A REPRESENTATION THEOREM FOR OPERATORS ON A SPACE OF INTERVAL FUNCTIONS

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(Received May 4, 1978)

ABSTRACT. Suppose \( N \) is a Banach space of norm \(|\cdot|\) and \( R \) is the set of real numbers. All integrals used are of the subdivision-refinement type. The main theorem [Theorem 3] gives a representation of \( TH \) where \( H \) is a function from \( R \times R \) to \( N \) such that \( H(p^+, p^+) \), \( H(p, p^+) \), \( H(p^-, p^-) \), and \( H(p^-, p) \) each exist for each \( p \) and \( T \) is a bounded linear operator on the space of all such functions \( H \). In particular we show that

\[
TH = \left( I \right)_{a}^{b} f_{H} \, d\alpha + \sum_{i=1}^{\infty} \left[ H(x_{i-1}^{+}, x_{i-1}^{+}) - H(x_{i-1}^{+}, x_{i-1}^{+}) \right] \beta(x_{i-1}) + \sum_{i=1}^{\infty} \left[ H(x_{i}^{-}, x_{i}^{-}) - H(x_{i}^{-}, x_{i}^{-}) \right] \Theta(x_{i-1}, x_{i})
\]

where each of \( \alpha \), \( \beta \), and \( \Theta \) depend only on \( T \), \( \alpha \) is of bounded variation, \( \beta \) and \( \Theta \) are 0 except at a countable number of points, \( f_{H} \) is a function from \( R \) to \( N \) depending on \( H \), and \( \{x_{i}\}_{i=1}^{\infty} \) denotes the points \( p \) in \([a, b] \) for which

\[
[H(p, p^+)-H(p^+, p^+)] \neq 0 \text{ or } [H(p^-, p)-H(p^-, p^-)] \neq 0.
\]

We also define an interior
interval function integral and give a relationship between it and the standard interval function integral.

1. INTRODUCTION.

Let $N$ be a Banach space of norm $|\cdot|$ and $R$ the set of real numbers. The purpose of this paper is to exhibit a representation of $TH$ where $H$ is a function from $R \times R$ to $N$ such that $H(p^+, p^+)$, $H(p, p^+)$, and $H(p^-, p^-)$, and $H(p^-, p)$ each exist for each $p$ and $T$ is a bounded linear operator on the space of all such functions $H$. Functions $H$ for which each of the four preceding limits exist have been used extensively in the study of both sum integration and multiplicative integration, (for example see [2]). In particular we show that

$$
TH = (1) \int_a^b f_H \, d\alpha + \sum_{i=1}^n \left[ H(x_{i-1}, x_{i-1}^+) - H(x_{i-1}^+, x_{i-1}^+) \right] \beta(x_{i-1}) \\
+ \sum_{i=1}^n \left[ H(x_i^-, x_i^-) - H(x_i^+, x_i^-) \right] \theta(x_{i-1}, x_i),
$$

where each of $\alpha$, $\beta$, $\theta$ depend only on $T$, $\alpha$ is of bounded variation, $\beta$ and $\theta$ are 0 except at a countable number of points, $f_H$ is a function from $R$ to $N$ depending on $H$, and $(x_i)_{i=1}^\infty$ denotes the points $p$ in $[a, b]$ for which $H(p, p^+) - H(p, p^+) \neq 0$ or $[H(p^-, p^-) - H(p^-, p^-)] \neq 0$. We also define an interior interval function integral and give a relationship between it and the standard interval function integral.

2. DEFINITIONS.

If $H$ is a function from $R \times R$ to $N$, then $H(p^+, p^+) = \lim_{x,y \to p} H(x, y)$ and similar meanings are given to $H(p, p^+)$, $H(p^-, p^-)$, and $H(p^-, p)$. The set of all functions for which each of the preceding four limits exist will be denoted by $O^0$. If $H$ is a function from $R \times R$ to $N$ then $H$ is said to be (1) of bounded variation on the interval $[a, b]$ and (2) bounded on $[a, b]$ if there exists a number $M$ and a subdivision $D$ of $[a, b]$ such that if $D' = \{x_i\}_{i=0}^N$ is a refinement of $D$ then
(1) \[ \sum_{i=1}^{n} |H(x_{i-1}, x_i)| < M \] and (2) if \( 0 < i \leq n \), then \( |H(x_{i-1}, x_i)| < M \), respectively.

