-IPT, PS, and PS-IPT for problem 1.

3.3. Problem 3. We consider the following BVP for an SPDDE: With boundary conditions at $\varepsilon = 10^{-2}$, we must obtain the solution to the problem by utilizing the proposed methodology, as it depends upon fitness function $\widehat{E}$ along with the step size $h = 0.1$. Here, we have determined the fitness function $N = 10$.

$$\widehat{E} = \frac{1}{10}\sum_{i=1}^{10}\left[\varepsilon\frac{d^2\widehat{u}_i}{dy^2} - 3\frac{d\widehat{u}_i}{dy} + \widehat{u}_i(y-1)\right]^2 + \frac{1}{2}\left[(\widehat{u}(y)-1)^2 + (\widehat{u}(2)-2)^2\right]. \quad (21)$$

To obtain the solution for equation (20) with boundary conditions, we have utilized the GA, PS, and hybrid technique of GA-IPT, PS-IPT by using the MATLAB built-in functions with the parameters set, given in Table 1 as IPT, PS, and GA, respectively. The residual error is presented in Figure 10 for problem 3. The group of prepared weights with individual fitness for PS, GA, PS-IPT, and GA-IPT algorithms are shown in Figure 11 as fitness functions are given in equation (21). Tables 6 and 7 present the values as "number of weights for PS, GA, PS-IPT, and GA-IPT" and residual errors of the proposed technique. The suggested solution $\widehat{u}(y)$ is given in equation (4), the proposed model is designed by utilizing optimal weights, and equations

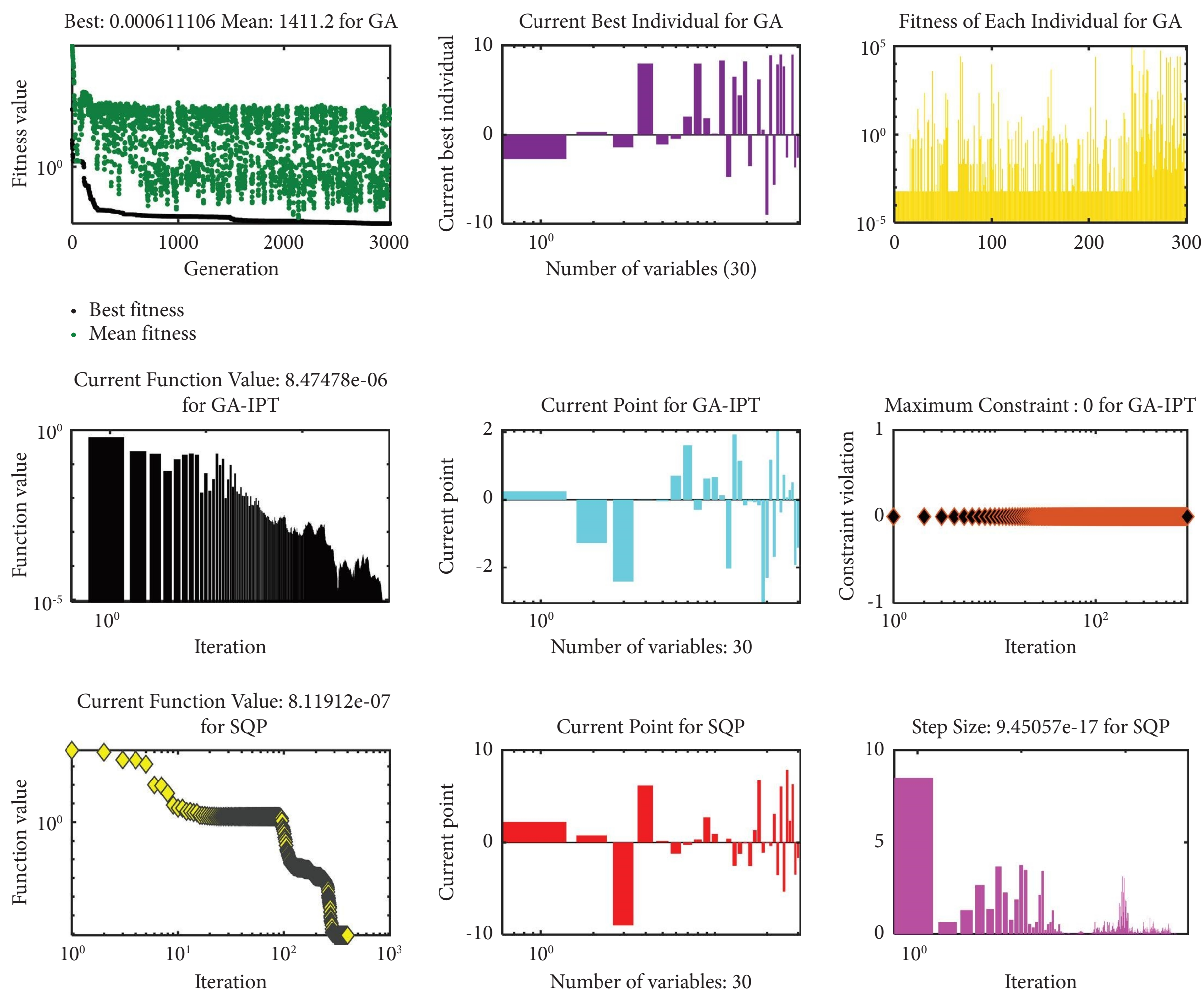

FIGURE 7: Learning curve of the proposed techniques GA, GA-IPT, and SQP for problem 2.

TABLE 4: Optimized weights toward ANN through the designed technique in problem 2.

Indexed	Optimized parameters with ANN models			
	j	GA	GA-IPT	SQP
α_j	1	−0.3066433	−0.5986779	0.0207156
	2	−1.2010241	8.1161527	1.4734028
	3	8.9842532	8.9999993	−0.3392609
	4	0.0773500	−0.9425981	−0.0010755
	5	−1.7397039	8.8649342	2.0380605
	6	8.9968926	−5.6430424	−9.0000000
	7	1.9807586	8.9754851	−1.9437645
	8	8.7069722	−0.4344950	0.5442588
	9	2.2906260	−7.4839824	−8.6412766
	10	−8.0764674	8.2281267	−1.2892910
β_j	1	−7.7304308	8.9999997	−0.3723538
	2	−2.3587874	4.6469365	2.5220127
	3	0.0344424	−0.0200903	−0.1267292
	4	−4.8840836	−1.6122136	−1.3105608
	5	8.8652874	0.0056376	0.0161680
	6	0.1108028	6.9869438	0.5443783
	7	−0.0156290	5.3994669	2.7784698
	8	−0.0972322	7.9333855	2.8628747
	9	0.8715110	7.4987751	1.7588169
	10	5.1672438	0.0627377	2.5552734

Table 4: Continued.

Indexed	Optimized parameters with ANN models			
	j	GA	GA-IPT	SQP
γ_j	1	−2.9660161	8.6280949	1.1310972
	2	−2.6745176	7.9161440	1.5718114
	3	2.4669648	−1.7590079	9.0000000
	4	−2.3066822	−2.6844709	1.5400303
	5	8.9609374	−1.3095311	0.1383488
	6	2.6876202	8.2425325	7.6122620
	7	0.0506930	8.1734344	6.4755557
	8	−2.8775411	7.0033038	2.6109926
	9	8.7434707	8.9745170	2.6863053
	10	7.0893938	2.8669696	1.5426654

Table 5: Residual error for the proposed techniques in problem 2.

