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Research Article

Some Intuitionistic Cubic Fuzzy Muirhead Mean Operators with Their Application to Multicriteria Decision Making

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Intuitionistic cubic fuzzy sets (ICFSs), as a hybrid fuzzy set consisting of interval-valued fuzzy sets and intuitionistic fuzzy sets, can deal with the uncertainty of information in decision problems more comprehensively. Muirhead mean (MM) operators can deal with the correlation between attributes in decision problems more realistically. However, there has been no research on MM operators under intuitionistic cubic fuzzy sets so far. In this study, some new operators are proposed under the intuitionistic cubic fuzzy set. It contains the intuitionistic cubic fuzzy MM (ICFMM) operator, intuitionistic cubic fuzzy weighted MM (ICFWMM) operator, intuitionistic cubic fuzzy dual MM (ICFDMM) operator, and intuitionistic cubic fuzzy dual weighted MM (ICFDWMM) operator. Their proof procedure, properties, and proof of properties are given. Two MCDM decision models based on ICFWMM and ICFDWMM operators are proposed, and the proposed decision models are applied to supplier selection, and the better reliability and accuracy of the proposed decision methods in this study are verified through comparative analysis with existing methods.

1. Introduction

Multicriteria decision making (MCDM) plays an important role in everyday activities such as economics, engineering, education, and healthcare [1–3]. In MCDM, a variety of different information sources are collected, a variance index is inferred for the degree of variation between criteria using language assessment, and the optimal result is selected through a process of comparison and aggregation based on the attribute values of different alternatives [4]. Due to the complexity of the decision problem, it is crucial to be able to provide the decision maker with the most reasonable decision information. Even if the decision maker has an extensive knowledge and experience with the decision problem, there will still be some uncertainty in the decision information. Zadeh [5] proposed the concept of fuzzy set (FS), which is commonly used on MCDM methods by assigning affiliation to the set elements on the interval [0,1], in order to be able to quantify the uncertainty present in the decision information. After using fuzzy theory in a variety of applications, Atanassov discovered that it has some flaws, one of which is the possibility of some degree of uncertainty in the process of providing decision information. This uncertainty is a major cause for concern. Atanassov [6] extended fuzzy set theory to Atanassov intuitionistic fuzzy sets (AIFSs). An ordered pair of numbers that includes an affiliation function and an unaffiliated function and whose sum is less than or equal to one constitutes an element of an AIFS. When compared to fuzzy sets, AIFS can more accurately depict the uncertainty of decision-making information. Researchers have expanded and successfully applied AIFS theory to a number of fields. In order to provide a method with adaptive search and adjustment for group decision making, Shen [7] proposed a super-ranking ranking method based on intuitionistic fuzzy sets. By transforming between AIFS and Pythagorean fuzzy sets, Tao [8] proposed an ORESTE method based on Pythagorean fuzzy sets for multiattribute decision making with Pythagorean fuzzy information. In order to solve the issue of moving object detection in complex situations, Giveki [9] applied AIFS to background detection and proposed a reliable moving object detection method incorporating Atanasoff's comprehensible 3D fuzzy Histon roughness index and texture features. In order to segment leukocytes in color images, Bouchet [10] proposed a new algorithm based on AIFS and fuzzy mathematical morphology. Zeng [11] proposed a hybrid averaging operator and geometric operator under intuitionistic fuzzy sets to solve the information fusion problem of digital reform. Deb PP [12] has applied intuitionistic fuzzy sets to enterprise resource planning decisions with some results. Chen CP [13] further proposed the quantitative method of Atanassov-type intuitive membership hierarchy.

IFSs can quantify to some extent the uncertainty in the MCDM process, which is the uncertainty in the decision information provided by decision makers based on their own experience and knowledge because of the complex conditions that make the decision information provided. The aggregation of decision information as a key step to solve the MCDM problem has been extensively studied by scholars. Based on IFSs, scholars have introduced many operators to aggregate intuitionistic fuzzy numbers. Gou et al. [14] proposed an information aggregation method for intuitionistic fuzzy numbers. Liu and Chen [15] proposed the intuitionistic fuzzy Archimedean Helen aggregation (IFAHA) operator and the intuitionistic fuzzy weighted Archimedean Helen aggregation (IFWAHA) operator. Herrera et al. [16] conducted a study on the application of ordered weighted ensemble operators in MCDM. Xu and Chen [17] proposed interval-valued intuitionistic fuzzy geometry operators, such as interval-valued intuitionistic fuzzy ordered weighted geometry (IVFOWG) operator and intuitionistic interval-valued fuzzy hybrid geometry (IVFHG) operator. Yuan et al. [18] proposed several linguistic intuitionistic fuzzy aggregation operators, including the linguistic intuitionistic fuzzy hybrid weighted arithmetical averaging operator and the linguistic intuitionistic fuzzy hybrid weighted geometric mean operator. Liu et al. [19] proposed the upward intuitionistic fuzzy preference weighted average (UIFPWA) operator, the upward intuitionistic fuzzy preference ordered weighted average (UIFPOWA) operator, and the upward intuitionistic fuzzy preference hybrid average (UIFPHA) operator, among others. Jia and Wang [20] proposed the Choquet integral-based intuitionistic fuzzy hybrid arithmetic aggregation operator (CIIFHAA) and gives the MCDM method with intuitionistic fuzzy evaluation.

However, intuitionistic fuzzy sets still have considerable limitations in dealing with the uncertainty of decision information. To better solve this problem, cubic sets are introduced to extend intuitionistic fuzzy. Cube sets are widely used as an effective means to extend fuzzy languages, such as neutrosophic set [21]. Extended intuitionistic cubic fuzzy set (ICFS) [22] can explain satisfactory, unsatisfactory, and unpredictable information, which is not explained by fuzzy sets and intuitionistic fuzzy sets. ICFS is a generalized form of AIFS, and such as AIFS, each element of ICFS is represented as a structure of ordered pairs, characterized by an affiliation function and an unaffiliated function. While the affiliation function is in interval form, the unaffiliated function resembles an ordinary fuzzy set. ICF is widely used in MCDM to express more reasonable information about uncertainty, which helps decision makers to make more reasonable decision schemes [23–25]. Among them, the aggregation operator is widely used in ICF environment. Muneeza and Abdullah [26] and Kaur and Garg [22] established a series of weighted aggregation operators under ICFS theory, containing intuitionistic cubic fuzzy weighted average (ICFWA) operator, intuitionistic cubic fuzzy ordered weighted average (ICFOWA) operator, intuitionistic cubic fuzzy weighted geometry (ICFWG) operator, intuitionistic cubic fuzzy order weighted geometry (ICFOWG) operator, intuitionistic cubic fuzzy mixed average (ICFMA) operator, and intuitionistic cubic fuzzy mixed geometry (ICFMG) operator. Qiyas [27, 28] and Liu et al. [29] further established a series of weighted clustering operators on linguistic intuitionistic cubic fuzzy sets (LICFS), including linguistic intuitionistic cubic fuzzy weighted average (LICFWA) operator, linguistic intuitionistic cubic fuzzy order weighted average (LICFOWA) operator, linguistic intuitionistic cubic fuzzy weighted geometry (LICFWG) operator, linguistic intuitionistic cubic fuzzy geometry (LICFOWG) operator, linguistic intuitionistic cubic fuzzy weighted geometry (LICFOWG) operator, linguistic intuitionistic cubic fuzzy hybrid average (LICFHA) operator, and linguistic intuitionistic cubic fuzzy hybrid geometry (LICFHG) operator. However, these aggregation operators do not consider the relationship between aggregated attributes, and to address this problem, Ates and Akay [30] conducted a study on the application of Bonferroni mean (BM) operators in MCDM. Kaur and Garg [31] proposed the Bonferroni mean (BM) operator and the weighted Bonferroni mean average operator under ICFS. Deli [32] studied the BM operator under the generalized hesitant fuzzy set and proposed the generalized trapezoidal hesitant fuzzy Bonferroni arithmetic mean operator and generalized trapezoidal hesitant fuzzy Bonferroni geometric mean operator. Kumar and Chen [33] proposed the intuitionistic fuzzy Hamacher weighted average (IFHWA) operator. The ICFBM operator and IFHWA operator cannot handle the relationship between three or more attributes although they consider the influence between the two aggregated attributes.

From the above literature, it can be seen that the existing literature focuses on intuitionistic fuzzy sets, interval intuitionistic fuzzy sets, and other single use of exact values or interval values to represent the fuzziness and uncertainty of decision information. However, in the real world, it is usually difficult to represent the fuzziness and uncertainty of decision information by a single exact value or interval value. Therefore, using a mixture of interval and exact values may be a very useful way to express the certainty and uncertainty of a person's indecision in a complex decision problem. Rationality and validity. Therefore, in order to maximize the response to the uncertainty of decision information and give the most reasonable decision alternatives, this study proposes a further study on decision making under ICFS. The processing of decision information in fuzzy environments often requires aggregation using aggregation operators for further applications; however, most of the current research work has focused on weighted average operators, geometric operators, and so on. Although these aggregation operators are able to aggregate fuzzy information, the information they aggregate cannot be correlated. However, in the real world, the decision making process often has certain correlation between the attributes of the decision objects, and the decision information given by ignoring the correlation between them is bound to be defective, which will seriously affect the decision results. In order to consider the correlation between the aggregated attributes as much as possible, this study proposes to use Muirhead mean operator for the aggregation of fuzzy information. The Muirhead mean (MM) [34] operator is an aggregation operator that can be used to deal with information aggregation problems that require considering the interrelationships between multiple attribute values by specifying any number of parameters through a vector of variables. It is a good approach to consider the interrelationships between the attributes of the evaluated objects in MCDM problems. After the MM operator was proposed, researchers further extended its application in different MCDMs [35–37]. In order to further consider the uncertainty of decision information and the interrelationship between the values of the attributes of the evaluation alternatives in the MCDM problem and to propose a more comprehensive and realistic decision method, this study proposes a series of aggregation operators that can consider the above information in combination with the MM operator under ICFS.

In this study, the fuzziness and uncertainty of decision information and the correlation among the attributes of decision objects are considered more comprehensively in order to be more relevant to the real world. We investigate MM operators on intuitionistic cubic fuzzy sets and propose some new aggregation operators, including intuitionistic cubic fuzzy MM (ICFMM) operator, intuitionistic cubic fuzzy weighted MM (ICFWMM) operator, intuitionistic cubic fuzzy dual MM (ICFSDMM) operator, and intuitionistic cubic fuzzy dual weighted MM (ICFDWMM) operator. The proof procedures of these operators are given in detail, the rational properties of these operators are studied, and the proof procedures of the corresponding properties are given. Compared with other existing existing operators, it is clear that our proposed operators have a great advantage in the treatment of uncertainty of information and the treatment of correlation between aggregated attributes. Finally, we propose the two comprehensive MCDM decision methods based on the proposed intuitionistic cubic fuzzy weighted MM (ICFWMM) operator and intuitionistic cubic fuzzy dual weighted MM (ICFDWMM) operator and apply them to supplier selection.

As far as we know, there is no research on MuirHead mean aggregation operators based on intuitionistic cubic fuzzy numbers in the existing literature. In order to fill this gap, the research content of this study is as follows: in the second section, we review some basic concepts and algorithms of fuzzy sets, intuitionistic fuzzy sets, intuitionistic cubic fuzzy sets, and MM operators. In Section 3, we propose the four kinds of aggregation operators, namely, intuitionistic cubic fuzzy MM operator (ICFMM), intuitionistic cubic fuzzy weighted MM operator (ICFWMM), intuitionistic cubic dual MM operator (ICFDMM), and intuitionistic cubic dual weighted MM operator (ICFDWMM). In addition, we give their derivation process and property proof. In Section 4, we propose the two new MCDM methods using intuitionistic cubic fuzzy weighted MM operator (ICFWMM) and intuitionistic cubic dual weighted MM operator (ICFDWMM). In Section 5, we describe the numerical application of the proposed new MCDM method in the intuitionistic cubic fuzzy environment through suppliers purchasing parts for a company and compare it with existing methods to analyze the proposed method. Section 6 concludes the whole study and puts forward some suggestions for future research.

