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MAPS OF MANIFOLDS WITH INDEFINITE METRICS PRESERVING CERTAIN GEOMETRICAL ENTITIES

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ABSTRACT. It is shown that (i) a diffeomorphism of manifolds with indefinite metrics preserving degenerate r-plane sections is conformal, (ii) a sectional curvature-preserving diffeomorphism of manifolds with indefinite metrics of dimension ≥ 4 is generically an isometry.

1. INTRODUCTION.

Let (M^n, g), $(\overline{M}^n, \overline{g})$ be pseudo-Riemannian manifolds. A diffeomorphism $f: M \to \overline{M}$ is said to be curvature-preserving if given $p \in M$ and a 2-dimensional plane section σ at p such that the sectional curvature $K(\sigma)$ is defined then at $f(p)$ the sectional curvature $\overline{K}(f_* \sigma)$ is defined and $K(\sigma) = \overline{K}(f_* \sigma)$. A point $p \in M$ is called isotropic if there exists a constant $c(p)$ such that $K(\sigma) = c(p)$ for any 2-plane section σ at p for which K is defined. I studied the notion of a curvature preserving map in the Riemannian case and showed

THEOREM 1. If $n \geq 4$ (M^n, g), $(\overline{M}^n, \overline{g})$ *Riemannian manifolds and non-isotropic* *points are dense in* M *then a curvature-preserving map* $f:M \to \overline{M}$ *is an isometry.*

cf. [1] and for this and other types of "Riemannian" analogues cf. [5], [6] [2], [3], [4]. The purpose of this note is to point out Theorem 2.

THEOREM 2. *Theorem 1 is valid for pseudo-Riemannian manifolds.*

Unlike certain local results in pseudo-Riemannian geometry Theorem 2 is not obtained from Theorem 1 by formal changes of signs. Its proof is actually simpler but for an entirely different reason which seems to be well worth pointing out. One of the main steps in Theorem 1 and its other analogues mentioned above is that a curvature-preserving map is necessarily conformal on the set of nonisotropic points. This step is automatic in the case of indefinite metrics due for the next result. Let us call a subspace A of a tangent space at a point in M *degenerate* (resp. nondegerate) if $g|_A$ is degenerate (resp. nondegenerate). Sectional curvature is defined only for nondegenerate 2-plane sections. So by definition a curvature-preserving map carries degenerate 2-plane sections into degenerate 2-plane sections.

THEOREM 3. Let (M^n,g), $(\overline{M}^n,\overline{g})$ *be indefinite pseudo-Riemannian manifolds,* $n \geq 3$. *Let* $r \geq 1$. *Let* $f:M \to \overline{M}$ *be a diffeomorphism which carries degenerate r-dimensional plane sections of* M *into those of* $\overline{M}$. *Then* f *is conformal*. (i.e. *there exists a nowhere vanishing smooth function* $\Phi:M \to \mathbb{R}$ *such that* $f^*\overline{g} = \Phi\cdot g$.)

Recall that a geodesic on (M,g) whose tangent vector field X satisfies $g(X,X) = 0$ is called a *light like* geodesic.

COROLLARY 1. Let (M^n,g), $(\overline{M}^n,\overline{g})$ *be indefinite pseudo-Riemannian manifolds. Then a diffeomorphism* $f:M \to \overline{M}$ *which preserves light-like geodesics is conformal.*

This is the case $r = 1$ of Theorem 3. Note that this corollary is an extension and "Geometrization" of H. Weyl's famous observation about the conformal invariance of Maxwell's equations.

2. PROOF OF THEOREMS 2 AND 3.

First we prove Theorem 3.