Y	Residual errors		
	GA	GA-IPT	SQP
0.0	1.10753$E-04$	9.85444$E-06$	1.30948$E-05$
0.1	3.51240$E-04$	8.68236$E-05$	2.10384$E-04$
0.2	1.26247$E-03$	3.27473$E-05$	4.36154$E-05$
0.3	6.06292$E-04$	4.92249$E-05$	1.13348$E-04$
0.4	1.30458$E-03$	8.89636$E-06$	5.44375$E-05$
0.5	1.49585$E-03$	2.70841$E-05$	2.99286$E-05$
0.6	1.73233$E-03$	4.02804$E-05$	3.69801$E-05$
0.7	1.28022$E-03$	5.45395$E-05$	5.48185$E-05$
0.8	9.57652$E-04$	1.66669$E-05$	3.51978$E-05$
0.9	1.18028$E-03$	2.29189$E-05$	2.00808$E-05$
1	1.58956$E-03$	2.72872$E-05$	2.07580$E-05$

(22)–(25) show the optimal solutions obtained from PS, GA, and PS-IPT and GA-IPT optimization solvers.

$$\hat{u}_{\mathrm{PS}}(y) = \frac{(-1.06670)}{1 + e^{-(5.26583\,y+1.00006)}} + \frac{(0.91810)}{1 + e^{-((-8.34331)y+(-1.66416))}} + \cdots + \frac{(8.99449)}{1 + e^{-((-0.16243)y+1.96485)}}, \tag{22}$$

$$\hat{u}_{\mathrm{GA}}(y) = \frac{(6.96770)}{1 + e^{-(1.99107\,y+3.51608)}} + \frac{(-0.19620)}{1 + e^{-(8.43611\,y+8.93885)}} + \cdots + \frac{(7.86031)}{1 + e^{-(5.62569\,y+8.63041)}}, \tag{23}$$

$$\hat{u}_{PS-IPT}(y) = \frac{(6.87138)}{1 + e^{-((-1.28699)y+4.84758)}} + \frac{(-8.99257)}{1 + e^{-((-0.84986)y+(-2.86486))}} + \cdots + \frac{(-1.42723)}{1 + e^{-((-0.11311)y+(-0.70787))}}, \tag{24}$$

$$\hat{u}_{\mathrm{GA-IPT}}(y) = \frac{(-0.12362)}{1 + e^{-((-0.23698)y+(-0.92593))}} + \frac{(0.15182)}{1 + e^{-((-0.97499)y+(-0.32822))}} + \cdots + \frac{(0.03446)}{1 + e^{-((-0.16695)y+0.03775)}}. \tag{25}$$

The proposed technique of the residual error of four individual solvers for problem 3 is shown in Figure 9 and the y-axes show that y lies in selected interval from 0 to 1 and the obtained graphical representation shows that the behaviour of the PS technique lies from 10^{-3} to 10^{-2}, and hybrid PS technique with the IPT lies from 10^{-5} to 10^{-4}, GA technique lies from 10^{-4} to 10^{-2}, and hybrid GA technique with the IPT lies from 10^{-8} to 10^{-7}. The sorted and scattered numbers of runs for problem 3 are presented in Figure 9, and the y-axes displays that y lies in the selected interval from 0 to 100. The 3-D display of best weights α, β, and γ with GA, GA-IPT, PS, and PS-IPT for problem 3 is displayed in Figure 10, and best weights together with parameters of GA and GA-IPT are presented in Figure 11. Furthermore, the learning curve which presents the scheme systematic performance for hybrid solver is shown in Figures 12 and 13. The interval confidence levels are expressed for the problems in Figures 14–16, respectively. Tables 8–10 present a comparison of HSU with best (MCB) simultaneous tests for three problems. It shows the comparison for different hybridized solvers for various

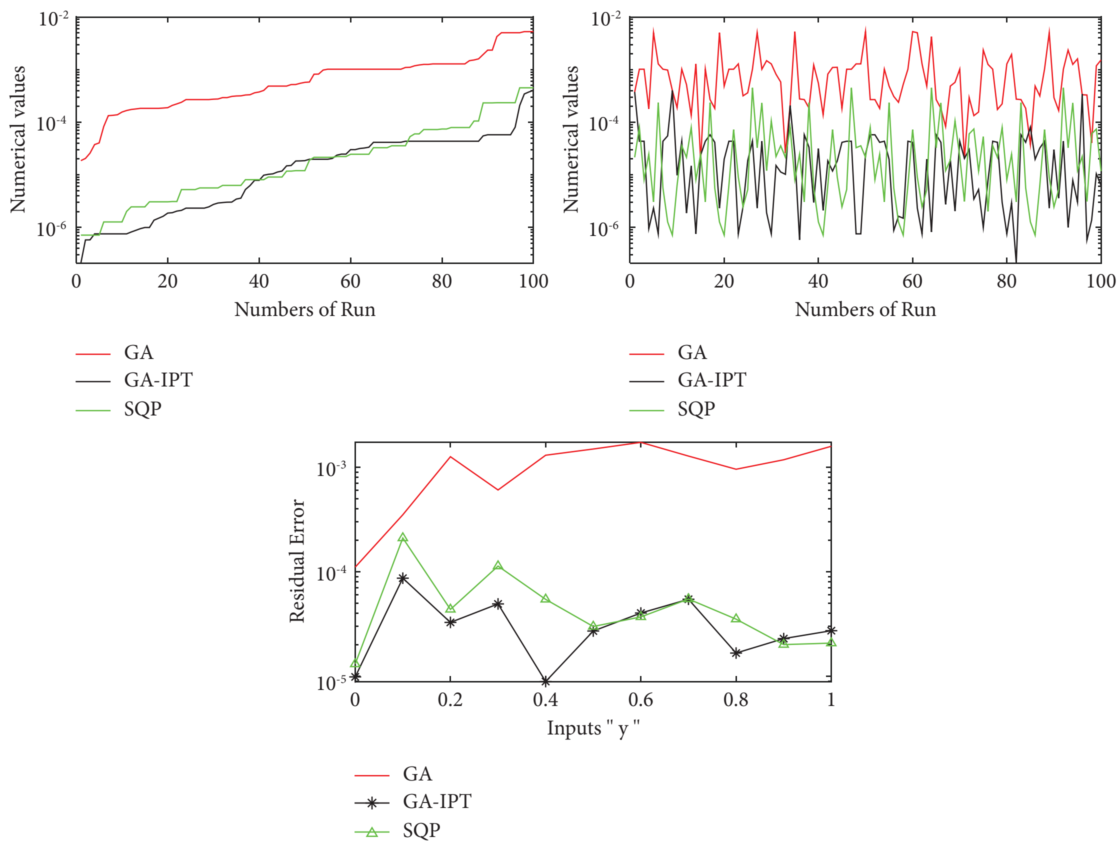

Figure 8: Sorted and scattered numbers of runs and the residual error for problem 2.