Further, \( H \) is said to be integrable on \([a,b]\) if there is a number \( A \) such that for each \( \varepsilon > 0 \) there is a subdivision \( D \) of \([a,b]\) such that if \( D' = \{x_i\}_{i=0}^{n} \) is a refinement of \( D \), then
\[
\left| \sum_{i=1}^{n} H(x_{i-1}, x_i) - A \right| < \varepsilon
\]
and \( A \) is denoted by \( \int_{a}^{b} H \) when such an \( A \) exists. In our development we will also find a slight modification of the preceding definition useful. If \( H(x_{i-1}, x_i) \) is replaced by \( H(r_i, s_i)G(x_{i-1}, x_i) \), \( x_{i-1} < r_i < s_i < x_i \), in the approximating sum of the preceding definition then the number \( A \) is denoted by \( (I_H)_{a}^{b} HG \) and termed the interior integral of \( H \) with respect to \( G \) on \([a,b]\). Also, if each of \( f \) and \( \alpha \) is a function from \( R \) to \( N \), then the interior integral of \( f \) with respect to \( \alpha \) exists means there is a number \( A \) such that if \( \varepsilon > 0 \) then there is a subdivision \( D \) of \([a,b]\) such that if \( D' = \{x_i\}_{i=0}^{n} \) is a refinement of \( D \) and for \( 0 < i \leq n \), \( x_{i-1} < t_i < x_i \),
\[
\left| \sum_{i=1}^{n} f(t_i)[\alpha(x_i) - \alpha(x_{i-1})] - A \right| < \varepsilon
\]
and \( A \) is denoted by \( (\int_{a}^{b} f)_{\alpha} \).

If \( \alpha \) is a function from \( R \) to \( N \), \( \alpha(p^+) = \lim_{x \to p^+} \alpha(x) \), \( \alpha(p^-) = \lim_{x \to p^-} \alpha(x) \), and \( d \alpha \) denotes the function \( H \) from \( RxR \) to \( N \) such that for \( x < y \), \( H(x,y) = \alpha(y) - \alpha(x) \).

If each of \( H, H_1, H_2, \ldots \) is a function from \( RxR \) to \( N \), then \( \lim_{n \to \infty} H_n = H \) uniformly on \([a,b]\) means if \( \varepsilon > 0 \) there is a positive integer \( N \) and a subdivision
\[ D = \{x_i\}_{i=0}^{n} \] of \([a,b]\) such that if \( n > N \) and \( x_{i-1} < r < s < x_i \) for some \( 0 < i \leq n \), then
\[
|H(r,s) - H_n(r,s)| < \varepsilon.
\]
If \( H \) is a function from \( RxR \) to \( N \), then \( H \) is bounded on \([a,b]\) means there is a number \( M \) and a subdivision \( D = \{x_i\}_{i=0}^{\infty} \) of \([a,b]\) such that if \( 0 < i \leq n \) and \( x_{i-1} < r < s < x_i \), then \( |H(r,s)| < M \). The norm of \( H \) on \([a,b]\) with respect to \( D \), \( \|H\|_D \) is then defined as the greatest lower bound of the set of all such \( M \)'s.
T is a linear operator on OL$^0$ means $T$ is a transformation from OL$^0$ to N such that if each of $H_1$ and $H_2$ are in OL$^0$ then

$$T[k_1H_1 + k_2H_2] = k_1TH_1 + k_2TH_2$$

for $k_1$, $k_2$ in R. $T$ is bounded on $[a,b]$ means there is a number $M$ such that

$$|TH| \leq M||H||_D$$

for some subdivision $D$ of $[a,b]$.

For convenience we adopt the following conventions for a function from RxR to N and R to N for some subdivision $D = \{x_i\}_{i=0}^n$ of $[a,b]$:

1. $H(a^-,a) = H(a^-,a^+) = H(b,b^+) = H(b^+,b^+) = 0$,
2. $H(x_{i-1},x_i) = H_i$, $0 < i < n$,
3. $\alpha(x_i) - \alpha(x_{i-1}) = \Delta \alpha_i$,
4. $\sum_{i=1}^n H(x_{i-1},x_i) = \sum_D H_1$.