2. Preliminaries

In the following, we introduce some basic concepts of fuzzy sets, intuitionistic fuzzy sets, intuitionistic cubic fuzzy sets, and MM operators and DMM operators.

2.1. Intuitionistic Cubic Fuzzy Set (ICFS)

Definition 1 (see [26]). The intuitionistic cubic fuzzy set (ICFS) I in the nonempty set X is defined as follows:

$$\begin{aligned} I &= \left\{ \left(x, \langle [e^-, e^+], \lambda\rangle, \langle [r^-, r^+], \delta\rangle\right) \,\middle|\, x \in X \right\} \\ &= \left\{ x, \langle c_I, c_I^{'}\rangle \,\middle|\, x \in X \right\}, \end{aligned} \tag{1}$$

where $c_I = \langle [e^-, e^+], \lambda\rangle$ is the membership affiliation of x and $c_I^{'} = \langle [r^-, r^+], \delta\rangle$ is the unaffiliatedness of x. Here, $[r^-, r^+] \subset [0,1]$, $[e^-, e^+] \subset [0,1]$, δ: $X \longrightarrow [0,1]$, λ: $X \longrightarrow [0,1]$, $r^+ + e^+ \leq 1$, and $\lambda + \delta \leq 1$.

Let $I_1 = (\langle [e_1^-, e_1^+], \lambda_1\rangle, \langle [r_1^-, r_1^+], \delta_1\rangle)$ and $I_2 = (\langle [e_2^-, e_2^+], \lambda_2\rangle, \langle [r_2^-, r_2^+], \delta_2\rangle)$ be the ICFS under two nonempty sets X with $k > 0$, we have the following algorithm:

(1) $I_1 \oplus I_2 = \left\{ \begin{array}{c} ([e_1^- + e_2^- - e_1^- e_2^-, e_1^+ + e_2^+ - e_1^+ e_2^+], \\ (\lambda_1 + \lambda_2 - \lambda_1\lambda_2) \\ ([r_1^- r_2^-, r_1^+ r_2^+], \delta_1\delta_2) \end{array} \right\}$

(2) $I_1 - I_2 = \left\{ \begin{array}{c} (\min\{[e_1^-, e_1^+], [r_2^-, r_2^+]\}, \max\{\delta_1, \lambda_2\}), \\ (\max\{[r_1^-, r_1^+], [e_2^-, e_2^+]\}, \min\{\delta_2, \lambda_1\}) \end{array} \right\}$

(3) $I_1 \otimes I_2 = \left\{ \begin{array}{c} ([e_1^- e_2^-, e_1^+ e_2^+], \lambda_1\lambda_2), ([r_1^- + r_2^- - r_1^- r_2^-) \\ (r_1^+ r_2^+ - r_1^+ r_2^+], \delta_1 + \delta_2 - \delta_1\delta_2) \end{array} \right\}$

(4) $kI = \{([1 - (1 - e^-)^k, 1 - (1 - e^+)^k], 1 - (1 - \lambda)^k), ([r^{-k}, r^{+k}], \delta^k)\}$

(5) $I^k = \{([e^{-k}, e^{+k}], \lambda^k), ([1 - (1 - r^-)^k, 1 - (1 - r^+)^k], 1 - (1 - \delta)^k)\}$

Definition 2 (see [26]). Let $I = \{(x, \langle [e^-, e^+], \lambda\rangle, \langle [r^-, r^+], \delta\rangle) \mid x \in X\}$ be the ICFS under nonempty sets X. Then,

(1) Score of I is denoted by $S(I)$ and is defined as follows:

$$S(I) = \left[\frac{(e^- + e^+ + \lambda - r^- - r^+ - \delta)}{3}\right], \quad (2)$$

Here $S(I) \in [0, 1]$, the function $S(I)$ can be used to calculate the score value of the cubic fuzzy set. The larger the value of $S(I)$, the larger the corresponding cubic fuzzy value I.

(2) The accuracy function of I is denoted by $H(I)$ and is defined as follows:

$$H(I) = \left[\frac{(e^- + e^+ + \lambda + r^- + r^+ + \delta)}{3}\right], \quad (3)$$

where $H(I) \in [0, 1]$ and the larger the value of $H(I)$, the higher the accuracy of the ICF-members.

Definition 3 (see [26]). Let $I_1 = (\langle [e_1^-, e_1^+], \lambda_1\rangle, \langle [r_1^-, r_1^+], \delta_1\rangle)$ and $I_2 = (\langle [e_2^-, e_2^+], \lambda_2\rangle, \langle [r_2^-, r_2^+], \delta_2\rangle)$ be the two ICF-numbers, and the score functions of I_1 and I_2 are represented as $S(I_1)$ and $S(I_2)$ and the accuracy functions be given as $H(I_1)$ and $H(I_2)$, respectively. Then,

(i) $S(I_2) < S(I_1) \Longrightarrow I_2 < I_1$

(ii) $S(I_2) = S(I_1)$

(a) $H(I_2) < H(I_1) \Longrightarrow I_2 < I_1$

(b) $H(I_2) = H(I_1) \Longrightarrow I_2 = I_1$

2.2. Muirhead Mean (MM) Operator. The MM (Muirhead mean) operator was originally proposed by Muirhead [38], which provides an aggregation mechanism positioned between various types of mean aggregation operators such as arithmetic average and geometric average, and are mainly used to solve the problem of aggregating ambiguous information about the existence of correlation of attributes. The definition is as follows.

Definition 4 (see [38]). Let $\alpha_j\,(j = 1, 2, \ldots, n)$ be a collection of nonnegative real numbers and $p = (p_1, p_2, \ldots, p_n) \in R^n$ be a parameter vector. The MM operator is defined as follows:

$$\mathrm{MM}^p(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left(\frac{1}{n!}\sum_{\theta \in S_n}\prod_{j=1}^{n}\alpha_{\theta(j)}^{p_j}\right)^{1/\sum_{j=1}^{n}p_j}, \quad (4)$$

where $\theta(j)(j = 1, 2, \ldots, n)$ is an permutations of $\{1, 2, \ldots, n\}$ and S_n is the collection of all permutations of $\{1, 2, \ldots, n\}$.

From the Definition 6 and the special case of the MM operator mentioned above, it is known that the advantage of the MM operator is that it captures the interrelationships between multiple aggregation parameters, and it is based on a generalization of most of the existing aggregation operators.

2.3. Dual Muirhead Mean (MM) Operator. The DMM (dual Muirhead mean) operator was proposed by Liu and Li [34] and is defined as follows.

Definition 5 (see [34]) Let $\alpha_j\,(j = 1, 2, \ldots, n)$ be a collection of nonnegative real numbers and $p = (p_1, p_2, \ldots, p_n) \in R^n$ be a parameter vector. The DMM operator is defined as follows:

$$\mathrm{DMM}^p(\alpha_1, \alpha_2, \cdots, \alpha_n) = \frac{1}{\sum_{j=1}^{n}p_j}\left(\prod_{\theta \in S_n}\sum_{j=1}^{n}p_j\alpha_{\theta(j)}\right)^{1/n!}, \quad (5)$$

where $\theta(j)(j = 1, 2, \ldots, n)$ is an permutations of $\{1, 2, \ldots, n\}$ and S_n is the collection of all permutations of $\{1, 2, \ldots, n\}$.

3. Proposed Operators

In this section, we propose the four aggregation operators using Muirhead mean (MM) operators in the intuitionistic cubic fuzzy set (ICFS) environment, namely, intuitionistic cubic fuzzy Muirhead mean (ICFMM) operator, intuitionistic cubic fuzzy weighted MM (ICFWMM) operator, intuitionistic cubic fuzzy dual MM (ICFDMM) operator, and intuitionistic cubic fuzzy dual weighted MM (ICFDWMM) operator. Let $I = \{(x, \langle [e^-, e^+], \lambda\rangle, \langle [r^-, r^+], \delta\rangle) \mid x \in X\}$ be the ICFS under nonempty sets X. Because $c_I = \langle [e^-, e^+], \lambda\rangle$ and $c_I' = \langle [r^-, r^+], \delta\rangle$. Here, c_I represents the affiliation of the ICFS, consisting of interval affiliation $[e^-, e^+]$ and affiliation λ, which have the same algorithm. Similarly, c_I' represents the unaffiliated relations of ICFS, consisting of interval unaffiliatedness $[r^-, r^+]$ and unaffiliatedness δ, and they have the same algorithm. Therefore, when the proof procedure and properties of the proposed operator in abbreviated form are correct, its complete form must also be correct. To facilitate the understanding of the proof procedure and properties of the proposed operator, we use the short form of ICFS for its proof.

3.1. The Intuitionistic Cubic Fuzzy Muirhead Mean (ICFMM) Operator

Definition 6. Let $I_i = \langle c_{I_i}, c_{I_i}'\rangle = (\langle [e_i^-, e_i^+], \lambda_i\rangle, \langle [r_i^-, r_i^+], \delta_i\rangle)\,(i = 1, 2, \ldots, n)$ be a collection of ICF-numbers and $p = (p_1, p_2, \cdots, p_n) \in R^n$ be a vector of parameters. Then, intuitionistic cubic fuzzy Muirhead mean (ICFMM) operator is denoted by ICFMM^p and is defined as follows:

$$\mathrm{ICFMM}^p(I_1, I_2, \ldots, I_n) = \left(\frac{1}{n!}\sum_{\theta \in S_n}\prod_{j=1}^{n}I_{\theta(j)}^{p_j}\right)^{1/\sum_{j=1}^{n}p_j}, \quad (6)$$

where $\theta(j)(j = 1, 2, \ldots, n)$ is any a permutation of $(1, 2, \ldots, n)$ and S_n is the collection of all permutations of $(1, 2, \ldots, n)$.

Theorem 1. *Let $I_i = \langle c_{I_i}, c_{I_i}'\rangle = (\langle [e_i^-, e_i^+], \lambda_i\rangle, \langle [r_i^-, r_i^+], \delta_i\rangle)\,(i = 1, 2, \ldots, n)$ be a collection of the ICF-numbers, then the aggregation result from Definition 6 is still an ICF-number, and it can be obtained as follows:*

$$\mathrm{ICFMM}^{P}(I_1, I_2, \ldots, I_n) = \begin{pmatrix} \left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n} c_{I_{\theta(j)}}^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n} p_j}, \\ 1-\left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-c'_{I_{\theta(j)}}\right)^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n} p_j} \end{pmatrix}$$

$$= \begin{pmatrix} \left\langle \left[\left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n} e^{-\,p_j}_{\theta(j)\theta(j)}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n} p_j}, \left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n} e^{+p_j}_{\theta(j)}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n} p_j}\right], \left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n} \lambda_{\theta(j)}^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n} p_j} \right\rangle, \\ \left[1-\left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-r^{-}_{\theta(j)}\right)^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n} p_j}, 1-\left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-r^{+}_{\theta(j)}\right)^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n} p_j}\right], \\ 1-\left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\delta_{\theta(j)}\right)^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n} p_j} \end{pmatrix}. \quad (7)$$

Proof. We need to prove the following: (1) equation (7) is kept and (2) equation (7) is an ICF-number. To facilitate the understanding of the proof procedure and properties of the proposed operator, we use the abbreviated form of ICFS for its proof.

(1) First, we prove (7) is kept. According to the operational laws of ICFS, we get the following equation:

$$\begin{aligned} I_{\theta(j)}^{p_j} &= \left(c_{I_{\theta(j)}}^{p_j}, 1-\left(1-c'_{I_{\theta(j)}}\right)^{p_j}\right), \\ \prod_{j=1}^{n} I_{\theta(j)}^{p_j} &= \left(\prod_{j=1}^{n} c_{I_{\theta(j)}}^{p_j}, 1-\prod_{j=1}^{n}\left(1-c'_{I_{\theta(j)}}\right)^{p_j}\right), \end{aligned} \quad (8)$$

Then,

$$\sum_{\theta\in S_n}\prod_{j=1}^{n} I_{\theta(j)}^{p_j} = \left(1-\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n} c_{I_{\theta(j)}}^{p_j}\right), \prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-c'_{I_{\theta(j)}}\right)^{p_j}\right)\right), \quad (9)$$

Furthermore,

$$\frac{1}{n!}\sum_{\theta\in S_n}\prod_{j=1}^{n} I_{\theta(j)}^{p_j} = \begin{pmatrix} 1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n} c_{I_{\theta(j)}}^{p_j}\right)\right)^{1/n!}, \\ \left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-c'_{I_{\theta(j)}}\right)^{p_j}\right)\right)^{1/n!} \end{pmatrix}, \quad (10)$$

So, we have

$$\left(\frac{1}{n!}\sum_{\theta\in S_n}\prod_{j=1}^{n} I_{\theta(j)}^{p_j}\right)^{1/\sum_{j=1}^{n} p_j} = \begin{pmatrix} \left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n} c_{I_{\theta(j)}}^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n} p_j} \\ 1-\left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-c_{I'_{\theta(j)}}\right)^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n} p_j} \end{pmatrix}, \quad (11)$$

i.e., (7) is kept.