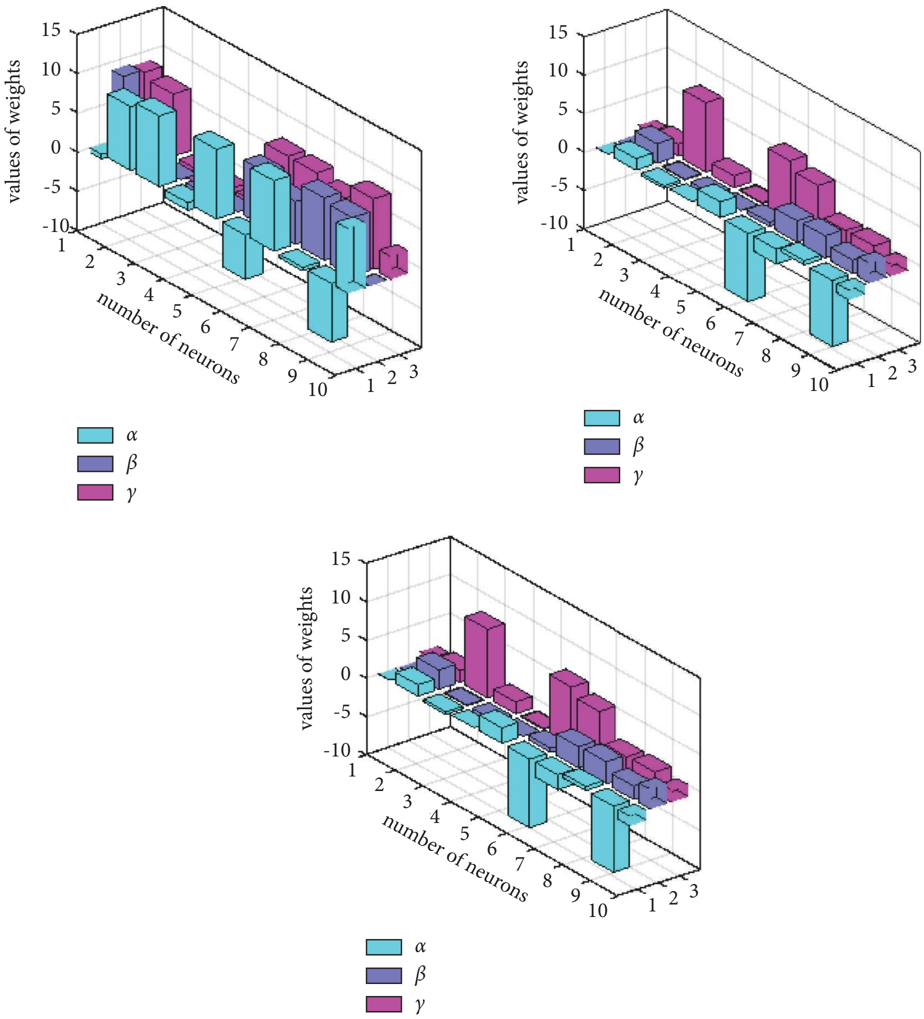

Figure 9: 3-D display of best weights of the GA, GA-IPT, and SQP for problem 2.

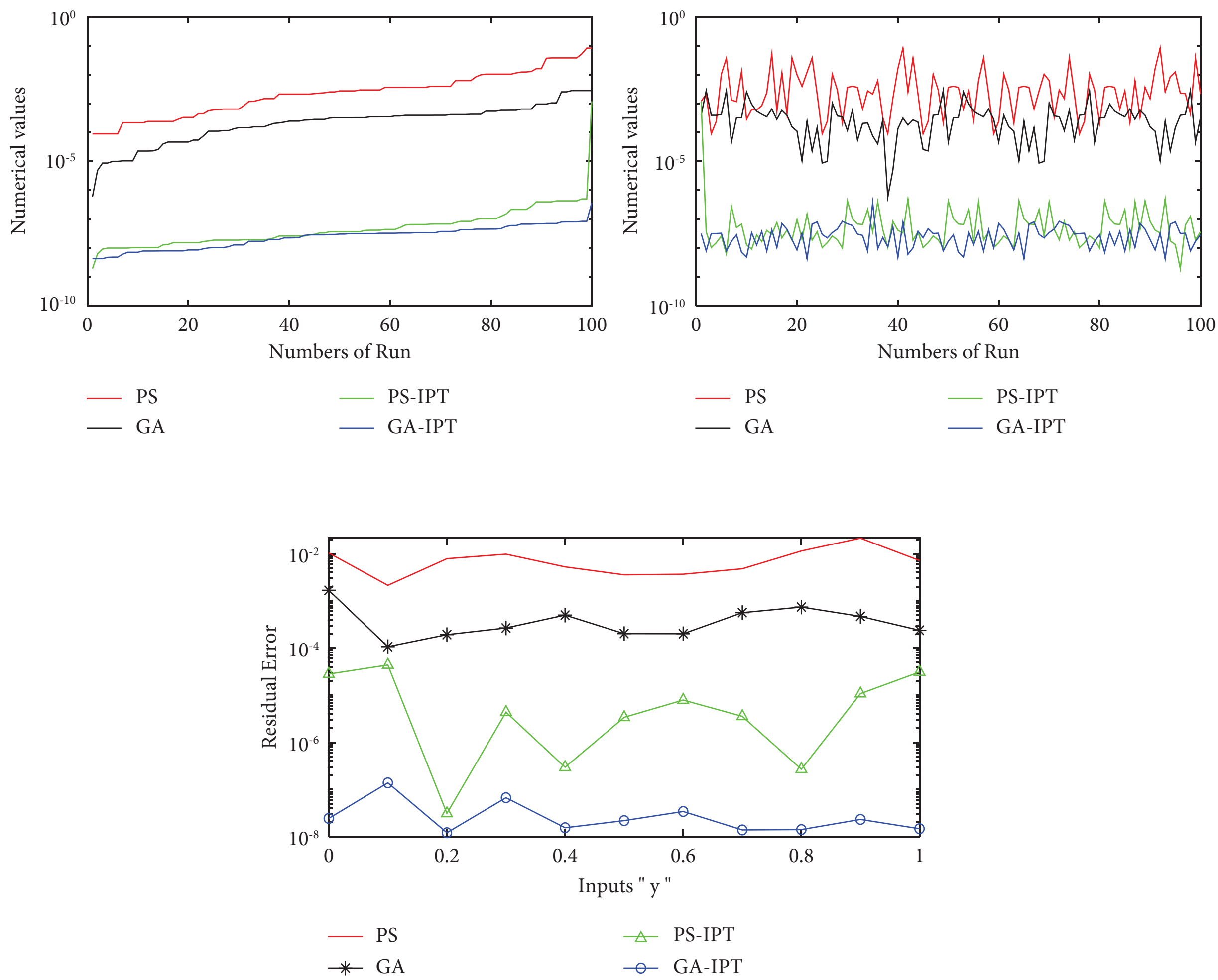

FIGURE 10: Sorted and scattered numbers of runs and the residual error for problem 3.

