A NOTE ON RINGS OF CONTINUOUS FUNCTIONS

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ABSTRACT. For a topological space X , and a topological ring A , let $C(X,A)$ be the ring of all continuous functions from X into A under the pointwise multiplication. We show that the theorem "there is a completely regular space Y associated with a given topological space X such that $C(Y,R)$ is isomorphic to $C(X,R)$ " may be extended to a fairly large class of topological rings, and that, in the study of algebraic structure of the ring $C(X,A)$, it is sufficient to study $C(X,R)$ if A is path connected.

For a topological space X and a topological ring A , let $C(X,A)$ denote the ring of all continuous functions from X into A under the pointwise multiplication. If A is the ring of real numbers R with the usual topology, $C(X,R)$ will simply be denoted by $C(X)$. In [2], the structure of the ring $C(X,A)$, where X is totally disconnected, is studied. The topologies for

$C(X,A)$ considered there are the topology of pointwise convergence, the compact-open topology, and the topology of uniform convergence. Under each of these topologies $C(X,A)$ is a topological ring.

In the study of rings of real-valued continuous functions on a topological space, it is usually assumed that X is completely regular. This assumption of complete regularity on X has no loss of generality as it can be seen in the following.

THEOREM 1. For every topological space X , there exists a completely regular space Y such that $C(Y)$ is (algebraically) isomorphic to $C(X)$.

The purpose of this note is to show that the above theorem may be extended to fairly large class of topological rings A , and that, in the study of algebraic structure of the ring $C(X,A)$, it is sufficient to study $C(X)$ if A is path connected. All topological spaces considered here are assumed to be Hausdorff.

The following definition is a modification of the one given in [4].

DEFINITION A pair (X,A) of a topological space X and a topological ring A is called an S -pair, if for each closed subset C of X and $x \notin C$, there exists $f \in C(X,A)$ such that $f(x) \neq 0$ and $Z(f) = \{x \mid f(x) = 0\} \supset C$, where 0 is the zero element of the ring A .

It is easy to see that if X is completely regular and A is path connected, or if X is 0-dimensional and A is any topological ring, then (X,A) is an S -pair.

REMARK If $\{(X_\alpha, A_\alpha) : \alpha \in I\}$ is a family of S -pairs, then $(\prod_{\alpha \in I} X_\alpha, \prod_{\alpha \in I} A_\alpha)$ is also an S -pair, where $\prod_{\alpha \in I} X_\alpha$ denotes the product space of the space X while $\prod_{\alpha \in I} A_\alpha$ denoted the direct product of the rings A_α .

PROOF: Let C be a closed subset of $X = \prod_{\alpha \in I} X_\alpha$, and let $x \notin C$. Then there exists some basic neighborhood of x

$$\Pi_{\alpha_1}^{-1}(U_1) \cap \Pi_{\alpha_2}^{-1}(U_2) \cap \dots \cap \Pi_{\alpha_n}^{-1}(U_n)$$

which is disjoint from C , where each U_i is open in X_{α_i} , $i = 1, 2, \dots, n$. For each i , $i = 1, 2, \dots, n$, let $f_i \in C(X_{\alpha_i}, A_{\alpha_i})$ such that $Z(f_i) \supset X_{\alpha_i} - U_i$, and $f(x_{\alpha_i}) \neq 0_{\alpha_i}$, where $x = (x_\alpha)$. Define $g: X \rightarrow A = \prod_{\alpha \in I} A_\alpha$ as follows: For $y = (y_\alpha) \in X$, let $g(y) = (t_\alpha)$ where $t_\alpha = f_i(y_{\alpha_i})$ if $\alpha = \alpha_i$ and $t_\alpha = 0_\alpha$ if $\alpha \neq \alpha_i$ for $i = 1, 2, \dots, n$. Then $g \in C(X, A)$, $Z(g) \supset C$, and $g(x) \neq 0$.