(2) Then, we will prove that (7) is an ICF-number.

Let

$$c_I=\left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}c_{I_{\theta(j)}}^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n}p_j},$$
$$c_I^{'}=1-\left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-c_{I_{\theta(j)}}^{'}\right)^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n}p_j}. \tag{12}$$

Then, we need to prove the following two conditions:

(a) $0\le c_I\le 1, 0\le c_I^{'}\le 1$
(b) $0\le c_I+c_I^{'}\le 1$

(i) Since $c_{I_{\theta(j)}}\in[0,1]$, we can get $c_{I_{\theta(j)}}^{p_j}\in[0,1]$ and $\prod_{j=1}^{n}c_{I_{\theta(j)}}^{p_j}\in[0,1]$
Then,

$$1-\prod_{j=1}^{n}c_{I_{\theta(j)}}^{p_j}\in[0,1],$$
$$\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}c_{I_{\theta(j)}}\right)\in[0,1], \tag{13}$$
$$\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}c_{I_{\theta(j)}}^{p_j}\right)\right)^{1/n!}\in[0,1],$$

Furthermore,

$$1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}c_{I_{\theta(j)}}^{p_j}\right)\right)^{1/n!}\in[0,1],$$
$$\left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}c_{I_{\theta(j)}}^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n}p_j}\in[0,1], \tag{14}$$

i.e., $\le c_I\le 1$. Similarly, we can get $0\le c_I^{'}\le 1$.So, condition (i) is met.

(ii) Since $c_{I_{\theta(j)}}+c_{I_{\theta(j)}}^{'}\le 1$, then $c_{I_{\theta(j)}}\le 1-c_{I_{\theta(j)}}^{'}$ we can get the following inequality:

$$\begin{aligned}c_{I_{\theta(j)}}+c_{I_{\theta(j)}}^{'}&=\left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}c_{I_{\theta(j)}}^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n}p_j}+1-\left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-c_{I_{\theta(j)}}^{'}\right)^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n}p_j}\\&\le\left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-c_{I_{\theta(j)}}^{'}\right)^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n}p_j}+1-\left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-c_{I_{\theta(j)}}^{'}\right)^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n}p_j}=1,\end{aligned} \tag{15}$$

i.e., $c_{I_{\theta(j)}}+c_{I_{\theta(j)}}^{'}\le 1$.

According to (i) and (ii), we can know the aggregation result from equation (32) is still an ICF-number. □

Property 1. The ICFMM^{p} operator has some properties as follows:

(1) (Idempotency). Let $I_i=I=(c_{I_i},c_{I_i}^{'})$, if all I_i ($i=1,2,\ldots,n$) are equal, then

$$\text{ICFMM}^{p}(I_1,I_2,\ldots,I_n)=I, \tag{16}$$

(2) (Monotonicity). Let $I_i=(c_{I_i},c_{I_i}^{'})$ and $L_i=(c_{L_i},c_{L_i}^{'})$ $(i=1,2,\ldots,n)$ be the two sets of ICF-numbers. If $c_{I_i}\ge c_{L_i}$ and $c_{I_i}^{'}\le c_{L_i}^{'}$, then

$$\text{ICFMM}^{p}(I_1,I_2,\ldots,I_n)\ge\text{ICFMM}^{p}(L_1,L_2,\ldots,L_n), \tag{17}$$

(3) (Boundedness). Let $I_i=(c_{I_i},c_{I_i}^{'})$ $(i=1,\ldots,n)$ be a collection of ICF-numbers and $I^{-}=(\min(c_{I_i}),\max(c_{I_i}^{'}))$ and $I^{+}=(\max(c_{I_i}),\min(c_{I_i}^{'}))$, then

$$I^{-} \leq \text{ICFMM}^{p}(I_1, I_2, \ldots, I_n) \leq I^{+}. \quad (18)$$

Proof. Since the proposed operator has the same properties in the abbreviated form and the full form of ICFS, we use the abbreviated form of ICFS to derive the properties for the sake of understanding.

(1) Since $I_i = I = (c_{I_i}, c'_{I_i})$, based on Theorem 1, we get the following equation:

$$\begin{aligned}
&\text{ICFMM}^{p}(I_1, I_2, \ldots, I_n) \\
&= \left(\begin{array}{c} \left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n} c_I^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n}p_j}, \\ 1-\left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-c_I'\right)^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n}p_j} \end{array} \right) = \left(\begin{array}{c} \left(1-\left(\prod_{\theta\in S_n}\left(1-c_I^{\sum_{j=1}^{n}p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n}p_j}, \\ 1-\left(1-\left(\prod_{\theta\in S_n}\left(1-\left(1-c_I'\right)^{\sum_{j=1}^{n}p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n}p_j} \end{array} \right) \\
&= \left(\begin{array}{c} \left(1-\left(\left(1-c_I\sum_{j=1}^{n}p_j\right)^{n!}\right)^{1/n!}\right)^{1/\sum_{j=1}^{n}p_j}, \\ 1-\left(1-\left(\left(\left(1-\left(1-c_I'\right)^{\sum_{j=1}^{n}p_j}\right)\right)^{n!}\right)^{1/n!}\right)^{1/\sum_{j=1}^{n}p_j} \end{array} \right) = \left(\begin{array}{c} \left(1-\left(1-c_I\sum_{j=1}^{n}p_j\right)\right)^{1/\sum_{j=1}^{n}p_j}, \\ 1-\left(1-\left(1-\left(1-c_I'\right)^{\sum_{j=1}^{n}p_j}\right)\right)^{1/\sum_{j=1}^{n}p_j} \end{array} \right) \quad (19) \\
&= \left(\begin{array}{c} \left(c_I^{\sum_{j=1}^{n}p_j}\right)^{1/\sum_{j=1}^{n}p_j}, \\ 1-\left(\left(1-c_I'\right)^{\sum_{j=1}^{n}p_j}\right)^{1/\sum_{j=1}^{n}p_j} \end{array} \right) = \left(c_I, 1-\left(1-c_I'\right)\right) = \left(c_I, c_I'\right),
\end{aligned}$$

(2) Let $M^{p}(I_1, I_2, \ldots, I_n) = (c_I, c_I')$ and $\text{ICFMM}^{p}(L_1, L_2, \ldots, L_n) = (c_{L_i}, c'_{L_i})$,

$$
\begin{aligned}
c_I &= \left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n} c_{I_{\theta(j)}}^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n}p_j},\\
c_L &= \left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n} c_{L_{\theta(j)}}^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n}p_j},\\
c_I' &= 1-\left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-c_I'\theta(j)\right)^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n}p_j},\\
c_L' &= 1-\left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-c_L'\theta(j)\right)^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n}p_j},
\end{aligned}
\tag{20}
$$

Since $c_{I_i} \geq c_{L_i}'$, we can get the following equation:

$$
\begin{aligned}
c_{I_{\theta(j)}}^{p_j} &\geq c_{L_{\theta(j)}}^{p_j},\\
\prod_{j=1}^{n} c_{I_{\theta(j)}}^{p_j} &\geq \prod_{j=1}^{n} c_{L_{\theta(j)}}^{p_j},\\
1-\prod_{j=1}^{n} c_{I_{\theta(j)}}^{p_j} &\leq 1-\prod_{j=1}^{n} c_{L_{\theta(j)}}^{p_j},\\
\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n} c_{I_{\theta(j)}}^{p_j}\right) &\leq \prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n} c_{L_{\theta(j)}}^{p_j}\right),\\
\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n} c_{I_{\theta(j)}}^{p_j}\right)\right)^{1/n!} &\leq \left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n} c_{L_{\theta(j)}}^{p_j}\right)\right)^{1/n!},
\end{aligned}
\tag{21}
$$

Futhermore,

$$
\begin{aligned}
1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n} c_{I_{\theta(j)}}^{p_j}\right)\right)^{1/n!} &\geq 1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n} c_{L_{\theta(j)}}^{p_j}\right)\right)^{1/n!},\\
\left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n} c_{I_{\theta(j)}}^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n}p_j} &\geq \left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n} c_{L_{\theta(j)}}^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n}p_j},
\end{aligned}
\tag{22}
$$

i.e., $c_I \geq c_L$. Similarly, we also have $c_I' \leq c_L'$. So, Property 2 is kept.

(3) According to Idempotency and Monotonicity, we have the following equation:

$$
\begin{aligned}
\mathrm{ICFMM}^{p}\left(I_1, I_2, \ldots, I_n\right) &\geq \mathrm{ICFMM}^{p}\left(I^-, I^-, \ldots, I^-\right) = I^-,\\
\mathrm{ICFMM}^{p}\left(I_1, I_2, \ldots, I_n\right) &\leq \mathrm{ICFMM}^{p}\left(I^+, I^+, \ldots, I^+\right) = I^+.
\end{aligned}
\tag{23}
$$

So, we have $I^- \leq \mathrm{ICFMM}^{p}\left(I_1, I_2, \ldots, I_n\right) \leq I^+$. Therefore, Property 3 holds.

Moreover, regarding the influence of the parameter vector p on the monotonicity of the ICFMM operator, the higher the control rate of the parameter vector, the larger the value of the aggregation operator result [34], for the proof procedure refer to reference [39, 40]. □

3.2. The Intuitionistic Cubic Fuzzy Weighted MM (ICFWMM) Operator. In practical decision making, the size of attribute weights will directly affect the decision result. However, the ICFWMM operator cannot consider attribute weights, so it is very important to consider the attribute weights of information. In this subsection, the following weighted ICFWMM operator will be proposed.

Definition 7. Let $I_i = \langle c_{I_i}, c'_{I_i}\rangle = (\langle [e_i^-, e_i^+], \lambda_i\rangle, \langle [r_i^-, r_i^+], \delta_i\rangle)\,(i = 1, 2, \ldots, n)$ be a collection of ICF-numbers and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of $I_i\,(i = 1, 2, \ldots, n)$, which satisfies $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1$, and let $p = (p_1, p_2, \cdots, p_n) \in R^n$ be a vector of parameters. If

$$\mathrm{ICFWMM}^p(I_1, I_2, \ldots, I_n) = \left(\frac{1}{n!}\sum_{\theta\in S_n}\prod_{j=1}^{n}\left(nw_{\theta(j)} I_{\theta(j)}\right)^{p_j}\right)^{1/\sum_{j=1}^n p_j}. \tag{24}$$

Then, we call *ICFWMM*^{p} the intuitionistic cubic fuzzy weighted MM (ICFWMM), where $\theta(j)(j = 1, 2, \ldots, n)$ is any a permutation of (1, 2, …, n), and S_n is the collection of all permutations of (1, 2, …, n).In addition, when $w = (1/n, 1/n, \ldots, 1/n)$, the ICFWMM operator changes to the ICFMM operator.