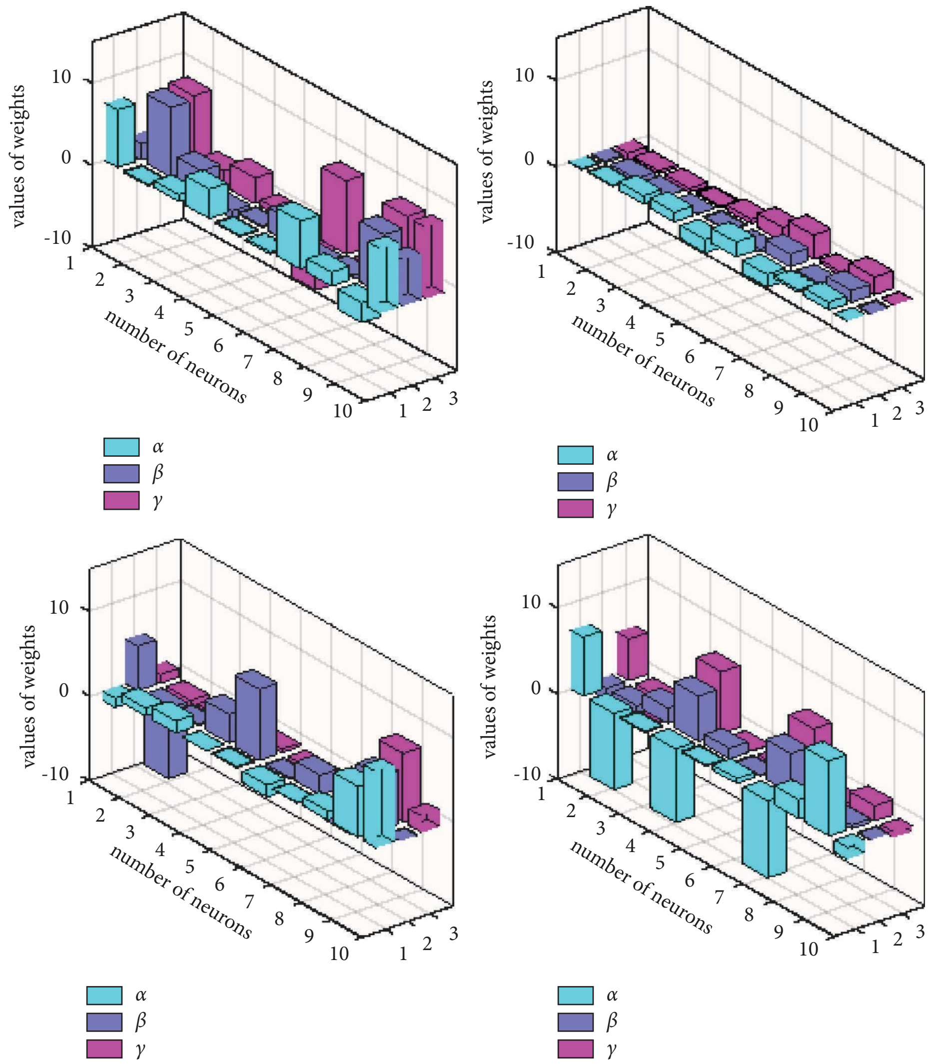

Figure 11: 3-D display of best weights of the GA, GA-IPT, PS, and PS-IPT for problem 3.

Table 6: Optimized weights toward ANN through the designed technique in problem 3.

Indices	Optimized parameters with ANN models				
	j	PS	GA	PS-IPT	GA-IPT
	1	−1.0667014	6.9677022	6.8713823	−0.1236203
	2	0.9181032	−0.1962030	−8.9925789	0.1518212
	3	1.4440710	0.8416656	−0.1578704	0.9692047
	4	−0.0290058	3.5331603	−8.6768061	1.1188702
α_j	5	0.1824522	−0.1795745	0.1650196	−1.5574211
	6	−1.3931810	−0.2637185	0.6222875	1.7603041
	7	−0.0845395	5.8502151	−8.9952144	−1.4745055
	8	1.0258214	1.6509182	2.2028382	0.2769833
	9	6.0202228	−2.3654917	8.8213978	0.8973661
	10	8.9944986	7.8603129	−1.4272394	0.0344642

Table 6: Continued.

Indices	Optimized parameters with ANN models				
	j	PS	GA	PS-IPT	GA-IPT
β_j	1	5.2658309	1.9910732	−1.2869910	−0.2369752
	2	−8.3433137	8.4361159	−0.8498656	−0.9749967
	3	0.2010970	2.8184493	1.8065413	0.0685090
	4	3.3951376	−0.8876038	5.2573940	0.0200204
	5	8.4006288	0.2282272	1.2128730	0.1556419
	6	−0.2286421	2.8250016	0.1366919	−0.3170663
	7	2.3361909	−0.9493155	4.7240429	1.4761096
	8	−2.6333540	0.3947841	0.2802119	−0.5567921
	9	6.2294457	7.4105844	0.4426208	1.3710782
	10	−0.1624363	5.6256926	−0.1131185	−0.1669521
γ_j	1	1.0000608	3.5160859	4.8475825	−0.9259315
	2	−1.6641645	8.9388543	−2.8648637	−0.3282290
	3	−0.5900346	1.8950385	0.4585321	−0.4450376
	4	−0.6921267	3.1092057	7.1949112	−0.1912964
	5	0.4070907	0.8495736	0.0419243	0.5343044
	6	−1.6702007	−6.4222482	−1.0026387	1.6625985
	7	0.0653843	8.9306464	6.7181488	2.8256164
	8	−1.2128472	−0.5848507	−0.6783619	0.3752158
	9	8.1286900	8.7420454	1.7816856	2.0496261
	10	1.9648559	8.6304174	−0.7078734	0.0377559

Table 7: Residual error for the proposed techniques in problem 3.

Y	Residual errors			
	PS	GA	PS-IPT	GA-IPT
0.0	$1.04193E-02$	$1.68507E-03$	$2.81351E-05$	$2.43471E-08$
0.1	$2.14534E-03$	$1.07289E-04$	$4.38149E-05$	$1.38542E-07$
0.2	$7.89241E-03$	$1.92880E-04$	$3.10187E-08$	$1.21096E-08$
0.3	$9.83712E-03$	$2.67722E-04$	$4.37483E-06$	$6.69807E-08$
0.4	$5.28494E-03$	$5.01788E-04$	$2.96411E-07$	$1.54421E-08$
0.5	$3.58451E-03$	$2.02807E-04$	$3.39506E-06$	$2.18717E-08$
0.6	$3.69571E-03$	$2.01873E-04$	$7.89721E-06$	$3.41485E-08$
0.7	$4.82142E-03$	$5.65441E-04$	$3.56027E-06$	$1.38857E-08$
0.8	$1.15101E-02$	$7.43195E-04$	$2.69433E-07$	$1.41226E-08$
0.9	$2.14956E-02$	$4.71775E-04$	$1.08002E-05$	$2.31872E-08$
1	$7.22192E-03$	$2.38969E-04$	$3.10677E-05$	$1.47303E-08$