3. THEOREMS.

We will begin by establishing a relationship between $\int_a^b H \, d\alpha$ and $(I_H)_{a}^{b} H \, d\alpha$ which will require the following lemmas.

**LEMMA 1.** If $H$ is in OL$^0$ and $\alpha$ is a function from R to N of bounded variation on $[a,b]$, then $\int_a^b H \, d\alpha$ exists.

This lemma is a special case of THEOREM 2 of [2].

**LEMMA 2.** Suppose $H$ is in OL$^0$, $[a,b]$ is an interval, $\epsilon > 0$, and $S_1$ and $S_2$ are sets such that $p$ is in $S_1$ if and only if $p$ is in $[a,b]$ and $|H(p,p^+)-H(p^+,p^+)| \geq \epsilon$ and $p$ is in $S_2$ if and only if $p$ is in $[a,b]$ and $|H(p^-,p)-H(p^-,p^-)| \geq \epsilon$. Then, each of $S_1$ and $S_2$ is a finite set. [2, lemma page 498].

We note that it follows from LEMMA 2 that if $S$ is the set such that $p$ is in $S$ if and only if $H(p,p^+)-H(p^+,p^+) \neq 0$ or $H(p^-,p)-H(p^-,p^-) \neq 0$ then $S$ is countable.

**LEMMA 3.** If $H$ is in OL$^0$ and $\alpha$ is a function from R to N of bounded variation on $[a,b]$ then (1) if $p$ is in $[a,b]$ each of $\alpha(p^+)$ and $\alpha(p^-)$ exists and (2) if $\{x_i\}_{i=1}^\infty$ is a sequence of numbers such that if $p$ is in $[a,b]$ and $H(p,p^+)-H(p^+,p^+) \neq 0$
or \(H(p^-,p^+)--H(p^-,p^-) \neq 0\), then there is an \(n\) such that \(p=x_n\), then

\[
(1) \sum_{i=1}^{\infty} \left[ H(x_i^+,x_i^+)-H(x_i^+,x_i^+) \right] \left[ \alpha(x_i^+)-\alpha(x_i^+) \right] \text{ exists}
\]

and \(2) \sum_{i=1}^{\infty} \left[ H(x_i^-,x_i^-)-H(x_i^-,x_i^-) \right] \left[ \alpha(x_i^-)-\alpha(x_i^-) \right] \text{ exists}.
\]

**INDICATION OF PROOF.** It follows from the bounded variation of \(\alpha\) that for \(p\) in \([a,b]\) each of \(\alpha(p^+)\) and \(\alpha(p^-)\) exists.

Since \(H\) is in \(OL^0\), it follows from the covering theorem that \(H\) is bounded on \([a,b]\) and that there is a number \(M_1\) such that for each positive integer \(i\),

\[
|H(x_i^+,x_i^+)-H(x_i^+,x_i^+)\left| < M_1,
\]

and, furthermore, for \(n\) a positive integer and \(0 < i \leq n\), let \(x_{pi} > x_i\) such that

\[
\sum_{i=1}^{n} \left| \alpha(x_i^+)-\alpha(x_{pi}) \right| < 1.
\]

Hence,

\[
\sum_{i=1}^{n} \left| \left[ H(x_i^+,x_i^+)-H(x_i^+,x_i^+) \right] \left[ \alpha(x_i^+)-\alpha(x_i) \right] \right| \leq M_1 \left( \sum_{i=1}^{n} \left| \alpha(x_i^+)-\alpha(x_{pi}) \right| + \sum_{i=1}^{n} \left| \alpha(x_{pi})-\alpha(x_i) \right| \right)
\]

\[
< M_1 \left( 1 + \sum_{D} \left| \alpha(x_i)-\alpha(x_{i-1}) \right| \right),
\]

where \(D\) is a subdivision of \([a,b]\) containing \(x_i\) and \(x_{pi}\) as consecutive points in \(D\) for each \(0 < i \leq n\). Hence, since \(\alpha\) is of bounded variation there is a number \(M\) such that

\[
\sum_{i=1}^{n} \left| \left[ H(x_i^+,x_i^+)-H(x_i^+,x_i^+) \right] \left[ \alpha(x_i^+)-\alpha(x_i) \right] \right| < M.
\]