A topological space X is called a V -space, [3], if for points p, q, x , and y of X , where $p \neq q$, there exists a continuous functions f of X into itself such that $f(p) = x$ and $f(q) = y$. It is shown [3] that every completely regular path connected space and every zero-dimensional space is a V -space. It is easy to see that if (x_α, A) is an S -pair for each $\alpha \in I$, then $(\prod_{\alpha \in I} X_\alpha, A)$ is also an S -pair if the underlying space of A is a V -space. One may ask the question that if A is a topological ring such that (A, A) is an S -pair, is A a V -space? The answer to this question is negative as the following example shows.

EXAMPLE 1. Let R_1 be the ring of real numbers with the usual topology, and let R_2 be the ring of integers with the discrete topology. Then R_1 is path connected while R_2 is zero-dimensional, thus (R_1, R_1) and (R_2, R_2) are S -pairs. Hence $(R_1 \times R_2, R_1 \times R_2)$ is also an S -pair by the remark above. Since $R_1 \times R_2$ is not connected with all components homeomorphic to R_1 , it follows from Theorem 3.5 of [3] that $R_1 \times R_2$ is not a V -space.

Now let X be a topological space, and A be a topological ring. For x and y in X , define $x \equiv_A y$ if and only if $f(x) = f(y)$ for each $f \in C(X, A)$. Then " $\equiv_A$ " is an equivalence relation in X . Let Y_A be the set of all equivalence classes, and let $T: X \rightarrow Y_A$ be the natural map. For each $f \in C(X, A)$, let $f_T: Y_A \rightarrow A$ be defined by $f_T([x]) = f(x)$. Then f_T is well-

defined, and $f_T \circ T = f$ for each $f \in C(X, A)$.

$$\begin{aligned} \text{Let } C_A &= \{f_T \in A^{Y_A} \mid f \in C(X, A)\} \\ &= \{g \in A^{Y_A} \mid g \circ T \in C(X, A)\} \end{aligned}$$

and let τ_A be the weak topology on Y_A induced by the family C_A . Note that the construction of the space Y_A is analogous to that of the space Y of Theorem 1.

THEOREM 2 (1) The topological space (Y_A, τ_A) is Hausdorff.

(2) (Y_A, τ_A) is completely regular.

(3) The mapping $T: X \rightarrow (Y_A, \tau_A)$ is continuous.

(4) The mapping $\phi: g \rightarrow g \circ T$ of $C(Y_A, A)$ onto $C(X, A)$ is a continuous isomorphism, where $C(Z, A)$ is assumed to have the compact-open topology.

PROOF: (3) and (4) are clear.

To show (1), let $y_1, y_2 \in Y_A$, where $y_1 = [x_1]$, $y_2 = [x_2]$, and $y_1 \neq y_2$. Then there exists $f \in C(X, A)$ such that $f(x_1) \neq f(x_2)$. Thus $f_T(y_1) \neq f_T(y_2)$. If V_1 and V_2 are open sets in A such that $f_T(y_i) \in V_i$ for $i = 1, 2$, and $V_1 \cap V_2 = \emptyset$, then $f_T^{-1}(V_1) \cap f_T^{-1}(V_2) = \emptyset$. Hence (Y_A, τ_A) is Hausdorff.

For (2), let $x \in U = f_{T_1}^{-1}(V_1) \cap f_{T_2}^{-1}(V_2) \cap \dots \cap f_{T_n}^{-1}(V_n)$, where each V_i is open in A , and $f_{T_i} \in C_A$, $i = 1, 2, \dots, n$. Then $f_{T_i}(x) \in V_i$ for each $i = 1, 2, \dots, n$. For each i , there exists $g_i \in C(A, [0, 1])$ such that $g_i(f_{T_i}(x)) \neq 0$ and $g_i(A - V_i) = 0$. If we let $h = (g_1 \circ f_{T_1})(g_2 \circ f_{T_2}) \dots (g_n \circ f_{T_n})$, then $h \in C(Y_A)$, and $h(x) \neq 0$ but $h(y) = 0$ for $y \notin U$. Hence (Y_A, τ_A) is completely regular.