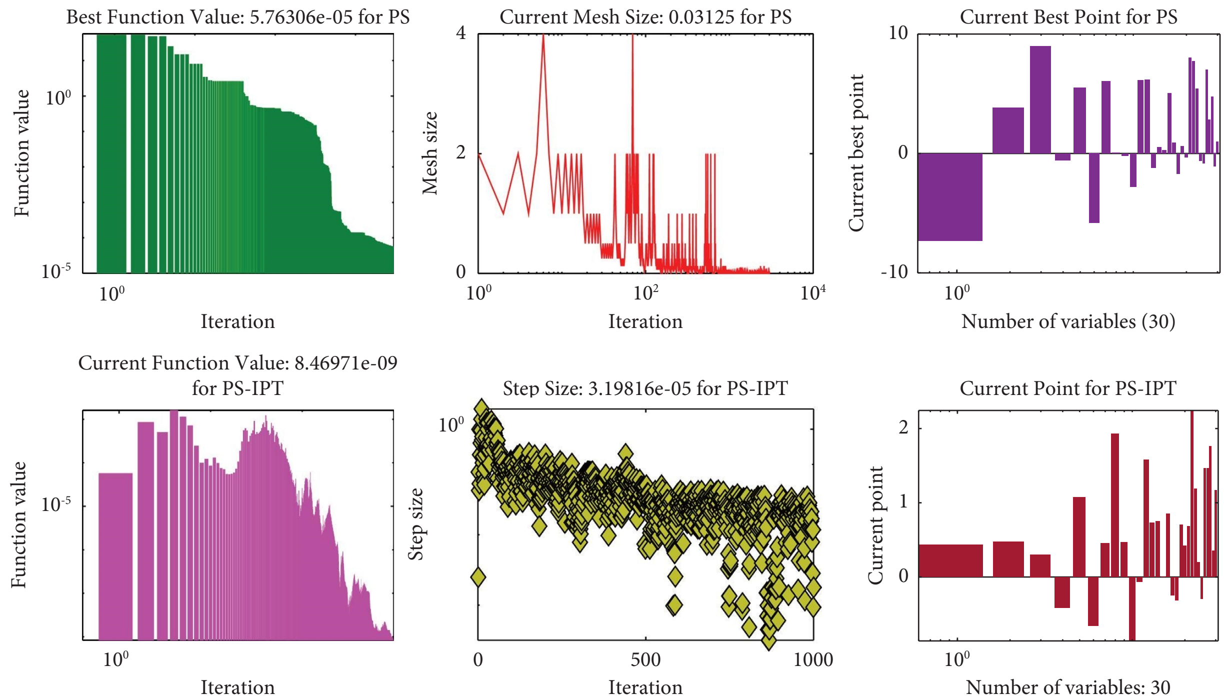

Figure 12: Learning curve of the proposed techniques PS and PS-IPT for problem 3.

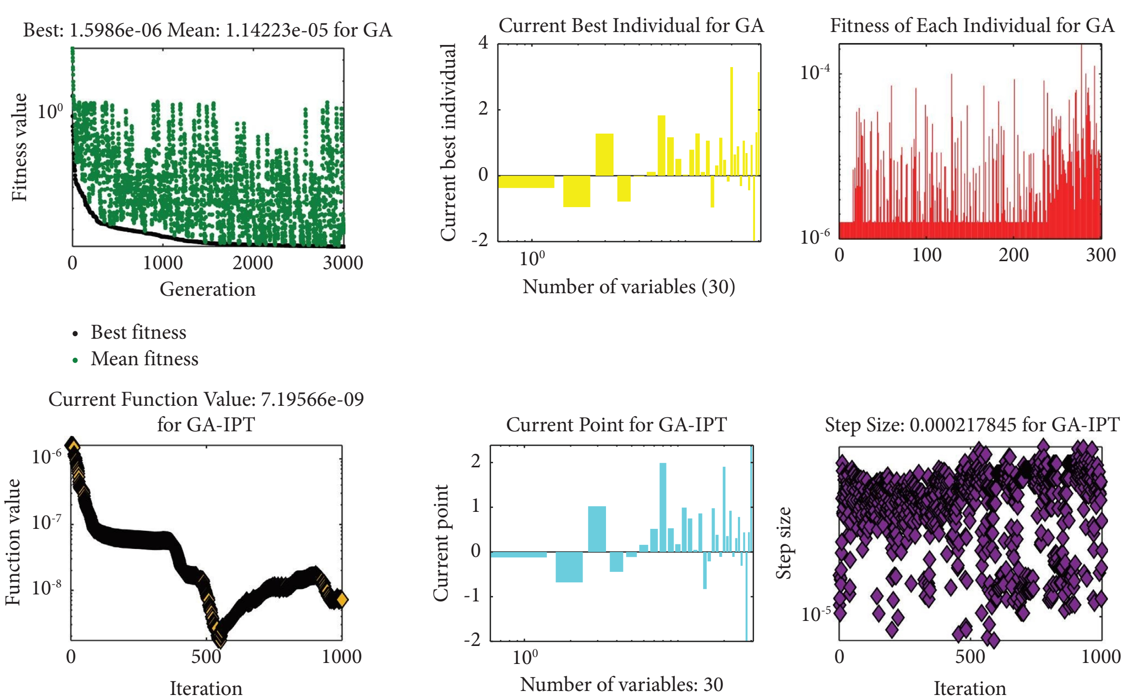

Figure 13: Learning curve of the proposed techniques GA and GA-IPT for problem 3.

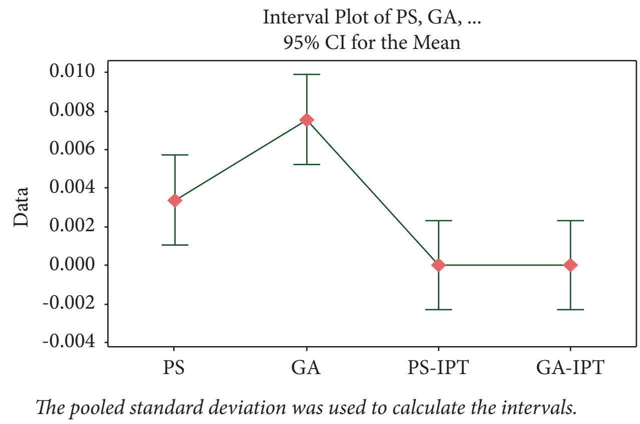

Figure 14: Interval plot at four techniques of the suggested methods for SPDDE in problem 1.

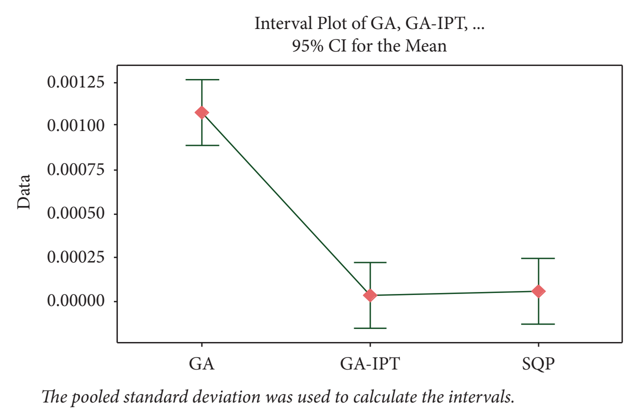

Figure 15: Interval plot at four techniques of the suggested methods for SPDDE in problem 2.

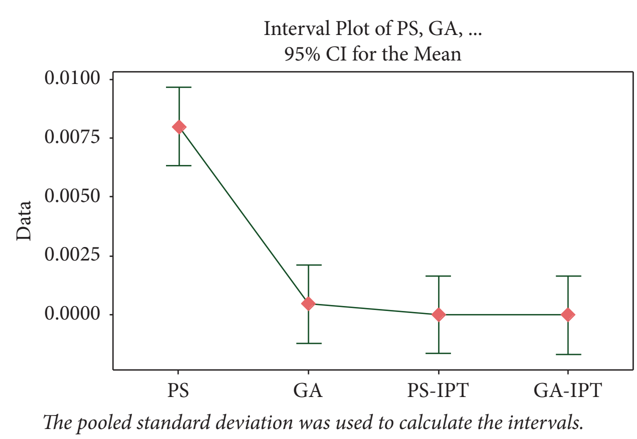

Figure 16: Interval plot at four techniques of the suggested methods for SPDDE in problem 3.

Table 8: HSU multiple comparisons with the best (MCB) simultaneous tests for problem 1.