Therefore,

\[
\sum_{i=1}^{n} \left[ H(x_i^+,x_i^+)-H(x_i^+,x_i^+) \right] \left[ \alpha(x_i^+)-\alpha(x_i) \right] \text{ exists. In a similar manner it may be shown that}
\]

\[
\sum_{i=1}^{n} \left[ H(x_i^+,x_i^+)-H(x_i^+,x_i^+) \right] \left[ \alpha(x_i^-)-\alpha(x_i^-) \right] \text{ exists.}
\]

**THEOREM 1.** If \(H\) is in \(OL^0\) and \(\alpha\) is a function from \(R\) to \(N\) of bounded variation on \([a,b]\), then \((I_H)_{a}^{b} H \alpha\) exists.
PROOF. If $\epsilon > 0$ then it follows from LEMMA 2 that each of the sets $A^+_{\epsilon}$ and $A^-_{\epsilon}$ to which $p$ belongs if and only if $p$ is in $[a, b]$ and $|H(p, p^+)-H(p^+, p^+)| > \epsilon$ or $|H(p^-, p)-H(p^-, p^-)| > \epsilon$, respectively, is a finite set. Let $A^+_0 = \{ c_i \}_{i=1}^{m_1}$, $A^-_0 = \{ d_i \}_{i=1}^{m_2}$, and $A^+$ and $A^-$ denote the sets to which $p$ belongs if and only if $p$ is in $[a, b]$ and $H(p, p^+)-H(p^+, p^+) < 0$ or $H(p^-, p)-H(p^-, p^-) > 0$, respectively. Since each of $A^+$ and $A^-$ is a countable set then let $A^+ + A^- = \{ y_i \}_{i=1}^{+\infty}$.

Since $\alpha$ is of bounded variation on $[a, b]$, then for each $c_i$, $0 < i < m_1$ and $d_i$, $0 < i < m_2$ there is an $e_i$ and an $f_i$ such that if $e_i < r_i < c_i$ and $d_i < s_i < d_i$ then $|\alpha(c_i) - \alpha(r_i)| < \frac{\epsilon}{16m_1}$ and $|\alpha(d_i) - \alpha(s_i)| < \frac{\epsilon}{16m_2}$.

From LEMMA 3, it follows that there is a positive integer $N$ such that if $n > N$, then

(1) $\sum_{i=1}^{N} \left[ H(y_i, y_i^+) - H(y_i^+, y_i^+) \right] \left[ \alpha(y_i^+) - \alpha(y_i) \right] - \sum_{i=1}^{\infty} \left[ H(y_i, y_i^+) - H(y_i^+, y_i^+) \right] \left[ \alpha(y_i^+) - \alpha(y_i) \right] < \frac{\epsilon}{8}$

and

(2) $\sum_{i=1}^{N} \left[ H(y_i, y_i^-) - H(y_i^-, y_i^-) \right] \left[ \alpha(y_i^-) - \alpha(y_i) \right] - \sum_{i=1}^{\infty} \left[ H(y_i, y_i^-) - H(y_i^-, y_i^-) \right] \left[ \alpha(y_i^-) - \alpha(y_i) \right] < \frac{\epsilon}{8}$.

Note that for some $y_i$'s, $\left[ H(y_i, y_i) - H(y_i^+, y_i^-) \right]$ may be zero.

Since, from LEMMA 1, $f^b_a Hda$ exists, then there is a number $M$ and a subdivision $D_1$ of $[a, b]$ such that if $D' = \{ x_i \}_{i=0}^{n}$ is a refinement of $D_1$, then

(3) $\Sigma_{D'} |\Delta \alpha_x| < M$,

(4) $\left| \int_a^b Hda - \Sigma_{D'} H \Delta \alpha_x \right| < \frac{\epsilon}{4}$,

and (5) if $0 < i \leq n$, then $|H(x_{i-1}, x_{i-1}^+) - H(x_{i-1}^+, x_{i-1}^+)| < M$ and $|H(x_{i-1}^-, x_{i-1}^-) - H(x_i^-, x_i^-)| < M$.