Theorem 2. *Let* $I_i = \langle c_{I_i}, c'_{I_i}\rangle = (\langle [e_i^-, e_i^+], \lambda_i\rangle, \langle [r_i^-, r_i^+], \delta_i\rangle)\,(i = 1, 2, \ldots, n)$ *be a collection of ICF-numbers, then the result from Definition 7 is an ICF-number, and it can be obtained as follows:*

$$\mathrm{ICFWMM}^p(I_1, I_2, \ldots, I_n) = \left(\begin{array}{c}\left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-c_{I_{\theta(j)}}\right)^{nw_{\theta(j)}}\right)^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^n p_j}, \\ 1-\left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-c'^{\,nw_{\theta(j)}}_{I_{\theta(j)}}\right)^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^n p_j}\end{array}\right)$$

$$= \left(\begin{array}{c}\left\langle\left[\begin{array}{c}\left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-e^-_{\theta(j)}\right)^{nw_{\theta(j)}}\right)^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^n p_j}, \\ \left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-e^+_{\theta(j)}\right)^{nw_{\theta(j)}}\right)^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^n p_j}\end{array}\right],\right\rangle, \\ \left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\lambda_{\theta(j)}\right)^{nw_{\theta(j)}}\right)^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^n p_j} \\ \left\langle\left[\begin{array}{c}1-\left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-r^{-\,nw_{\theta(j)}}_{\theta(j)}\right)^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^n p_j}, \\ 1-\left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-r^{+\,nw_{\theta(j)}}_{\theta(j)}\right)^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^n p_j}\end{array}\right],\right\rangle \\ 1-\left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\delta^{nw_{\theta(j)}}_{\theta(j)}\right)^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^n p_j}\end{array}\right). \tag{25}$$

To facilitate the understanding of the proof procedure and properties of the proposed operator, we use the abbreviated form of ICFS for its proof.

Proof. Because $nw_{\theta(j)} I_{\theta(j)} = (1-(1-I_{\theta(j)})^{nw_{\theta(j)}}, c'^{\,nw_{\theta(j)}}_{I_{\theta(j)}})$, we can replace $I_{\theta(j)}$ in equation (7) with $nw_{\theta(j)} I_{\theta(j)}$, and then we can get equation (25).

Because $I_{\theta(j)}$ is an ICF-number, $nw_{\theta(j)}I_{\theta(j)}$ is also an ICF-number. By (7), we know ICFWMM$^P(I_1,I_2,...,I_n)$ is an ICF-number. □

Property 2. The ICFWMM^P operator has some properties as follows.

(1) (Monotonicity). Let $I_i = (c_{I_i}, c'_{I_i})$ and $L_i = (c_{L_i}, c'_{L_i})$ $(i = 1, 2, \ldots, n)$ be two sets of ICF-numbers. If $c_{I_i} \geq c_{L_i}$ and $c'_{I_i} \leq c'_{L_i}$ for all i, then

$$\begin{aligned}&\mathrm{ICFWMM}^P(I_1, I_2, \ldots, I_n)\\&\quad\geq \mathrm{ICFSWMMM}^P(I'_1, I'_2, \ldots, I'_n),\end{aligned} \tag{26}$$

(2) (Boundedness). Let $I_i = (c_{I_i}, c'_{I_i})\,(i = 1, \ldots, n)$ be a collection of ICF-numbers and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of $I_i\,(i = 1, 2, \ldots, n)$, which satisfies $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. $I^- = (\min(c_{I_i}), \max(c'_{I_i}))$ and $I^+ = (\max(c_{I_i}), \min(c'_{I_i}))$, then

$$\left(c_{I^-}, c'_{I^-}\right) \leq \mathrm{ICFWMM}^P(I_1, I_2, \ldots, I_n) \leq \left(c_{I^+}, c'_{I^+}\right), \tag{27}$$

where

$$\begin{aligned}
c_{I^-} &= \left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\min\left(c_{I_i}\right)\right)^{nw_{\theta(j)}}\right)^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n}p_j},\\
c'_{I^-} &= 1-\left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\max\left(c'_{I_i}\right)^{nw_{\theta(j)}}\right)^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n}p_j},\\
c_{I^+} &= \left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\max\left(c_{I_i}\right)\right)^{nw_{\theta(j)}}\right)^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n}p_j},\\
c'_{I^+} &= 1-\left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\min\left(c'_{I_i}\right)^{nw_{\theta(j)}}\right)^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n}p_j}.
\end{aligned} \tag{28}$$

Proof. Since the proposed operator has the same properties in the short form and the full form of ICFS, we use the abbreviated form of ICFS to derive the properties for the sake of understanding.

(1) The Proof is similar to that of ICFMM operator, it is omitted here.

(2) According to Monotonicity, we have the following equation:

$$\begin{aligned}\mathrm{ICFWMM}^P(I^-, I^-, \ldots, I^-) &\leq \mathrm{ICFWMM}^P(I_1, I_2, \ldots, I_n)\\&\leq \mathrm{ICFWMM}^P(I^+, I^+, \ldots, I^+).\end{aligned} \tag{29}$$

According to (25), we have the following equation:

$$\begin{aligned}
\mathrm{ICFWMM}^P(I^-, I^-, \ldots, I^-) &= \left(\begin{array}{c}\left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\min\left(c_{I_i}\right)\right)^{nw_\theta(j)}\right)^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n}p_j},\\ 1-\left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\max\left(c'_{I_i}\right)^{nw_{\theta(j)}}\right)^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n}p_j}\end{array}\right),\\
\mathrm{ICFWMM}^P(I^+, I^+, \ldots, I^+) &= \left(\begin{array}{c}\left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\max\left(c_{I_i}\right)\right)^{nw_{\theta(j)}}\right)^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n}p_j},\\ 1-\left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\min\left(c'_{I_i}\right)^{nw_{\theta(j)}}\right)^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n}p_j}\end{array}\right).
\end{aligned} \tag{30}$$

So,

$$\left(c_{I_{I^-}}, c'_{I_{I^-}}\right) \leq \text{ICFWMM}^p\left(I_1, I_2, \ldots, I_n\right) \leq \left(c_{I_{I^+}}, c'_{I_{I^+}}\right). \tag{31}$$

□

3.3. The Intuitionistic Cubic Fuzzy Dual MM (ICFDMM) Operator. Based on the DMM operator, this study proposes the pairwise MM operator for intuitionistic cubic fuzzy sets, as follows.

Definition 8. Let $I_i = \langle c_{I_i}, c'_{I_i}\rangle = (\langle [e_i^-, e_i^+], \lambda\rangle_i, \langle [r_i^-, r_i^+], \delta_i\rangle)\,(i = 1, 2, \ldots, n)$ be a collection of ICF-numbers, and let $p = (p_1, p_2, \cdots, p_n) \in R^n$ be a vector of parameters. If

$$\text{ICFDMM}^p\left(I_1, I_2, \ldots, I_n\right) = \frac{1}{\sum_{j=1}^n p_j}\left(\prod_{n\in S_n}\sum_{j=1}^n\left(p_j I_{\theta(j)}\right)\right)^{1/n!}. \tag{32}$$

Then, we call ICFDMM^p the intuitionistic cubic fuzzy dual MM (IFDMM), where $\theta(j)(j = 1, 2, \ldots, n)$ is any a permutation of (1, 2, . . ., n), and S_n is the collection of all permutations of (1, 2, . . ., n).

Theorem 3. *Let $I_i = \langle c_{I_i}, c'_{I_i}\rangle = (\langle [e_i^-, e_i^+], \lambda\rangle_i, \langle [r_i^-, r_i^+], \delta_i\rangle)\,(i = 1, 2, \ldots, n)$ be a collection of ICF-numbers, then the result from Definition 8 is also an ICF-number, and it can be obtained as follows:*

$$\begin{aligned}
\text{ICFDMM}^p\left(I_1, I_2, \ldots, I_n\right) &= \begin{pmatrix} 1-\left(1-\prod_{\theta\in S_n}\left(1-\prod_{j=1}^n\left(1-c_{I_{\theta(j)}}\right)^{p_j}\right)\right)^{1/n!\;1/\sum_{j=1}^n p_j}, \\ \left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^n c'^{\,p_j}_{I_{\theta(j)}}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^n p_j} \end{pmatrix} \\
&= \begin{pmatrix} \left\langle\left[1-\left(1-\prod_{\theta\in S_n}\left(1-\prod_{j=1}^n\left(1-e^-_{\theta(j)}\right)^{p_j}\right)^{1/n!}\right)^{1/\sum_{j=1}^n p_j}, 1-\left(1-\prod_{\theta\in S_n}\left(1-\prod_{j=1}^n\left(1-e^+_{\theta(j)}\right)^{p_j}\right)^{1/n!}\right)^{1/\sum_{j=1}^n p_j}\right],\right\rangle, \\ 1-\left(1-\prod_{\theta\in S_n}\left(1-\prod_{j=1}^n\left(1-\lambda_{\theta(j)}\right)^{p_j}\right)^{1/n!}\right)^{1/\sum_{j=1}^n p_j} \\ \left\langle\left[\left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^n r^{-\,p_j}_{\theta(j)}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^n p_j}, \left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^n r^{+\,p_j}_{\theta(j)}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^n p_j}\right],\right\rangle \\ \left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^n \delta^{p_j}_{\theta(j)}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^n p_j} \end{pmatrix}.
\end{aligned} \tag{33}$$

To facilitate the understanding of the proof procedure and properties of the proposed operator, we use the abbreviated form of ICFS for its proof.

Proof. We need to prove (1) equation (33) is kept and (2) equation (33) is an ICF-number.

(1) First, we prove that equation (33) is kept. According to the operational laws of ICF-numbers, we get the following equation:

$$\begin{aligned}
p_j I_{\theta(j)} &= \left(1-\left(1-c_{I_{\theta(j)}}\right)^{p_j}, c'^{\,p_j}_{I_{\theta(j)}}\right), \\
\sum_{j=1}^n\left(p_j c_{I_{\theta(j)}}\right) &= \left(1-\prod_{j=1}^n\left(1-c_{I_{\theta(j)}}\right)^{p_j}, \prod_{j=1}^n c'^{\,p_j}_{I_{\theta(j)}}\right),
\end{aligned} \tag{34}$$

Then,

$$\prod_{\theta\in S_n}\sum_{j=1}^n\left(p_j c_{I_{\theta(j)}}\right) = \left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^n\left(1-c_{I_{\theta(j)}}\right)^{p_j}\right), 1-\prod_{\theta\in S_n}\left(1-\prod_{j=1}^n c'^{\,p_j}_{I_{\theta(j)}}\right)\right), \tag{35}$$

Furthermore,

$$\left(\prod_{\theta\in S_n}\sum_{j=1}^{n}\left(p_j c_{I_{\theta(j)}}\right)\right)^{1/n!}=\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-c_{I_{\theta(j)}}\right)^{p_j}\right)^{1/n!},1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}c_{I_{\theta(j)}}^{'p_j}\right)\right)^{1/n!}\right), \tag{36}$$

So,

$$\frac{1}{\sum_{j=1}^{n}p_j}\left(\prod_{\theta\in S_n}\sum_{j=1}^{n}\left(p_j c_{I_{\theta(j)}}\right)\right)^{1/n!}=\left(1-\left(1-\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-c_{I_{\theta(j)}}\right)^{p_j}\right)^{1/n!}\right)^{1/\sum_{j=1}^{n}p_j},\left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}c_{I_{\theta(j)}}^{'}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n}p_j}\right), \tag{37}$$

i.e., equation (33) is kept

(2) Then, we will prove that equation (33) is an ICF-number

Let

$$c_I=1-\left(1-\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-c_{I_{\theta(j)}}\right)^{p_j}\right)^{1/n!}\right)^{1/\sum_{j=1}^{n}p_j},$$
$$c_I^{'}=\left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}c_{I_{\theta(j)}}^{'p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n}p_j}, \tag{38}$$

Then, we need prove the following two conditions:

(a) $0\le c_I\le 1, 0\le c_I^{'}\le 1$

(b) $0\le c_I+c_I^{'}\le 1$

(i) Since $c_{I_{\theta(j)}}\in[0,1]$, we can get the following equation:

$$\left(1-c_{I_{\theta(j)}}\right)\in[0,1],$$
$$\left(1-c_{I_{\theta(j)}}\right)^{p_j}\in[0,1],$$
$$\prod_{j=1}^{n}\left(1-c_{I_{\theta(j)}}\right)^{p_j}\in[0,1], \tag{39}$$

Then,

$$\left(1-\prod_{j=1}^{n}\left(1-c_{I_{\theta(j)}}\right)^{p_j}\right)\in[0,1],$$
$$\left(1-\prod_{j=1}^{n}\left(1-c_{I_{\theta(j)}}\right)^{p_j}\right)^{1/n!}\in[0,1],$$
$$\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-c_{I_{\theta(j)}}\right)^{p_j}\right)^{1/n!}\in[0,1], \tag{40}$$

Furthermore,

$$\left(1-\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-c_{I_{\theta(j)}}\right)^{p_j}\right)^{1/n!}\right)\in[0,1],$$
$$\left(1-\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-c_{I_{\theta(j)}}\right)^{p_j}\right)^{1/n!}\right)^{1/\sum_{j=1}^{n}p_j}\in[0,1], \tag{41}$$

i.e., $0\le c_I\le 1$. Similarly, we can get $0\le c_I^{'}\le 1$. So, condition (i) is met.