Differences of levels	Differences of means	SEs of the difference	95% CI	T-value	Adjusted P value
PS-GA	−0.00419	0.00163	(−0.00766, 0.00000)	−2.56	0.019
GA-PS	0.00419	0.00163	(0.00001, 0.00766)	2.56	0.019
PS-IPT-GA	−0.00757	0.00163	(−0.01104, 0.00000)	−4.63	0.001
GA-IPT-GA	−0.00757	0.00163	(−0.01104, 0.00000)	−4.63	0.001

Table 9: HSU multiple comparisons with the best (MCB) simultaneous tests for problem 2.

Differences of levels	Differences of means	SEs of the difference	95% CI	T-value	Adjusted P value
GA-SQP	0.001022	0.000129	(0.000001, 0.001278)	7.93	0.001
GA-IPT-GA	−0.001045	0.000129	(−0.001301, 0.000001)	−8.11	0.001
SQP-GA	−0.001022	0.000129	(−0.001278, 0.000001)	−7.93	0.001

Table 10: HSU multiple comparisons with the best (MCB) simultaneous tests for problem 3.

Differences of levels	Differences of means	SEs of the difference	95% CI	T-value	Adjusted P value
PS-GA	0.00752	0.00116	(0.00001, 0.00999)	6.48	0.001
GA-PS	−0.00752	0.00116	(−0.00999, 0.00001)	−6.48	0.001
PS-IPT-PS	−0.00798	0.00116	(−0.01045, 0.00001)	−6.88	0.001
GA-IPT-PS	−0.00799	0.00116	(−0.01046, 0.00002)	−6.89	0.001

scenarios of P value and T-values, ensuring the CI on or above 95%. Moreover, the differences of means and SE are tabulated for all the cases of hybridized solvers.

4. Conclusions and Recommendations

In this research work, the numerical solution of second-order SPDDE is presented by an innovative "artificial neural network" (ANN). This study involves the use of a log-sigmoid function incorporated with optimization solvers such as the GA, PS, and SQP and with hybridized solvers such as GA-IPT, GA-AST, PS-IPT, and PS-AST. It is concluded that the proposed optimum technique has estimated the numerical results efficiently and accurately, and the results are also very fast convergent. A comparative study is presented based on residual errors that are compared with the GA, GA-IPT, and SQP for problems 1, 2, and 3 in the form of tables and graphs. To further enhance and ensure the stability and accuracy of the presented results, HSU multiple comparisons with the best (MCB) simultaneous tests for problems 1, 2, and 3 are presented in tabular forms, strengthening the statistical analysis.

It is observed that the presented research work has better optimum accuracy than other numerical techniques at present. The hybrid optimum solvers have approximated the numerical solution of the proposed problem in lesser time, ensuring the accuracy and reliability of the obtained results. Through the statistical analysis, it is concluded that GA-IPT hybridization has achieved better results as compared to PS, GA, SQP, and PS-IPT.

In the future, one may explore the higher-order, singular, and nonlinear kind of partial differential equations [73–78] by using ANNs.

Data Availability

The datasets generated during and/or analyzed during the current study are available on request from the corresponding author.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] G. A. Mensah, A. Orchini, and J. P. Moeck, "Perturbation theory of nonlinear, non-self-adjoint eigenvalue problems: simple eigenvalues," *Journal of Sound and Vibration*, vol. 473, Article ID 115200, 2020.

[2] R. Roy, M. A. Akbar, A. R. Seadawy, and D. Baleanu, "Search for adequate closed-form wave solutions to space-time fractional nonlinear equations," *Partial Differential Equations in Applied Mathematics*, vol. 3, Article ID 100025, 2021.

[3] S. Asghar, J. A. Haider, and N. Muhammad, "The modified KdV equation for a nonlinear evolution problem with perturbation technique," *International Journal of Modern Physics B*, vol. 36, no. 24, Article ID 2250160, 2022.

[4] Z. Yan, B. Fakieh, and R. I. Ismail, "Financial accounting measurement model based on numerical analysis of rigid normal differential equation and rigid functional equation," *Applied Mathematics and Nonlinear Sciences*, vol. 7, no. 2, pp. 69–78, 2021.

[5] N. Sharma and A. Kaushik, "A hybrid finite difference method for singularly perturbed delay partial differential equations with discontinuous coefficient and source," *Journal of Marine Science and Technology*, vol. 30, no. 3, pp. 217–236, 2022.

[6] X. An, R. Yang, D. M. Alghazzawi, and N. R. Joseph, "Mathematical function data model analysis and synthesis system based on short-term human movement," *Applied Mathematics and Nonlinear Sciences*, vol. 7, no. 2, pp. 49–58, 2021.

[7] Y. Li, H. Gao, C. Sun, and A. Sedki, "Asymptotic stability problem of predator–prey system with linear diffusion," *Applied Mathematics and Nonlinear Sciences*, vol. 8, no. 1, pp. 15–26, 2023.

[8] M. Lalu and K. Phaneendra, "A numerical approach for singularly perturbed nonlinear delay differential equations using a trigonometric spline," *Computational and Mathematical Methods*, 2022.

[9] M. Luo, B. Fakieh, and H. Hasan, "Children's cognitive function and mental health based on finite element nonlinear mathematical model," *Applied Mathematics and Nonlinear Sciences*, vol. 7, no. 2, pp. 59–68, 2021.

[10] M. M. Rashidi, M. A. Sheremet, M. Sadri et al., "Semi-analytical solution of two-dimensional viscous flow through expanding/contracting gaps with permeable walls," *Mathematical and Computational Applications*, vol. 26, no. 2, p. 41, 2021.

[11] Z. Sabir, D. Baleanu, M. A. Z. Raja, and E. Hincal, “A hybrid computing approach to design the novel second order singular perturbed delay differential Lane-Emden model,” *Physica Scripta*, vol. 97, no. 8, Article ID 85002, 2022.

[12] K. S. Roshni and M. N. Mohamedunni, “Problems of online mathematics teaching and learning during the pandemic: a reverberation into the perception of prospective teachers,” *3c Empresa: Investigación Y Pensamiento Crítico*, vol. 11, no. 2, pp. 153–162, 2022.

[13] S. E. Fadugba, J. V. Shaalini, O. M. Ogunmiloro, J. T. Okunlola, and F. H. Oyelami, “Analysis of exponential–polynomial single step method for singularly perturbed delay differential equations,” in *Journal of Physics: Conference Series*vol. 2199, no. 1, Bristol, UK, IOP Publishing, Article ID 12007, 2022, February.

[14] K. R. Singh, A. D. Gaikwad, and S. D. Kamble, “Empirical analysis of machine learning-based energy efficient cloud load balancing architectures: a quantitative perspective,” *3c Empresa: Investigación Y Pensamiento Crítico*, vol. 11, no. 2, pp. 232–248, 2022.

[15] M. K. Kadalbajoo and Y. Reddy, “Asymptotic and numerical analysis of singular perturbation problems: a survey,” *Applied Mathematics and Computation*, vol. 30, no. 3, pp. 223–259, 1989.

[16] R. Mahendran and V. Subburayan, “Uniformly convergent finite difference method for reaction-diffusion type third order singularly perturbed delay differential equation,” *Turkish Journal of Mathematics*, vol. 46, no. 2, pp. 360–376, 2022.

[17] V. B. Kiran Kumar, “Essential spectrum of discrete laplacian-revisited,” *3 c TIC: Cuadernos de desarrollo aplicados a las TIC*, vol. 11, no. 2, pp. 52–59, 2022.