It is noted that the map T need not be a quotient map as the following example, [1], shows.

EXAMPLE 2. Let S denote the subspace of $\mathbb{R}^2$ obtained by deleting $(0, 0)$ and all points $(\frac{1}{n}, y)$ with $y \neq 0$ and $n \in \mathbb{N}$. Define $\pi(x, y) = x$ for all $(x, y) \in S$.

Let X be the quotient space of S induced by the mapping π then X can be identified as the set of real numbers endowed with the largest topology for which the mapping π is continuous. It is demonstrated in [1] that X is Hausdorff, not completely regular, Y_R is the space of real numbers, and that the mapping T is not a quotient map.

THEOREM 3 If the ring A is path connected, then

- (1) $((Y_A, \tau_A), A)$ is an S -pair
- (2) $Y_A = Y_R$
- (3) $\tau_A = \tau_R$.

PROOF: Since A is assumed to be path connected while (Y_A, τ_A) is completely regular by Theorem 2, (1) is clear.

To show (2) it is sufficient to show that $x \equiv_R y$ if and only if $x \equiv_A y$ whenever $x, y \in X$. Let $x \equiv_R y$ but suppose that $x \not\equiv_A y$. Then there exists $f \in C(X, A)$ such that $f(x) \neq f(y)$. Let $g \in C(A)$ such that $g \circ f(x) \neq g \circ f(y)$. This would imply that $x \not\equiv_R y$ since $g \circ f \in C(X)$, a contradiction. Conversely, if $x \equiv_A y$ but $x \not\equiv_R y$. Then there exists $f \in C(X)$ such that $f(x) \neq f(y)$. Then there exists $h \in C(R, A)$ such that $h(f(x)) = 0$ but $h(f(y)) = t \neq 0$. If $g = h \circ f$, then $g \in C(X, A)$ but $g(x) \neq g(y)$ which leads to a contradiction again.

Finally we shall prove (3). Since A is completely regular $C(A)$ separates points from closed sets in A , thus sets of the form $k^{-1}(V)$, where $k \in C(A)$ and V open in R , form a subbase for the topology of A . Let $f_T^{-1}(U)$ be a subbasic open set in τ_A . Then U is open in A , hence we may let $U = \bigcap_{i=1}^n k_i^{-1}(V_i)$, where for each $i = 1, 2, 3, \dots, n$, $k_i \in C(A)$ and V_i open in R . Thus $f_T^{-1}(U) = f_T^{-1}(\bigcap_{i=1}^n k_i^{-1}(V_i)) = \bigcap_{i=1}^n f_T^{-1}(k_i^{-1}(V_i)) = \bigcap_{i=1}^n (k_i \circ f_T)^{-1}(V_i)$. Since $k_i \circ f_T \in R^A$ and $Y_A = Y_R$ by (2), this implies that $\tau_A \subset \tau_R$. Conversely, let

$h^{-1}(U)$ be a subbasic open set in τ_R , where U is open in R and $h \circ T \in C(X)$. Let $y \in h^{-1}(U)$. Since (Y_R, τ_R) is completely regular, $((Y_R, \tau_R), A)$ is an S-pair, hence there exists $f \in C(Y_R, A)$ such that $f(y) \neq 0$ but $f(Y_R - h^{-1}(U)) = 0$. Then $y \in f^{-1}(A - \{0\}) \subset h^{-1}(U)$. Since $f \circ T \in C(X, A)$ and $f \in A^Y_A$, this shows that $h^{-1}(U) \in \tau_A$. Hence $\tau_R \subset \tau_A$.

The above theorem shows that, within the category of path connected topological rings, the space Y_A is independent of the ring A .

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