Further, since $H$ is in $OL^0$, using the covering theorem we may obtain a subdivision $D_2$ of $[a, b]$ such that if $D' = \{ x_i \}_{i=0}^{n}$ is a refinement of $D_2$, $0 < i < n$, and
Let \( D = D_1 + D_2 + A^+ + A^- + \sum_{i=1}^{m_1} \{ e_i \} + \sum_{i=1}^{m_2} \{ f_i \} + \sum_{i=1}^{N} \{ y_i \} \), \( D' = \{ x_i \}^n \) be a refinement of \( D \), and for each \( 0 < i \leq n \), \( x_{i-1} < r_{i-1} < s_j < x_i \). Choose \( m > N \) such that for each \( x_i \), \( 0 < i \leq n \), in \( D' \cdot (A^+ + A^-) \) there exists a positive integer \( z \) such that

Let \( y_z = x_i \). Hence, for \( x_i \), \( 0 < i \leq n \), in \( D' \) such that neither \( x_{i-1} \) nor \( x_i \) is in \( (A^+ + A^-) \), it follows from (6)-(9) that \( |H(x_{i-1}, x_i)| < \frac{\epsilon}{32M} \).

If \( W_i = H(y_{i-1}, y_i) - H(x_{i-1}, x_i) \) and \( Q = \{ y_1, y_2, \ldots, y_m \} \) for \( 0 < i \leq m \) then

\[
\sum_{i=1}^{m} W_i [\alpha(y_{i-1}) - \alpha(y_i)] = H(x_{i-1}, x_i) - H(y_{i-1}, y_i) \Delta \alpha_i
\]

\[
\sum_{Q \cdot A^+} [\alpha(y_{i-1}) - \alpha(y_i)]
\]

\[
\sum_{D' \cdot A^+} [H(x_{i-1}, x_i) - H(y_{i-1}, y_i)] \Delta \alpha_i
\]

\[
\sum_{D' \cdot (A^+ - A^-)} [H(x_{i-1}, x_i) - H(y_{i-1}, y_i)] \Delta \alpha_i
\]

\[
|\sum_{A^+} W_i | + |\alpha(y_{i-1}) - \alpha(y_i)| + \sum_{Q \cdot A^+} |W_1| \cdot |\alpha(y_{i-1}) - \alpha(y_i)|
\]

\[
< M \sum_{A^+} \frac{\epsilon}{16M} \sum_{D' \cdot (A^+ - A^-)} |\alpha(y_{i-1}) - \alpha(y_i)| + \frac{\epsilon}{16M} \sum_{D' \cdot (A^+ - A^-)} |\Delta \alpha_i|
\]

\[
< \frac{3\epsilon}{16}
\]

Hence
and in a similar manner it may be shown that

$$\sum_{i=1}^{m} W_i \left[ \alpha(y_{i_1}) - \alpha(y_{i_1}) \right] - \sum_{i=1}^{m} Z_i \left[ H(x_{i_1}^{-}, x_{i_1}^{-}) - H(x_{i_1}^{-}, x_{i_1}^{-}) \right] \Delta \alpha_i < \frac{3\varepsilon}{16},$$

where $Z_i = H(y_{i_1}^{-}, y_{i_1}^{-}) - H(y_{i_1}^{-}, y_{i_1}^{-}).$

Using inequalities (10) and (11) we are able to complete the proof of the theorem. In the following manipulations $W_i$ and $Z_i$ are as defined for (10) and (11) and $P_i = H(x_{i_1}^{-}, x_{i_1}^{-}) - H(x_{i_1}^{-}, x_{i_1}^{-})$ and $Q_i = H(x_{i_1}^{-}, x_{i_1}^{-}) - H(x_{i_1}^{-}, x_{i_1}^{-}).$

Hence, we have a relationship established between $(I_H)_{a}^{b} \alpha d\alpha$ and $\int_{a}^{b} \alpha d\alpha$ which will be used in the proof of the principal theorem.

**THEOREM 2.** If $\{H_i\}_{i=0}^{\infty}$ is a sequence of functions from $S \times S$ to $N$, such that for each $i$, $H_i$ is in $OLO$, $\lim_{n \to \infty} H_n = H_0$ uniformly on $[a, b]$, and $T$ is a bounded linear operator on $OLO$ then $\lim_{n \to \infty} TH_n = TH_0$.

The proof of this theorem is straightforward and we omit it.