[18] E. Prajisha and P. Shaini, “FG-COUPLEDfixed-point theorems in parti-ally ordered S metric spaces,” *3 c TIC: Cuadernos de desarrollo aplicados a las TIC*, vol. 11, no. 2, pp. 81–97, 2022.

[19] Y. Valentin, G. Fail, and U. Pavel, “Shapley values to explain machine learning models of school STUDENT'S academic performance during COVID-19,” *3 c TIC*, vol. 11, no. 2, pp. 136–144, 2022.

[20] R. Praveena and M. J. Paramasivam, “First order parameter uniform numerical method for a system of two singularly perturbed delay differential equations with robin initial conditions,” *Journal Of Algebraic Statistics*, vol. 13, no. 2, pp. 1063–1071, 2022.

[21] N. Minorsky, *Nonlinear Oscillations*, John Wiley and Sons, Hoboken, NJ, USA, 1962.

[22] R. D. Driver, “Linear differential systems with small delays,” *Journal of Differential Equations*, vol. 21, no. 1, pp. 148–166, 1976.

[23] A. D. Myshkis, “General theory of differential equations with retarded arguments,” *Uspekhi Matematicheskikh Nauk*, vol. 4, no. 5, pp. 99–141, 1949.

[24] R. E. Bellman and K. L. Cooke, *Differential-Difference Equations*, Academic Press, Cambridge, MA, USA, 1963.

[25] H. Chen, B. Fakieh, and B. M. Muwafak, “Differential equation model of financial market stability based on Internet big data,” *Applied Mathematics and Nonlinear Sciences*, vol. 7, no. 2, pp. 171–180, 2021.

[26] C. Chen, A. Albarakati, and Y. Hu, “Application of B-theory for numerical method of functional differential equations in the analysis of fair value in financial accounting,” *Applied Mathematics and Nonlinear Sciences*, vol. 7, no. 2, pp. 193–202, 2021.

[27] Z. Diab, M. T. de Bustos, M. . Á. López, and R. Martínez, “Limit cycles of perturbed global isochronous center,” *3C Tecnología_Glosas de innovación aplicadas a la pyme*, vol. 11, no. 2, pp. 25–36, 2022.

[28] G. G. Kiltu, G. F. Duressa, and T. A. Bullo, “Computational method for singularly perturbed delay differential equations of the reaction-diffusion type with negative shift,” *Journal of Ocean Engineering and Science*, vol. 6, no. 3, pp. 285–291, 2021.

[29] L. Xu, Z. Xu, W. Li, and S. Shi, “Renormalization group approach to a class of singularly perturbed delay differential equations,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 103, Article ID 106028, 2021.

[30] Q. Chen, A. Albarakati, and L. Gui, “Research on motion capture of dance training pose based on statistical analysis of mathematical similarity matching,” *Applied Mathematics and Nonlinear Sciences*, vol. 7, no. 2, pp. 127–138, 2021.

[31] H. Guo, “Nonlinear strategic human resource management based on organisational mathematical model,” *Applied Mathematics and Nonlinear Sciences*, vol. 7, no. 2, pp. 163–170, 2022.

[32] R. C. Dharmik, S. Chavhan, and S. R. Sathe, “Deep learning based missing object detection and person identification: an application for smart CCTV,” *3C Tecnología_Glosas de innovación aplicadas a la pyme*, vol. 11, no. 2, pp. 51–57, 2022.

[33] A. Parkhi and A. S. Khobragade, “Review on deep learning-based techniques for person re-identification,” *3 c TIC: cuadernos de desarrollo aplicados a las TIC*, vol. 11, no. 2, pp. 208–223, 2022.

[34] F. A. Rihan, “Delay differential equations in biosciences: parameter estimation and sensitivity analysis,” in *Recent Advances in Applied Mathematics and Computational Methods: Proceedings of the 2013 International Conference on Applied Mathematics and Computational Methods*, pp. 50–58, University of Potsdam, Venice, Italy, 2013.

[35] Q. Liu, B. Dai, I. Katib, and M. A. Alhamami, “Financial accounting measurement model based on numerical analysis of rigid normal differential equation and rigid generalised functional equation,” *Applied Mathematics and Nonlinear Sciences*, vol. 7, no. 1, pp. 541–548, 2022.

[36] L. Tao, L. Xu, and H. J. Sulaimani, “Nonlinear differential equations based on the BSM model in the pricing of derivatives in financial markets,” *Applied Mathematics and Nonlinear Sciences*, vol. 7, no. 2, pp. 91–102, 2021.

[37] A. Longtin and J. G. Milton, “Complex oscillations in the human pupil light reflex with “mixed” and delayed feedback,” *Mathematical Biosciences*, vol. 90, no. 1-2, pp. 183–199, 1988.

[38] G. A. Bocharov and F. A. Rihan, “Numerical modelling in biosciences using delay differential equations,” *Journal of Computational and Applied Mathematics*, vol. 125, no. 1-2, pp. 183–199, 2000.

[39] P. W. Nelson and A. S. Perelson, “Mathematical analysis of delay differential equation models of HIV-1 infection,” *Mathematical Biosciences*, vol. 179, no. 1, pp. 73–94, 2002.

[40] A. R. Kanth and P. M. M. Kumar, “Numerical method for a class of nonlinear singularly perturbed delay differential equations using parametric cubic spline,” *International Journal of Nonlinear Sciences and Numerical Stimulation*, vol. 19, no. 3-4, pp. 1–9, 2018.

[41] E. Sekar and A. Tamilselvan, “Finite difference scheme for singularly perturbed system of delay differential equations with integral boundary conditions,” *Journal of the Korean Society for Industrial and Applied Mathematics*, vol. 22, no. 3, pp. 201–215, 2018.

[42] F. Erdogan and Z. Cen, "A uniformly almost second order convergent numerical method for singularly perturbed delay differential equations," *Journal of Computational and Applied Mathematics*, vol. 333, pp. 382–394, 2018.

[43] M. K. Vaid and G. Arora, "Solution of second order singular perturbed delay differential equation using trigonometric B-spline," *International Journal of Mathematical, Engineering and Management Sciences*, vol. 4, no. 2, pp. 349–360, 2019.

[44] K. Phaneendra and M. Lalu, "Numerical solution of singularly perturbed delay differential equations using gaussion quadrature method," in *Journal of Physics: Conference Series*vol. 1344, no. 1, Bristol, UK, IOP Publishing, Article ID 12013, 2019.

[45] P. Pramod Chakravarthy, S. Dinesh Kumar, and R. Nageshwar Rao, "An exponentially fitted finite difference scheme for a class of singularly perturbed delay differential equations with large delays," *Ain Shams Engineering Journal*, vol. 8, no. 4, pp. 663–671, 2017.

[46] S. Kumar and M. Kumar, "A second order uniformly convergent numerical scheme for parameterized singularly perturbed delay differential problems," *Numerical Algorithms*, vol. 76, no. 2, pp. 349–360, 2017.

[47] P. P. Chakravarthy and K. Kumar, "A novel method for singularly perturbed delay differential equations of reaction-diffusion type," *Differential Equations and Dynamical Systems*, vol. 29, no. 3, pp. 723–734, 2017.

[48] V. Subburayan, "An hybrid initial value method for singularly perturbed delay differential equations with interior layers and weak boundary layer," *Ain Shams Engineering Journal*, vol. 9, no. 4, pp. 727–733, 2018.