(ii) Since $c_{I_{\theta(j)}}+c_{I_{\theta(j)}}^{'}\le 1$, then $c_{I_{\theta(j)}}\le 1-c_{I_{\theta(j)}}^{'}$, we can get the following inequality:

$$\begin{aligned}c_I+c_I^{'}&=1-\left(1-\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-c_{I_{\theta(j)}}\right)^{p_j}\right)^{1/n!}\right)^{1/\sum_{j=1}^{n}p_j}+\left(1-\left(\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}c_{I_{\theta(j)}}^{'p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n}p_j}\\&\le 1-\left(1-\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-c_{I_{\theta(j)}}\right)^{p_j}\right)^{1/n!}\right)^{1/\sum_{j=1}^{n}p_j}+\left(1-\prod_{\theta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-c_{I_{\theta(j)}}\right)^{p_j}\right)^{1/n!}\right)^{1/\sum_{j=1}^{n}p_j}=1,\end{aligned} \tag{42}$$

i.e., $0 \le c_I + c_I^{'} \le 1$, we can know the aggregation result from (33) is still an ICF-number □

Property 3. Similar to the properties of ICFMM operator, the *ICFDMM*^{p} operator has some properties as follows.

(1) (Idempotency). If all $I_i\,(i = 1, 2, \ldots, n)$ are equal, i.e., $I_i = (c_{I_i}, c_{I_i}^{'})\,(i = 1, \ldots, n)$, then

$$\mathrm{ICFDMM}^p(I_1, I_2, \ldots, I_n) = I, \tag{43}$$

(2) (Monotonicity). Let $I_i = (c_{I_i}, c_{I_i}^{'})$ and $L_i = (c_{L_i}, c_{L_i}^{'})\,(i = 1, 2, \ldots, n)$ be the two sets of ICF-numbers. If $c_{I_i} \ge c_{L_i}$ and $c_{I_i}^{'} \le c_{L_i}^{'}$ for all i, then

$$\mathrm{ICFDMM}^p(I_1, I_2, \ldots, I_n) \ge \mathrm{ICFDMM}^p\left(I_1^{'}, I_2^{'}, \ldots, I_n^{\ '}\right), \tag{44}$$

(3) (Boundedness). Let $I_i = (c_{I_i}, c_{I_i}^{'})\,(i = 1, \ldots, n)$ be a collection of ICF-numbers, and $I^- = (\min(c_{I_i}), \max(c_{I_i}^{'}))$ and $I^+ = (\max(c_{I_i}), \min(c_{I_i}^{'}))$, then

$$I^- \le \mathrm{ICFDMM}^p(I_1, I_2, \ldots, I_n) \le I^+. \tag{45}$$

Same as ICFMM operator, regarding the influence of the parameter vector p on the monotonicity of the ICFMM operator, the higher the control rate of the parameter vector the larger the value of the aggregation operator result [34], for the proof procedure refer to reference [39, 40].

3.4. The Intuitionistic Cubic Fuzzy Dual Weighted MM (ICFDWMM) Operator. Similar to ICFWMM operator, we will propose intuitionistic cubic fuzzy dual weighted MM (ICFDWMM) operator so as to consider the weight vector of the attribute values, which is defined as follows.

Definition 9. Let $I_i = \langle c_{I_i}, c_{I_i}^{'}\rangle = (\langle [e_i^-, e_i^+], \lambda\rangle_i, \langle [r_i^-, r_i^+], \delta_i\rangle)\,(i = 1, 2, \ldots, n)$ be a collection of ICF-numbers and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of $I_i\,(i = 1, 2, \ldots, n)$, which satisfies $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1$, and let $p = (p_1, p_2, \cdots, p_n) \in R^n$ be a vector of parameters. If

$$\mathrm{ICFDWMM}(I_1, I_2, \ldots, I_n) = \frac{1}{\sum_{j=1}^n p_j}\left(\prod_{\theta \in S_n}\sum_{j=1}^n\left(p_j c_{I_{\theta(j)}}^{nw_{\theta(j)}}\right)\right)^{1/n!}. \tag{46}$$

Then, we call ICFDWMM^p the intuitionistic cubic fuzzy dual weighted MM (ICFDMM), where $\theta(j)(j = 1, 2, \ldots, n)$ is any a permutation of (1, 2, ..., n) and S_n is the collection of all permutations of (1, 2, ..., n).

When $w = (1/n, 1/n, \ldots, 1/n)$, the ICFDWMM operator changes to the ICFDMM operator.

Theorem 4. *Let $I_i = \langle c_{I_i}, c_{I_i}^{'}\rangle = (\langle [e_i^-, e_i^+], \lambda\rangle_i, \langle [r_i^-, r_i^+], \delta_i\rangle)\,(i = 1, 2, \ldots, n)$ be a collection of ICF-numbers, then the result from Definition 9 is also an ICF-number, and it can be obtained as follows:*

$$\mathrm{ICFDWMM}^p(I_1, I_2, \ldots, I_n) = \begin{pmatrix} 1 - \left(1 - \left(\prod_{\theta \in S_n}\left(1 - \prod_{j=1}^n\left(1 - c_{I_{\theta(j)}}^{nw_{\theta(j)}}\right)^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^n p_j}, \\ \left(1 - \left(\prod_{\theta \in S_n}\left(1 - \prod_{j=1}^n\left(1 - \left(1 - c_{I_{\theta(j)}}^{'}\right)^{nw_{\theta(j)}}\right)^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^n p_j} \end{pmatrix}$$
$$= \begin{pmatrix} \left\langle \begin{bmatrix} 1 - \left(1 - \left(\prod_{\theta \in S_n}\left(1 - \prod_{j=1}^n\left(1 - e_{\theta(j)}^{-\,nw_{\theta(j)}}\right)^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^n p_j} \\ 1 - \left(1 - \left(\prod_{\theta \in S_n}\left(1 - \prod_{j=1}^n\left(1 - e_{\theta(j)}^{+\,nw_{\theta(j)}}\right)^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^n p_j} \end{bmatrix} \right\rangle \\ 1 - \left(1 - \left(\prod_{\theta \in S_n}\left(1 - \prod_{j=1}^n\left(1 - \lambda_{\theta(j)}^{nw_{\theta(j)}}\right)^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^n p_j} \\ \left\langle \begin{bmatrix} \left(1 - \left(\prod_{\theta \in S_n}\left(1 - \prod_{j=1}^n\left(1 - \left(1 - r_{\theta(j)}^-\right)^{nw_{\theta(j)}}\right)^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^n p_j} \\ \left(1 - \left(\prod_{\theta \in S_n}\left(1 - \prod_{j=1}^n\left(1 - \left(1 - r_{\theta(j)}^+\right)^{nw_{\theta(j)}}\right)^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^n p_j} \end{bmatrix} \right\rangle \\ \left(1 - \left(\prod_{\theta \in S_n}\left(1 - \prod_{j=1}^n\left(1 - \left(1 - \delta_{\theta(j)}\right)^{nw_{\theta(j)}}\right)^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^n p_j} \end{pmatrix}. \tag{47}$$

To facilitate the understanding of the proof procedure and properties of the proposed operator, we use the abbreviated form of ICFS for its proof.

Proof. Because $I_{\theta(j)}^{nw_{\theta(j)}} = (c_{I_{\theta(j)}}^{nw_{\theta(j)}}, 1-(1-c'_{I_{\theta(j)}})^{nw_{\theta(j)}})$, we can replace $I_{\theta(j)}$ in equation (33) with $nw_{\theta(j)}I_{\theta(j)}$, and then we get equation (47). Because $I_{\theta(j)}$ is an ICF-number, $nw_{\theta(j)}z_{\theta(j)}$ is also an ICF-number. By equation (33), we know *ICFDWMM*$^p(I_1, I_2, \ldots, I_n)$ is an ICF-number. □

Property 4. The ICFDWMM^p operator has some properties as follows:

(1) (Monotonicity). Let $I_i = (c_{I_i}, c'_{I_i})$ and $L_i = (c_{L_i}, c'_{L_i})\,(i = 1, 2, \ldots, n)$ be the two sets of ICF-numbers and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of $I_i\,(i = 1, 2, \ldots, n)$, which satisfies $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$. If $c_{I_i} \geq c_{L_i}$ and $c'_{I_i} \leq c'_{L_i}$, then

$$\text{ICFDWMM}^p(I_1, I_2, \ldots, I_n) \geq \text{ICFDWMM}^p(I'_1, I'_2, \ldots, I'_n), \tag{48}$$

(2) (Boundedness). Let $I_i = (c_{I_i}, c'_{I_i})\,(i = 1, \ldots, n)$ be a collections of ICF-numbers and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of $I_i\,(i = 1, 2, \ldots, n)$, which satisfies $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$. Let $I^- = (\min(c_{I_i}), \max(c'_{I_i}))$ and $I^+ = (\max(c_{I_i}), \min(c'_{I_i}))$, then

$$\left(c_{I^-}, c'_{I^-}\right) \leq \text{ICFDWMM}^p(I_1, I_2, \ldots, I_n) \leq \left(c_{I^-}, c'_{I^-}\right) \tag{49}$$

where

$$\begin{aligned}
c_{I^-} &= 1 - \left(1 - \left(\prod_{\theta \in S_n}\left(1 - \prod_{j=1}^{n}\left(1 - \max\left(c_{I_i}\right)^{nw_{\theta(j)}}\right)^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n} p_j},\\
c'_{I^-} &= \left(1 - \left(\prod_{\theta \in S_n}\left(1 - \prod_{j=1}^{n}\left(1 - \left(1 - \min\left(c'_{I_i}\right)\right)^{nw_{\theta(j)}}\right)^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n} p_j},\\
c_{I^+} &= 1 - \left(1 - \left(\prod_{\theta \in S_n}\left(1 - \prod_{j=1}^{n}\left(1 - \min\left(c_{I_i}\right)^{nw_{\theta(j)}}\right)^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n} p_j},\\
c'_{I^+} &= \left(1 - \left(\prod_{\theta \in S_n}\left(1 - \prod_{j=1}^{n}\left(1 - \left(1 - \max\left(c'_{I_i}\right)\right)^{nw_{\theta}(1)}\right)^{p_j}\right)\right)^{1/n!}\right)^{1/\sum_{j=1}^{n} p_j}.
\end{aligned} \tag{50}$$

Proof. The proof process is the same as that of ICFWMM, which is omitted here. □

4. Application of the Proposed Operator

4.1. Two MCDM Methods Based on ICFWMM Operator and ICFDWMM. In this section, the two new MCDM methods are developed based on the proposed ICFWMM or ICFDWMM operator, which are described as follows.

Suppose there are q decision makers $\{X_1, X_2, \cdots, X_q\}$ to evaluate m alternatives $\{S_1, S_2, \cdots, S_m\}$ with respect to n attributes $\{A_1, A_2, \cdots, A_n\}$ in a MCDM problem, where the weight vector of the attributes is $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$ satisfying $w_k \geq 0, k = 1, 2, \cdots, q, \sum_{k=1}^{q} w_k = 1$, and the weight vector of decision makers is $w = (w_1, w_2, \cdots, w_q)^T$ and satisfying $w_k \geq 0, k = 1, 2, \cdots, q, \sum_{k=1}^{q} w_k = 1$. $R^k = [r_{ij}^k]m \times n$ is the given decision matrix of this decision problem, and R^k is the ICFN given by the decision maker X_k for the alternative S_i of attribute A_j, where $r_{ij}^k = (c_{I_{ij}}^k, c'_{I_{ij}})$, $c_{I_{ij}}{}^k = \langle [e_{ij}^- k, e_{ij}^+ k], \lambda_{ij}^k \rangle$, $c'_{I_{ij}} k = \langle [r_{ij}^- k, r_{ij}^+ k], \delta_{ij}^k \rangle$, $[r_{ij}^- k, r_{ij}^+ k] \subset [0,1]$, $[e_{ij}^- k, e_{ij}^+ k] \subset [0,1]$, $\delta_{ij}^k \subset [0,1]$, $\lambda_{ij}{}^k \subset [0,1]$, and $r_{ij}^+ k + e_{ij}^+ k \leq 1, \lambda_{ij}{}^k + \delta_{ij}{}^k \leq 1$. Then, the goal is to rank the alternatives.