The case $r = 2$ contains the essential ideas so we prove the theorem only in this case leaving the general case to the reader. Let $T_p(M)$ denote the tangent space to M at p etc. It clearly suffices to show that for each p in M $f_{*_p}: T_p(M) \to T_{f(p)}(\overline{M})$ is a homothety. Let $\{e_i, e_j, e_\alpha\}$ be an orthonormal set of vectors so that

$$<e_i, e_i> = <e_j, e_j> = -<e_\alpha, e_\alpha>$$

Let $f_* e_i = \overline{e_i}$ and g or $<,>$ also denote the canonically induced metric in all tensor powers and similarly for $\overline{g}$. Let $x^2 + y^2 = 1$. Then the 2-dimensional plane $\sigma = \text{span}\ \{xe_i + ye_j + e_\alpha, -ye_i + xe_j\}$ is degenerate. Hence by hypothesis $f_*\sigma$ is degenerate i.e.

$$o = \overline{g}\,((x\,\overline{e_i} + y\,\overline{e_j} + \overline{e_\alpha}) \wedge (-y\,\overline{e_i} + x\,\overline{e_j}),\ (x\,\overline{e_i} + y\,\overline{e_j} + \overline{e_\alpha}) \wedge (-y\,\overline{e_i} + x\,\overline{e_j}))$$

$$= \overline{g}\,(\overline{e_i} \wedge \overline{e_j} + x\,\overline{e_\alpha} \wedge \overline{e_j} - y\,\overline{e_\alpha} \wedge \overline{e_i},\ \overline{e_i} \wedge \overline{e_j} + x\,\overline{e_\alpha} \wedge \overline{e_j} - y\,\overline{e_\alpha} \wedge \overline{e_i})$$

$$= \{\overline{g}(\overline{e_i} \wedge \overline{e_j}, \overline{e_i} \wedge \overline{e_j}) + x^2\overline{g}(\overline{e_\alpha} \wedge \overline{e_j}, \overline{e_\alpha} \wedge \overline{e_j}) + y^2\overline{g}(\overline{e_\alpha} \wedge \overline{e_i}, \overline{e_\alpha} \wedge \overline{e_i}) -$$

$$- 2xy\ \overline{g}(\overline{e_\alpha} \wedge \overline{e_i}, \overline{e_\alpha} \wedge \overline{e_j})\} + \{2x\ \overline{g}(\overline{e_i} \wedge \overline{e_j}, \overline{e_\alpha} \wedge \overline{e_j}) - 2y\ g(\overline{e_i} \wedge \overline{e_j}, \overline{e_\alpha} \wedge \overline{e_i})\}.$$

A similar expression with (x,y) replaced by $(-x,-y)$ is also true. Hence each $\{,\}$ is separately zero and since (x,y) are subject to the only relation $x^2 + y^2 = 1$ it follows that

$$o = \overline{g}(\overline{e_i} \wedge \overline{e_j}, \overline{e_\alpha} \wedge \overline{e_i}) = \overline{g}(\overline{e_i} \wedge \overline{e_j}, \overline{e_\alpha} \wedge \overline{e_j}) = \overline{g}(\overline{e_i} \wedge \overline{e_\alpha}, \overline{e_j} \wedge \overline{e_\alpha})$$

and

$$\overline{g}(\overline{e_i} \wedge \overline{e_j}, \overline{e_i} \wedge \overline{e_j}) = -\overline{g}(\overline{e_i} \wedge \overline{e_\alpha}, \overline{e_i} \wedge \overline{e_\alpha}) = -\overline{g}(\overline{e_j} \wedge \overline{e_\alpha}, \overline{e_j} \wedge \overline{e_\alpha})$$

i.e. $\{\overline{e}_i \wedge \overline{e}_j, \overline{e}_i \wedge \overline{e}_\alpha, \overline{e}_j \wedge \overline{e}_\alpha\}$ is an orthogonal basis of the second exterior power $\Lambda^2(\text{span } \{\overline{e}_i, \overline{e}_j, \overline{e}_\alpha\})$. This means that f induces a homothetic map of $\Lambda^2(\text{span } \{e_i, e_j, e_\alpha\})$ onto $\Lambda^2(\text{span } \{\overline{e}_i, \overline{e}_j, \overline{e}_\alpha\})$. It is then easy to see that f induces a homothety of span $\{e_i, e_j, e_\alpha\}$ onto span $\{e_i, e_j, e_\alpha\}$. By varying the set $\{e_i, e_j, e_\alpha\}$ it is clear that f_{*_p} is a homothety. This finishes the proof. QED

PROOF OF THEOREM 2. By Theorem 3 we have $f^*\bar{g} = \Phi \cdot g$ where Φ is a nowhere vanishing function on M. Now the proof that f is an isometry i.e. $\Phi = 1$ is exactly as in [1] or [4] §7. QED

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