**THEOREM 3.** Suppose $H$ is in $OLO$, $T$ is a bounded linear operator on $OLO$.

Then,

$$TH = (I)_{a}^{b} f \alpha d\alpha \sum_{i=1}^{\infty} \left[ H(x_{i_1}^{-}, x_{i_1}^{-}) - H(x_{i_1}^{-}, x_{i_1}^{-}) \right] \beta(x_{i_1})$$

$$+ \sum_{i=1}^{\infty} \left[ H(x_{i_1}^{-}, x_{i_1}^{-}) - H(x_{i_1}^{-}, x_{i_1}^{-}) \right] \delta(x_{i_1}).$$
where each of $\alpha$, $\beta$, and $\Theta$ depend only on $T$, $\alpha$ is of bounded variation, $\beta$ and $\Theta$ are 0 except at a countable number of points, $f_H$ is a function from $\mathbb{R}$ to $\mathbb{R}$ depending on $H$, and $\{x_i\}_{i=1}^{\infty}$ denote the points in $[a, b]$ for which

$$[H(x_i, x_i^+) - H(x_i^-, x_i^+)] \neq 0 \text{ or } H(x_i, x_i^+) - H(x_i^-, x_i^-) \neq 0, \quad i=1, 2, \ldots, n.$$  

PROOF. We first define a sequence of functions converging uniformly to a given function $H$ in $OL^0$ and then apply THEOREM 2 to establish THEOREM 3. We first define functions $g$ and $h$ for each pair of numbers $t, x, a < t < b, a < x < b$ such that

$$g(t,x) = \begin{cases} 1, & \text{if } t=x \\ 0, & \text{if } t \neq x \end{cases} \quad \text{and} \quad h(t,x) = \begin{cases} 1, & \text{if } a \leq t \leq x \\ 0, & \text{if } x < t < b, \end{cases}$$

and using these functions and the operator $T$ define functions $\alpha, \beta, \gamma,$ and $\Theta$ such that

$$\alpha(x) = TH(\cdot, x); \quad \beta(x) = Tg(\cdot, x); \quad \gamma(x) = Tg(x, \cdot); \quad \Theta(x, y) = Tg(\cdot, x)g(y, \cdot);$$

and $\Theta(x, y) = \gamma(y) - \Theta(x, y)$ for $x$ and $y$ in $[a, b]$. Clearly, $\alpha$ is of bounded variation on $[a, b]$ and we see from

$$\sum_{D'} |\Theta(x_{i-1}, x_i)| = \sum_{D'} \Theta^2 = \sum_{D'} \text{sgn}(x_{i-1}, x_i)g(x_i, \cdot)$$

$$\leq M \left| \sum_{D'} \text{sgn}(x_{i-1}, x_i)g(x_i, \cdot) \right| \leq M,$$

for $D'$ a refinement of a subdivision $D$ of $[a, b]$, it follows that $\sum_{i=1}^{\infty} |\Theta(x_{i-1}, x_i)|$ exists and in a similar manner that each of $\sum_{i=1}^{\infty} |\beta(x_i)|$ and $\sum_{i=1}^{\infty} |(x_i)|$ exists. Hence, $\sum_{i=1}^{\infty} |\Theta(x_{i-1}, x_i)|$ exists.

Each of our approximating functions $H_n$ will be defined in terms of a subdivision $D_n$ of $[a, b]$ determined in the following manner.
Since $\alpha$ is of bounded variation on $[a,b]$ and $H$ is in $OL^0$ then from THEOREM 1, $\int_a^b Hd\alpha$ exists and there is a subdivision $K_n$ of $[a,b]$ such that if $K' = \{x_i\}_{i=1}^m$ is a refinement of $K_n$, then $\left| (I_{K'}^b Hd\alpha - \sum_{i=1}^m H(x_i, s_i) \Delta x_i) \right| < \frac{1}{n}$ where for $0 < i < m$,

$x_{i-1} < r_i < s_i < x_i$. It follows from the covering theorem and the existence of the limits $H(p, p^+)$ and $H(p^+, p^+)$ that there is a subdivision $I_n = \{x_i\}_{i=0}^m$ of $[a,b]$ such that if $x_{i-1} < x < r < s < y < x_i$, $0 < i < m$, then $|H(x, y) - H(r, s)| < \frac{1}{n}$.