In the following section, the two MCDM methods are proposed using the ICFWMM and ICFDWMM aggregation operators proposed in this study, and the detailed decision steps are shown in Figure 1:

Step 1. Normalize the intuitionistic cubic fuzzy decision matrix $R^k = [r_{ij}^k]m \times n$

Step 2. Aggregate all individual decision matrices $R^k\,(k = 1, 2, \cdots, q)$ into an aggregate matrix R by the ICFWMM or ICFDWMM operator, as in equations (51) and (52) as follows:

$$r_{ij} = \text{ICFWMM}\left(r_{ij}^1, r_{ij}^2, \cdots, r_{ij}^q\right), \tag{51}$$

or

$$r_{ij} = \text{ICFWMM}\left(r_{ij}^1, r_{ij}^2, \cdots, r_{ij}^q\right), \tag{52}$$

Step 3. By ICFWMM or ICFDWMM operators, all attribute values $r_{ij}\,(j = 1, 2, \cdots, n)$ are aggregated into the composite value z_i as in equations (53) and (54) as follows:

$$z_i = \text{ICFWMM}\left(r_{i1}, r_{i2}, \cdots, r_{in}\right), \tag{53}$$

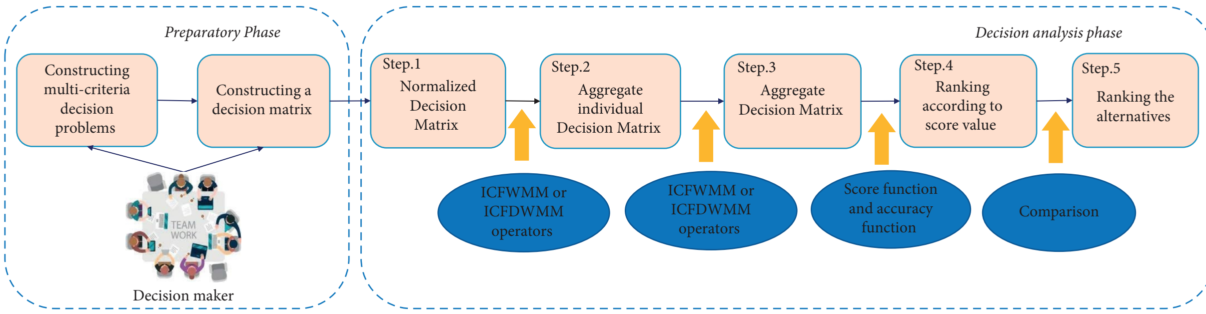

Figure 1: Decision-making process of the proposed MCDM method.

or

$$z_i = \text{ICFDWMM}(r_{i1}, r_{i2}, \cdots, r_{in}), \quad (54)$$

Step 4. Ranking according to the score function and the accuracy function of Definition 2

Step 5. Rank all alternatives. The larger the z_i, the better the alternative S_i

5. Case Study

To illustrate the application of the method in this study, an example from the literature [34] on investment selection decisions is cited. An investment firm wants to choose one of five candidates (S_1, S_2, S_3, S_4, and S_5) to invest in. In order to make a scientific decision and avoid investment loss, three experts (X_1, X_2, and X_3) are invited to evaluate five candidate companies from four attributes (A_1, A_2, A_3, and A_4), where A_1 means risk assessment, A_2 means growth assessment, A_3 means socio-political impact assessment, and A_4 means environmental impact assessment. Based on the existing experience and knowledge, the investment company set the attribute weight vector $\omega = (0, 2, 0.1, 0.3, 0.4)^T$ and the expert weight vector $w = (0.35, 0.40, 0.25)^T$. The three decision makers (X_1, X_2, and X_3) assessed information on all attributes according to the ICFS concept, and the evaluation matrices obtained are shown in Tables 1–3, respectively. The ICFS concept referenced for information evaluation in practical decision making is as follows: evaluate information results as $r_{ij}^k = (c_{I_{ij}}^k, c_{I_{ij}}^{'}k)$, where $c_{I_{ij}}{}^k = \langle [e_{ij}^-k, e_{ij}^+k], \lambda_{ij}^k \rangle$ represents the degree of affiliation, $[e_{ij}^-k, e_{ij}^+k]$ is the interval intuitionistic fuzzy value of the degree of affiliation given by the decision maker, and λ_{ij}^k represents the intuitionistic fuzzy value of the affiliation given by the decision maker. $c_{I_{ij}}^{'}k = \langle [r_{ij}^-k, r_{ij}^+k], \delta_{ij}^k \rangle$ represents the unaffiliated degree, $[r_{ij}^-k, r_{ij}^+k]$ is the interval value intuitionistic fuzzy value of the unaffiliated degree given by the decision maker, and δ_{ij}^k represents the intuitionistic fuzzy value of the unaffiliated degree given by the decision maker. They need to meet the following conditions: $[r_{ij}^-k, r_{ij}^+k] \subset [0,1]$, $[e_{ij}^-k, e_{ij}^+k] \subset [0,1]$, $\delta_{ij}^k \subset [0,1]$, $\lambda_{ij}{}^k \subset [0,1]$, and $r_{ij}^+k + e_{ij}^+k \leq 1, \lambda_{ij}{}^k + \delta_{ij}{}^k \leq 1$. For example, for the information evaluation of attribute A1 of alternative S1, according to the concept of ICFS, the decision maker gives the evaluation information as $r_{ij}^k = \begin{pmatrix} ([0.1, 0.5], 0.5) \\ ([0.2, 0.4], 0.4) \end{pmatrix}$, where ([0.1, 0.5], 0.5) represents the affiliation degree, [0.1, 0.5] is the interval value intuitionistic fuzzy value of the affiliation degree given by the decision maker, and 0.5 is the intuitionistic fuzzy value of the affiliation degree given by the decision maker.([0.2, 0.4],0.4) represents the unaffiliated degree, [0.2, 0.4] is the interval value of the unaffiliated degree intuitionistic fuzzy value given by the decision maker, and 0.4 is the intuitionistic fuzzy value of the unaffiliated degree given by the decision maker. Also, meet the conditions $[0.1, 0.5] \subset [0, 1]$, $[0.2, 0.4] \subset [0, 1]$, $0.5 \subset [0, 1]$, $0.4 \subset [0, 1]$, and $0.5 + 0.4 \leq 1, 0.5 + 0.4 \leq 1$.

5.1. *The Decision Making Steps.*

To get the best alternative(s), the steps are shown in the following:

Step 1. Normalizing the attribute values.

Since all property values in this case are of the same type, no normalization is required.

Step 2. All individual decision matrices R^k ($k = 1, 2, 3$) are aggregated into an ensemble matrix R by the ICFWMM and ICFDWMM operators as shown in Tables 4 and 5.

Step 3. The ICFWMM operator and the ICFDWMM operator are used to aggregate all the attribute values r_{ij} (j = 1, 2, $\cdots$ n) into a composite value z_i as shown in Table 6.

Step 4. Calculating the score function $S(z_i)$ (i = 1, 2, $\cdots$ n) of the collective overall values z_i ($i = 1, 2, \ldots, 5$) produced by ICFWMM or ICFDWMM operators shown in Table 7.

Step 5. Ranking all the alternatives.

According to the score function $S(z_i)$ (i = 1, 2, $\cdots$ n), we can rank the alternatives {S_1, S_2, S_3, S_4, and S_5} shown in Table 8. From Table 8, we can see the best alternative is S_2

5.2. *The Influence of the Parameter Vector p on the Decision Result of This Example.*

In order to illustrate the influence of the parameter vector p on the decision of this example, different parameter vectors p are set to discuss the ranking results in this study, and the results are shown in Tables 9 and 10.

Table 1: Decision matrix R^1 for decision maker X_1.

	A_1	A_2	A_3	A_4
S_1	$\begin{pmatrix} ([0.1, 0.5], 0.5) \\ ([0.2, 0.4], 0.4) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.5], 0.5) \\ ([0.2, 0.4], 0.3) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.6], 0.6) \\ ([0.1, 0.4], 0.2) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.5], 0.4) \\ ([0.2, 0.5], 0.4) \end{pmatrix}$
S_2	$\begin{pmatrix} ([0.4, 0.7], 0.7), \\ ([0.2, 0.3], 0.3) \end{pmatrix}$	$\begin{pmatrix} ([0.3, 0.7], 0.7), \\ ([0.1, 0.3], 0.3) \end{pmatrix}$	$\begin{pmatrix} ([0.2, 0.7], 0.6), \\ ([0.1, 0.3], 0.2) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.6], 0.6), \\ ([0.2, 0.4], 0.2) \end{pmatrix}$
S_3	$\begin{pmatrix} ([0.1, 0.5], 0.5) \\ ([0.2, 0.4], 0.4) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.6], 0.6) \\ ([0.2, 0.4], 0.4) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.6], 0.6) \\ ([0.1, 0.3], 0.2) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.5], 0.5), \\ ([0.2, 0.4], 0.3) \end{pmatrix}$
S_4	$\begin{pmatrix} ([0.1, 0.8], 0.8), \\ ([0.1, 0.2], 0.2) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.8], 0.7), \\ ([0.1, 0.2], 0.2) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.5], 0.4), \\ ([0.1, 0.2], 0.2) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.5], 0.5) \\ ([0.2, 0.4], 0.2) \end{pmatrix}$
S_5	$\begin{pmatrix} ([0.1, 0.5], 0.4) \\ ([0.2, 0.4], 0.3) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.5], 0.4), \\ ([0.2, 0.4], 0.2) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.4], 0.4), \\ ([0.2, 0.5], 0.5) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.4], 0.4) \\ ([0.2, 0.6], 0.6) \end{pmatrix}$

Table 2: Decision matrix R^2 for decision maker X_2.

	A_1	A_2	A_3	A_4
S_1	$\begin{pmatrix} ([0.1, 0.4], 0.4), \\ ([0.2, 0.5], 0.5) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.6], 0.6) \\ ([0.2, 0.4], 0.2) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.5], 0.5), \\ ([0.2, 0.4], 0.4) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.5], 0.5) \\ ([0.2, 0.4], 0.3) \end{pmatrix}$
S_2	$\begin{pmatrix} ([0.1, 0.5], 0.5) \\ ([0.2, 0.4], 0.4) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.6], 0.6), \\ ([0.2, 0.4], 0.2) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.6], 0.6) \\ ([0.2, 0.4], 0.3) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.7], 0.7) \\ ([0.2, 0.3], 0.3) \end{pmatrix}$
S_3	$\begin{pmatrix} ([0.1, 0.4], 0.4), \\ ([0.2, 0.5], 0.5) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.4], 0.3), \\ ([0.2, 0.5], 0.5) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.5], 0.4) \\ ([0.2, 0.4], 0.4) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.3], 0.2) \\ ([0.2, 0.6], 0.6) \end{pmatrix}$
S_4	$\begin{pmatrix} ([0.1, 0.5], 0.5), \\ ([0.2, 0.4], 0.4) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.7], 0.7) \\ ([0.2, 0.3], 0.2) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.5], 0.4) \\ ([0.2, 0.4], 0.4) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.7], 0.6), \\ ([0.2, 0.4], 0.2) \end{pmatrix}$
S_5	$\begin{pmatrix} ([0.1, 0.6], 0.6), \\ ([0.2, 0.4], 0.3) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.7], 0.7) \\ ([0.2, 0.4], 0.2) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.5], 0.4) \\ ([0.2, 0.4], 0.2) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.7], 0.7) \\ ([0.2, 0.3], 0.2) \end{pmatrix}$

Table 3: Decision matrix R^3 for decision maker X_3.