[49] Z. Bartoszewski and A. Baranowska, "Solving boundary value problems for second order singularly perturbed delay differential equations by ε-approximate fixed-point method," *Mathematical Modelling and Analysis*, vol. 20, no. 3, pp. 369–381, 2015.

[50] S. Sethurathinam, S. Veerasamy, R. Arasamudi, and R. P. Agarwal, "An asymptotic streamline diffusion finite element method for singularly perturbed convection-diffusion delay differential equations with point source," *Computational and Mathematical Methods*, vol. 3, no. 6, p. 1201, 2021.

[51] M. M. Woldaregay and G. F. Duressa, "Uniformly convergent numerical method for singularly perturbed delay parabolic differential equations arising in computational neuroscience," *Kragujevac Journal of mathematics*, vol. 46, no. 1, pp. 65–54, 2022.

[52] G. Babu and K. Bansal, "A high order robust numerical scheme for singularly perturbed delay parabolic convection diffusion problems," *Journal of Applied Mathematics and Computing*, vol. 68, no. 1, pp. 363–389, 2022.

[53] I. T. Daba and G. F. Duressa, "A Robust computational method for singularly perturbed delay parabolic convection-diffusion equations arising in the modeling of neuronal variability," *Computational Methods for Differential Equations*, vol. 10, no. 2, pp. 475–488, 2022.

[54] F. Z. Geng and S. P. Qian, "Piecewise reproducing kernel method for singularly perturbed delay initial value problems," *Applied Mathematics Letters*, vol. 37, pp. 67–71, 2014.

[55] S. Cengizci, "An asymptotic-numerical hybrid method for solving singularly perturbed linear delay differential equations," *International Journal of Differential Equations*, 2017.

[56] K. W. Chang and F. A. Howes, *Nonlinear Singular Perturbation Phenomena: Theory and Applications*, Springer Science and Business Media, Heidelberg, Germany, 2012.

[57] R. Vulanovic, P. A. Farrell, and P. Lin, "Numerical solution of nonlinear singular perturbation problems modeling chemical reactions," 1993, https://www.researchgate.net/publication/2245678_Numerical_Solution_of_Nonlinear_Singular_Perturbation_Problems_Modeling_Chemical_Reactions.

[58] M. K. Kadalbajoo and K. K. Sharma, "Numerical treatment for singularly perturbed nonlinear differential difference equations with negative shift," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 63, no. 5-7, pp. e1909–e1924, 2005.

[59] M. K. Kadalbajoo, "Parameter uniform numerical method for a boundary-value problem for singularly perturbed nonlinear delay differential equation of neutral type," *International Journal of Computer Mathematics*, vol. 81, no. 7, pp. 845–862, 2004.

[60] M. J. Kabeto and G. F. Duressa, "Robust numerical method for singularly perturbed semilinear parabolic differential difference equations," *Mathematics and Computers in Simulation*, vol. 188, pp. 537–547, 2021.

[61] M. K. Kadalbajoo and D. Kumar, "Fitted mesh B-spline collocation method for singularly perturbed differential–difference equations with small delay," *Applied Mathematics and Computation*, vol. 204, no. 1, pp. 90–98, 2008.

[62] P. Angappan, S. Thangiah, and S. Subbarayan, "Taguchi-based grey relational analysis for modeling and optimizing machining parameters through dry turning of Incoloy 800H," *Journal of Mechanical Science and Technology*, vol. 31, no. 9, pp. 4159–4165, 2017.

[63] G. R. Kusi, A. H. Habte, and T. A. Bullo, "Layer resolving numerical scheme for singularly perturbed parabolic convection-diffusion problem with an interior layer," *MethodsX*, vol. 10, Article ID 101953, 2023.

[64] L. Bogachev, G. Derfel, S. Molchanov, and J. Ochendon, "On bounded solutions of the balanced generalized pantograph equation," *Topics in stochastic analysis and nonparametric estimation*, vol. 145, pp. 29–49, 2008.

[65] S. Elango, "Second order singularly perturbed delay differential equations with non-local boundary condition," *Journal of Computational and Applied Mathematics*, vol. 417, Article ID 114498, 2023.

[66] M. Achache and N. Tabchouche, "A full-Newton step feasible interior-point algorithm for monotone horizontal linear complementarity problems," *Optimization Letters*, vol. 13, no. 5, pp. 1039–1057, 2019.

[67] Z. Darvay and P. R. Rigó, "New interior-point algorithm for symmetric optimization based on a positive-asymptotic barrier function," *Numerical Functional Analysis and Optimization*, vol. 39, no. 15, pp. 1705–1726, 2018.

[68] R. Hooke and T. A. Jeeves, "Direct Search" Solution of numerical and statistical problems," *Journal of the ACM*, vol. 8, no. 2, pp. 212–229, 1961.

[69] A. Rof V and A. Krishnamoorthy, "On a queueing inventory with common lifetime and reduction sale consequent to increase in age," *3c Empresa: Investigación Y Pensamiento Crítico*, vol. 11, no. 02, pp. 15–31, 2022.

[70] L. Zhang, X. Tian, and Z. Chabani, "Application of higher order ordinary differential equation model in financial investment stock price forecast," *Applied Mathematics and Nonlinear Sciences*, vol. 7, no. 1, pp. 893–900, 2022.

[71] I. Ahmad, S. I. Hussain, M. Usman, and H. Ilyas, "On the solution of Zabolotskaya–Khokhlov and Diffusion of Oxygen equations using a sinc collocation method," *Partial*

Differential Equations in Applied Mathematics, vol. 4, Article ID 100066, 2021.

[72] Y. Jiang and M. M. Abdeldayem, "Prediction and analysis of ChiNext stock price based on linear and non-linear composite model," *Applied Mathematics and Nonlinear Sciences*, vol. 10, no. 1, 2022.

[73] I. Ahmad, H. Ilyas, K. Kutlu, V. Anam, S. I. Hussain, and J. L. G. Guirao, "Numerical computing approach for solving Hunter-Saxton equation arising in liquid crystal model through sinc collocation method," *Heliyon*, vol. 7, no. 7, 2021.

[74] I. Ahmad, S. I. Hussain, H. Ilyas et al., "Numerical solutions of Schrödinger wave equation and Transport equation through Sinc collocation method," *Nonlinear Dynamics*, vol. 105, no. 1, pp. 691–705, 2021.

[75] I. Ahmad, S. I. Hussain, M. A. Z. Raja, and M. Shoaib, "Transportation of hybrid MoS 2–SiO 2/EG nanofluidic system toward radially stretched surface," *Arabian Journal for Science and Engineering*, vol. 48, pp. 1–14, 2022.

[76] I. Ahmad, S. U. I. Ahmad, K. Kutlu, H. Ilyas, S. I. Hussain, and F. Rasool, "On the dynamical behavior of nonlinear Fitz-hugh–Nagumo and Bateman–Burger equations in quantum model using Sinc collocation scheme," *The European Physical Journal Plus*, vol. 136, no. 11, p. 1108, 2021.

[77] S. I. Hussain, I. Ahmad, M. Z. Umer, and M. A. Z. Raja, "A computational convection analysis of SiO2/water and MOS2–SiO2/water based fluidic system in an inverted cone," *Authorea Preprint*, 2023.

[78] P. Paikrao, D. Doye, M. Bhalerao, and M. Vaidya, "Near-lossless compression scheme using hamming codes for non-textual important regions in document images," *3 c TIC*, vol. 11, no. 2, pp. 225–237, 2022.