Further, let $J_n$ denote the set such that $p$ is in $J_n$ if $p$ is in $[a,b]$ and $|H(p, p^+) - H(p^+, p^+)| > \frac{1}{n}$ or $|H(p^-, p) - H(p^-, p^-)| > \frac{1}{n}$ and $D_n = K_n + J_n + I_n$. For each positive integer $n$, let $H_n$ be a function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{N}$ determined by

$$D_n = \{x_i\}_{i=1}^m$$

in the following manner:

$$H_n(x, y) = \sum_{i=1}^m H(r_i, s_i) \left[ h(x, x_i) - h(x, x_{i-1}) + \sum_{i=1}^m [H(x_i, x_{i-1}) - H(r_i, s_i)] g(x_i, y) \right]$$

$$+ \sum_{i=1}^m [H(x_i^+, x_i) - H(r_i, s_i)] g(x, x_i)$$

$$- \sum_{i=1}^m [H(x_i^-, x_i) - H(r_i, s_i)] g(x_i, y)$$

for each $(x, y)$ such that $x_{i-1} < x < y < x_i$, for some $0 < i < m$, and for each $[x_{i-1}, x_i]$, $0 < i < m$, $x_{i-1} < r_i < s_i < x_i$.

It is evident that $\lim_{n \to \infty} H_n = H$ uniformly on $[a,b]$ for if $\varepsilon > 0$, $\frac{1}{n} < \varepsilon$,

$$D = D_n = \{x_i\}_{i=1}^m$$

and $x_{p-1} < x < y < x_p$ for some $0 < p < m$, then

$$H_n(x_{p-1}, x_p) = H(x_{p-1}, x_p), H_n(x, x_p) = H(x, x_p), H_n(x_{p-1}, y) = H(x_{p-1}, y),$$

$$H_n(x, y) = H(r, s).$$

Hence $\lim_{n \to \infty} H_n = H$ uniformly on $[a,b]$.

Since $\lim_{n \to \infty} H_n = H$ uniformly on $[a,b]$, applying THEOREM 2, we have
\[ \text{TH} = \lim_{n \to \infty} \text{TH}_n \]
\[ = \lim_{n \to \infty} \sum_{D_n} H(r_1, s_1) \left[ \text{TH}(\cdot, x_1) - \text{TH}(\cdot, x_{1-1}) \right] \]
\[ + \lim_{n \to \infty} \sum_{D_n} \left[ H(x_{1-1}^+, x_{1-1}) - H(r_1, s_1) \right] Tg(\cdot, x_{1-1}) \]
\[ + \lim_{n \to \infty} \sum_{D_n} \left[ H(x_1^-, x_1) - H(r_1, s_1) \right] Tg(x_1, \cdot) \]
\[ + \lim_{n \to \infty} \left[ -H(x_1^-, x_1) + H(x_1, s_1) \right] Tg(\cdot, x_{1-1}) e(x_{1-1}, \cdot) \]
\[ = \left( \int_a^b \text{H} d\alpha \right) + \sum_{i=1}^{\infty} \left[ H(x_{1-1}^+, x_{1-1}^+) - H(x_{1-1}^+, x_{1-1}^-) \right] \beta(x_{1-1}) \]
\[ + \sum_{i=1}^{\infty} \left[ H(x_1^-, x_1^+) - H(x_1^-, x_1^-) \right] \gamma(x_1) \]
\[ + \sum_{i=1}^{\infty} \left[ H(x_1^-, x_1) - H(x_1^-, x_1) \right] \theta(x_{1-1}, x_1) \]
\[ + \sum_{i=1}^{\infty} H(x_1^-, x_1^-) \Theta(x_{1-1}, x_1) \]

where the existence of each of the infinite sums is assured by LEMMA 3 and the equality of the last two expressions follows from the definition of \( D_n \).

All that remains to complete the proof of THEOREM 3 is to show that \( \int_a^b \text{H} d\alpha \) may be represented by \( \left( \int_a^b f_H d\alpha \right) \) where \( f_H \) is a function from \( \mathbb{R} \) to \( \mathbb{N} \). If we let \( f_H \) be the function such that for each \( p \) in \( [a, b] \) \( f_H(p) = H(p^+, p^+) \) then it follows that \( \left( \int_a^b f_H d\alpha \right) \) exists and is \( \left( \int_a^b \text{H} d\alpha \right) \).
REFERENCES


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