	A_1	A_2	A_3	A_4
S_1	$\begin{pmatrix} ([0.1, 0.5], 0.4) \\ ([0.2.0.4], 0.2) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.5], 0.5) \\ ([0.2, 0.4], 0.2) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.5], 0.5), \\ ([0.2, 0.4], 0.3) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.5], 0.5), \\ ([0.2, 0.4], 0.2) \end{pmatrix}$
S_2	$\begin{pmatrix} ([0.1, 0.5], 0.5), \\ ([0.2, 0.4], 0.3) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.5], 0.5) \\ ([0.2, 0.4], 0.3) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.6], 0.6), \\ ([0.2, 0.3], 0.2) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.7], 0.7) \\ ([0.2, 0.3], 0.2) \end{pmatrix}$
S_3	$\begin{pmatrix} ([0.1, 0.5], 0.4) \\ ([0.2, 0.4], 0.4) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.5], 0.3) \\ ([0.2, 0.4], 0.4) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.5], 0.4), \\ ([0.2, 0.4], 0.3) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.5], 0.3) \\ ([0.2, 0.4], 0.3) \end{pmatrix}$
S_4	$\begin{pmatrix} ([0.1, 0.5], 0.5)) \\ ([0.2, 0.4], 0.3) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.5], 0.5), \\ ([0.2, 0.4], 0.3) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.4], 0.3) \\ ([0.2, 0.6], 0.5) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.5], 0.5), \\ ([0.2, 0.3], 0.2) \end{pmatrix}$
S_5	$\begin{pmatrix} ([0.1, 0.6], 0.6) \\ ([0.2, 0.4], 0.4) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.6], 0.6), \\ ([0.2, 0.4], 0.4) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.5], 0.4) \\ ([0.2, 0.4], 0.4) \end{pmatrix}$	$\begin{pmatrix} ([0.1, 0.5], 0.6) \\ ([0.2, 0.4], 0.4) \end{pmatrix}$

As can be seen from Tables 9 and 10, the values of the score functions using different parameter vectors p are different, and the ranking results are slightly different accordingly, where the ICFWMM operator will become the intuitionistic cubic fuzzy weighted average (ICFWA) operator and the ICFDWMM operator will become the intuitionistic cubic fuzzy weighted geometry (ICFWG) operator when $p = (1, 0, 0, 0)$. When $p = (1, 1, 0, 0)$, the ICFWMM operator will become the intuitionistic cubic fuzzy weighted Bonferroni (ICFWBM) operator. For the ICFWMM aggregation operator, it can be found that the more interrelationships among the attributes considered in

Table 4: Collective matrix R by ICFWMM operator.

	A_1	A_2	A_3	A_4
S_1	$\begin{pmatrix} ([0.10, 0.46], 0.42) \\ ([0.21, 0.44], 0.37) \end{pmatrix}$	$\begin{pmatrix} ([0.10, 0.52], 0.52), \\ ([0.21, 0.41], 0.25) \end{pmatrix}$	$\begin{pmatrix} ([0.10, 0.52], 0.52), \\ ([0.18, 0.41], 0.31) \end{pmatrix}$	$\begin{pmatrix} ([0.10, 0.49], 0.46), \\ ([0.21, 0.44], 0.31) \end{pmatrix}$
S_2	$\begin{pmatrix} ([0.16, 0.55], 0.55) \\ ([0.21, 0.38], 0.34) \end{pmatrix}$	$\begin{pmatrix} ([0.14, 0.58], 0.58) \\ ([0.18, 0.38], 0.29) \end{pmatrix}$	$\begin{pmatrix} ([0.12, 0.62], 0.59) \\ ([0.18, 0.34], 0.24) \end{pmatrix}$	$\begin{pmatrix} ([0.10, 0.65], 0.65), \\ ([0.21, 0.34], 0.24) \end{pmatrix}$
S_3	$\begin{pmatrix} ([0.10, 0.46], 0.42) \\ ([0.21, 0.44], 0.44) \end{pmatrix}$	$\begin{pmatrix} ([0.10, 0.49], 0.37) \\ ([0.21, 0.44], 0.44) \end{pmatrix}$	$\begin{pmatrix} ([0.10, 0.52], 0.45) \\ ([0.18, 0.38], 0.31) \end{pmatrix}$	$\begin{pmatrix} ([0.10, 0.42], 0.30) \\ ([0.21, 0.48], 0.42) \end{pmatrix}$
S_4	$\begin{pmatrix} ([0.10, 0.57], 0.57) \\ ([0.18, 0.35], 0.31) \end{pmatrix}$	$\begin{pmatrix} ([0.10, 0.63], 0.61), \\ ([0.18, 0.32], 0.25) \end{pmatrix}$	$\begin{pmatrix} ([0.10, 0.45], 0.35) \\ ([0.18, 0.44], 0.40) \end{pmatrix}$	$\begin{pmatrix} ([0.10, 0.54], 0.52) \\ ([0.21, 0.37], 0.21) \end{pmatrix}$
S_5	$\begin{pmatrix} ([0.10, 0.56], 0.52) \\ ([0.21, 0.41], 0.35) \end{pmatrix}$	$\begin{pmatrix} ([0.10, 0.58], 0.54) \\ ([0.21, 0.41], 0.30) \end{pmatrix}$	$\begin{pmatrix} ([0.10, 0.46], 0.40) \\ ([0.21, 0.44], 0.40) \end{pmatrix}$	$\begin{pmatrix} ([0.10, 0.54], 0.54) \\ ([0.21, 0.46], 0.44) \end{pmatrix}$

Table 5: Collective matrix R by ICFDWMM operator.

	A_1	A_2	A_3	A_4
S_1	$\begin{pmatrix} ([0.11, 0.48], 0.44) \\ ([0.20, 0.42], 0.33) \end{pmatrix}$	$\begin{pmatrix} ([0.11, 0.54], 0.54) \\ ([0.20, 0.39], 0.22) \end{pmatrix}$	$\begin{pmatrix} ([0.11, 0.54], 0.54) \\ ([0.16, 0.39], 0.28) \end{pmatrix}$	$\begin{pmatrix} ([0.11, 0.51], 0.48) \\ ([0.20, 0.42], 0.28) \end{pmatrix}$
S_2	$\begin{pmatrix} ([0.22, 0.58], 0.58), \\ ([0.20, 0.36], 0.32) \end{pmatrix}$	$\begin{pmatrix} ([0.18, 0.61], 0.61) \\ ([0.16, 0.36], 0.26) \end{pmatrix}$	$\begin{pmatrix} ([0.14, 0.64], 0.60) \\ ([0.16, 0.32], 0.22) \end{pmatrix}$	$\begin{pmatrix} ([0.11, 0.68], 0.68) \\ ([0.20, 0.32], 0.22) \end{pmatrix}$
S_3	$\begin{pmatrix} ([0.11, 0.48], 0.44) \\ ([0.20, 0.42], 0.42) \end{pmatrix}$	$\begin{pmatrix} ([0.11, 0.52], 0.42) \\ ([0.20, 0.42], 0.42) \end{pmatrix}$	$\begin{pmatrix} ([0.11, 0.54], 0.48) \\ ([0.16, 0.36], 0.28) \end{pmatrix}$	$\begin{pmatrix} ([0.11, 0.46], 0.36) \\ ([0.20, 0.45], 0.37) \end{pmatrix}$
S_4	$\begin{pmatrix} ([0.11, 0.64], 0.63) \\ ([0.16, 0.31], 0.28) \end{pmatrix}$	$\begin{pmatrix} ([0.11, 0.69], 0.65) \\ ([0.16, 0.29], 0.23) \end{pmatrix}$	$\begin{pmatrix} ([0.11, 0.47], 0.37) \\ ([0.16, 0.36], 0.34) \end{pmatrix}$	$\begin{pmatrix} ([0.11, 0.58], 0.54) \\ ([0.20, 0.35], 0.20) \end{pmatrix}$
S_5	$\begin{pmatrix} ([0.11, 0.58], 0.55) \\ ([0.20, 0.39], 0.33) \end{pmatrix}$	$\begin{pmatrix} ([0.11, 0.61], 0.59) \\ ([0.20, 0.39], 0.25) \end{pmatrix}$	$\begin{pmatrix} ([0.11, 0.48], 0.41) \\ ([0.20, 0.42], 0.34) \end{pmatrix}$	$\begin{pmatrix} ([0.11, 0.59], 0.59) \\ ([0.20, 0.40], 0.36) \end{pmatrix}$

Table 6: The comprehensive value z_i by ICFWMM and ICFDWMM operators.

Operator	S_1	S_2	S_3	S_4	S_5
ICFWMM	$\begin{pmatrix} ([0.06, 0.31], 0.43) \\ ([0.48, 0.64], 0.55) \end{pmatrix}$	$\begin{pmatrix} ([0.08, 0.38], 0.52) \\ ([0.47, 0.60], 0.54) \end{pmatrix}$	$\begin{pmatrix} ([0.06, 0.30], 0.34), \\ ([0.48, 0.64], 0.63) \end{pmatrix}$	$\begin{pmatrix} ([0.06, 0.35], 0.46) \\ ([0.46, 0.60], 0.55) \end{pmatrix}$	$\begin{pmatrix} ([0.06, 0.34], 0.44) \\ ([0.48, 0.64], 0.59) \end{pmatrix}$
ICFDWMM	$\begin{pmatrix} ([0.40, 0.70], 0.69) \\ ([0.12, 0.26], 0.17) \end{pmatrix}$	$\begin{pmatrix} ([0.46, 0.77], 0.76), \\ ([0.11, 0.21], 0.16) \end{pmatrix}$	$\begin{pmatrix} ([0.40, 0.69], 0.64) \\ ([0.12, 0.26], 0.23) \end{pmatrix}$	$\begin{pmatrix} ([0.40, 0.76], 0.73) \\ ([0.10, 0.20], 0.16) \end{pmatrix}$	$\begin{pmatrix} ([0.40, 0.73], 0.71), \\ ([0.12, 0.25], 0.20) \end{pmatrix}$

Table 7: The score function $S(z_i)$ of the comprehensive value z_i by two operators.

Operator	S_1	S_2	S_3	S_4	S_5
ICFWMM	−0.2897	−0.2101	−0.3503	−2.465	−2.907
ICFDWMM	0.4125	0.5000	0.3723	0.4718	0.4225

Table 8: The ranking results of five alternatives by two operators.

Operator	Ranking results
ICFWMM	$S_2 > S_4 > S_1 > S_5 > S_3$
ICFDWMM	$S_2 > S_4 > S_5 > S_1 > S_3$

this study, the larger the value of the score function will be. The stronger the control ability of the parameter vector p, the larger the value of the score function. However, for the ICFDWMM aggregation operator, the result is the opposite, the more interrelationships between attributes are considered, the smaller the value of the score function, the greater the control ability of the parameter vector p, and the value of the score function will become larger. In addition, as more relationships between attributes are considered, the values of the ranking results of ICFWMM operator and ICFDWMM operator tend to be uniform, and the more relationships between attributes are considered, the less the resulting ranking results differ from the results of considering all attributes.

Table 9: Ranking results by utilizing the different parameter vector p in the ICFWMM operator.

Parameter vector p	The score function $S(z_i)$	Ranking results
$p=(1, 0, 0, 0)$	$S(z_1)=-0.8016$, $S(z_2)=-0.7882$, $S(z_3)=-0.8108$, $S(z_4)=-0.7723$, $S(z_5)=-0.7939$	$S_4>S_2>S_5>S_1>S_3$
$p=(1, 1, 0, 0)$	$S(z_1)=-0.7478$, $S(z_2)=-0.7384$, $S(z_3)=-0.7724$, $S(z_4)=-0.7230$, $S(z_5)=-0.7490$	$S_4>S_2>S_1>S_5>S_3$
$p=(1, 1, 1, 0)$	$S(z_1)=-0.6388$, $S(z_2)=-0.6227$, $S(z_3)=-0.6614$, $S(z_4)=-0.6228$, $S(z_5)=-0.6479$	$S_2>S_4>S_1>S_5>S_3$
$p=(1, 1, 1, 1)$	$S(z_1)=-0.2879$, $S(z_2)=-0.2101$, $S(z_3)=-0.3503$, $S(z_4)=-2.465$, $S(z_5)=-2.907$	$S_2>S_4>S_1>S_5>S_3$
$p=(0.25, 0.25, 0.25, 0.25)$	$S(z_1)=-0.7835$, $S(z_2)=-0.7695$, $S(z_3)=-0.8004$, $S(z_4)=-0.7698$, $S(z_5)=-0.7930$	$S_2>S_4>S_1>S_5>S_3$
$p=(2, 0, 0, 0)$	$S(z_1)=-0.6858$, $S(z_2)=-0.6675$, $S(z_3)=-0.6986$, $S(z_4)=-0.6476$, $S(z_5)=-0.6753$	$S_4>S_2>S_5>S_1>S_3$
$p=(3, 0, 0, 0)$	$S(z_1)=-0.6238$, $S(z_2)=-0.6031$, $S(z_3)=-0.6385$, $S(z_4)=-0.5814$, $S(z_5)=-0.6119$	$S_4>S_2>S_5>S_1>S_3$

Table 10: Ranking results by utilizing the different parameter vector p in the ICFDWMM operator.

Parameter vector p	The score function $S(z_i)$	Ranking results
$p=(1, 0, 0, 0)$	$S(z_1)=0.7118$, $S(z_2)=0.7775$, $S(z_3)=0.7030$, $S(z_4)=0.7675$, $S(z_5)=0.7383$	$S_2>S_4>S_5>S_1>S_3$
$p=(1, 1, 0, 0)$	$S(z_1)=0.6731$, $S(z_2)=0.7264$,$S(z_3)=0.6434$, $S(z_4)=0.7280$, $S(z_5)=0.6931$	$S_4>S_2>S_5>S_1>S_3$
$p=(1, 1, 1, 0)$	$S(z_1)=0.5337$, $S(z_2)=0.6012$, $S(z_3)=0.5040$, $S(z_4)=0.5805$, $S(z_5)=0.5385$	$S_2>S_4>S_1>S_5>S_3$
$p=(1, 1, 1, 1)$	$S(z_1)=0.4125$, $S(z_2)=0.5000$, $S(z_3)=0.3723$, $S(z_4)=0.4718$, $S(z_5)=0.4225$	$S_2>S_4>S_5>S_1>S_3$
$p=(0.25, 0.25, 0.25, 0.25)$	$S(z_1)=0.7265$, $S(z_2)=0.7763$, $S(z_3)=0.7053$, $S(z_4)=0.7543$, $S(z_5)=0.7316$	$S_2>S_4>S_5>S_1>S_3$
$p=(2, 0, 0, 0)$	$S(z_1)=0.5391$, $S(z_2)=0.6263$, $S(z_3)=0.5251$, $S(z_4)=0.6263$, $S(z_5)=0.5789$	$S_4>S_2>S_5>S_1>S_3$
$P=(3, 0, 0, 0)$	$S(z_1)=0.4469$, $S(z_2)=0.5445$, $S(z_3)=0.4299$, $S(z_4)=0.5512$, $S(z_5)=0.4937$	$S_4>S_2>S_5>S_1>S_3$

Table 11: Ranking results by different methods.

Aggregation operator	Parameter value	Ranking
Xu [41]	No	$S_2>S_4>S_5>S_1>S_3$
Muneeza and Abdullah [26]	No	$S_4>S_2>S_5>S_1>S_3$
Kaur and Garg [31]	$p=q=1$	$S_4>S_2>S_5>S_1>S_3$
ICFWMM	$p=(0.25, 0.25\ 0.25, \text{and } 0.25)$	$S_2>S_4>S_5>S_1>S_3$
ICFDWMM	$p=(0.25, 0.25, 0.25, \text{and } 0.25)$	$S_2>S_4>S_5>S_1>S_3$

5.3. Comparing with the Other Methods. To further demonstrate the effectiveness and outstanding advantages of the method developed in this study, three existing MCDM methods were used to solve the same example, including the intuitionistic fuzzy weighted average (IFWA) operator proposed by Xu [41], the intuitionistic cubic weighted average operator (ICFWA) proposed by Muneeza [26], and the intuitionistic cubic weighted fuzzy Bonferroni mean (ICFWBM) operator proposed by Kaur [31], ranking The results are shown in Table 11. The three methods are also compared with the proposed method in this study in terms of some characteristics, and the results are shown in Table 12.

Further analysis for Tables 11 and 12 leads to the following conclusions:

(1) Compared with the IFWA operator, the proposed method in this study has the following advantages: (1) it can more effectively communicate the information that is predictable, unpredictably, and satisfactorily present in the input data as well as more effectively address the input data's uncertainty and (2) the proposed operator in this study can not only consider the interrelationship between attributes of aggregated information but can also control the interrelationship between attributes more flexibly through the MM aggregation, in contrast to the IFWA operator, which does not take into account the interrelationship between attributes when aggregating information.

(2) Compared with ICFWA, the proposed aggregation operator in this study has high flexibility and generality. One the one hand, it can take into account how aggregation attributes are related to one another. On the other hand, the ICFWMM operator proposed in this study can be seen as a special case of the ICFWA operator when $p=(1, 0, ..., 0, 0)$. It is clear that the new method is more flexible and general than the ICFWA operator, particularly when the decision-maker needs to take into account on how the input attribute parameters interact with one another.

(3) Compared with ICFWBM, which can capture the interrelationship between two attribute parameters but often has to consider the interrelationship between two and more attributes in a practical decision-making environment, the proposed method in this study addresses this drawback of ICFWBM well.

Through comparative analysis, it can be seen that the proposed ICFWMM operator and ICFDWMM operator have the following advantages: (1) they can more comprehensively solve the problem of uncertainty brought by decision makers in making judgments and (2) they can not only consider the interrelationship between any multiple attributes but also deal with the interrelationship between attributes more conveniently and flexibly through the parameter vector p.

Table 12: The comparisons of different methods.

Methods	Whether captures interrelationship of two attributes	Whether captures interrelationship of multiple attributes	Whether makes the method flexible by the parameter vector
Xu ZS [41]	No	No	No
Muneeza [26]	No	No	No
Kaur G [31]	Yes	No	No
ICFWMM	Yes	Yes	Yes
ICFDWMM	Yes	Yes	Yes

6. Conclusions

In recent years, the study of aggregation operators under fuzzy sets has become a hot issue and an important tool in the field of decision making. More and more aggregation operators based on fuzzy sets are applied to solve MCDM problems because of their advantages in dealing with decision information. However, the decision information in real situations is often vague and uncertain, and most of the existing fuzzy sets use a single affiliation or interval value to express uncertainty, which is often poor in expressing the uncertainty of information. Similarly, there is often a certain correlation between the attribute values of the alternatives in realistic situations, which is not taken into account by most of the existing aggregation operators and seriously affects the accuracy of the decision scheme. So far, no researcher has given an MCDM method that can precisely consider the uncertainty of decision information and take into account the correlation between the standard attribute values. To address the above problems, in this study, we propose a series of aggregation operators on intuitionistic stereo-fuzzy sets using MM operators that consider the interrelationships between attributes, namely, the intuitionistic cubic fuzzy MM (ICFMM) operator, the intuitionistic cubic fuzzy weighted MM (ICFWMM) operator, the intuitionistic cubic fuzzy dual MM (ICFDMM) operator, and the intuitionistic cubic fuzzy dual weighted MM (ICFDWMM) operator. In addition, we give the proof procedure of the proposed operators and some ideal properties as well as the proof procedure of these properties. Finally, based on the proposed ICFWMM operator and ICFDWMM operator, we propose the two new MCDM methods, and a comparative study with the existing methods shows that the proposed new operator-based methods can represent the decision information more accurately. At the same time, the interrelationship between decision attributes can be considered more flexibly through the parameter vector p, which can provide a more stable and accurate choice for decision makers. Therefore, we can conclude that the proposed operator-based MCDM method is a comprehensive and effective method for solving decision problems.

In future studies, we should further explore the extension and application of the method proposed in this study. First, we should further explore the application of some new popular fuzzy information in MCDM, and we should propose a more realistic solution to the decision problem. For example, this study combines Q-rung Orthopair fuzzy preference information with cubic fuzzy set concept to further solve the uncertainty of MCDM problem. Second, we should further consider the incomplete probability linguistic preference information, and we should apply the Muirhead mean operator to the MCDM problem under the incomplete probability linguistic preference information. Third, further explore the influence of weight on the results of the proposed decision-making method, and consider using the current popular weight calculation methods such as analytic hierarchy process and best-worst method, for weight calculation, further give more accurate decision results.

Nomenclature

Acronyms

ICFS: Intuitionistic cubic fuzzy sets
MM: MuirHead mean
ICFMM: Intuitionistic cubic fuzzy MuirHead mean
ICFWMM: Intuitionistic cubic fuzzy weighted MuirHead mean
ICFDMM: Intuitionistic cubic fuzzy dual MuirHead mean
ICFDWMM: Intuitionistic cubic fuzzy dual weighted MuirHead mean
AIFS: Atanassov intuitionistic fuzzy sets
BM: Bonferroni mean

Notation

X: Nonempty set
I: The intuitionistic cubic fuzzy set
$c_I = \langle [e^-, e^+], \lambda \rangle$: The membership affiliation of x
$c_I^{'} = \langle [r^-, r^+], \delta \rangle$: The unaffiliatedness of x
$[e^-, e^+]$: Interval membership
$[r^-, r^+]$: Interval nonmembership degree
λ: Membership value
δ: Nonmembership value
$d(I_1, I_2)$: The distance between I_1 and I_2
$S(I)$: Score of I
$H(I)$: The accuracy function of I
$\alpha_j\,(j = 1, 2, \ldots, n)$: A collection of nonnegative real numbers

p:	A parameter vector of MM
$\theta(j)\,(j = 1, 2, \ldots, n)$:	An permutations of {1, 2, ..., n}
S_n:	The collection of all permutations of {1, 2, ..., n}.
I_i:	A collection of the ICF-numbers
$I_{\theta(j)}^{p_j}$:	An intuitionistic fuzzy set with angular permutation $\theta(j)$ and vector p_j
$\mathrm{MM}^p(\alpha_1, \alpha_2, \ldots, \alpha_n)$:	MM aggregation operator with nonempty set α_j and parameter vector p
$\mathrm{ICFMM}^p(I_1, I_2, \ldots, I_n)$:	ICFMM aggregation operator with a collection of ICF-numbers $(I_1, I_2, \ldots, I_n)$ and parameter vector p
$\mathrm{ICFWMM}^p(I_1, I_2, \ldots, I_n)$:	ICFWMM aggregation operator with a collection of ICF-numbers $(I_1, I_2, \ldots, I_n)$ and parameter vector p
$w = (w_1, w_2, \ldots, w_n)^T$:	The weight vector of $I_i\,(i = 1, 2, \ldots, n)$
$\mathrm{ICFDMM}^p(I_1, I_2, \ldots, I_n)$:	ICFDMM aggregation operator with a collection of ICF-numbers $(I_1, I_2, \ldots, I_n)$ and parameter vector p
$\mathrm{ICFDWMM}^p(I_1, I_2, \ldots, I_n)$:	ICFDWMM aggregation operator with a collection of ICF-numbers $(I_1, I_2, \ldots, I_n)$ and parameter vector p.

Data Availability

The original data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Yan Liu conceptualized the study, proposed the methodology, wrote the original draft, reviewed and edited the manuscript, was responsible for software, validated the study, performed formal analysis, investigated the study, visualized the study, supervised the study, and was responsible for project administration. Jialong He was responsible for resources, supervised the study, reviewed and edited the manuscript, was responsible for project administration, and was involved in funding acquisition. Zhaojun Yang was responsible for resources, supervised the study, reviewed and edited the manuscript, was responsible for project administration, and was involved in funding acquisition. LiJuan Yu was responsible for resources, supervised the study, reviewed and edited the manuscript, and was involved in funding acquisition. Yuan Zhong provided the software, investigated the study, and reviewed and edited the manuscript.

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