THE SPACE OF ENTIRE FUNCTIONS OF TWO VARIABLES AS A METRIC SPACE

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Section 1. Introduction.

Let Γ^2 denote the space of entire functions of two variables.

If $f(z,w) \in \Gamma^2$, $f(z,w) = \sum_{m,n=0}^{\infty} a_{m,n} z^m w^n$, the series converging

absolutely for all (z,w) and uniformly in every

bicylinder centered at $(0,0)$, [2]. Here, a metric is defined on

Γ^2 and three classes of linear functionals on Γ^2 are characterized.

We use the following notation.

$$(1) \quad \langle a_{m,n} \rangle_{m+n=k}^{\infty} \equiv \langle a_{k,0}, a_{k-1,1}, \dots, a_{0,k}, a_{k+1,0}, \dots \rangle.$$

$$(2) \quad \sum_{m+n=0}^{\infty} a_{m,n} \equiv a_{0,0} + a_{1,0} + a_{0,1} + a_{2,0} + a_{1,1} + \dots$$

$$\equiv \lim_{N \rightarrow \infty} \sum_{m+n=0}^N a_{m,n}.$$

Definition 1.1. The sequence $\langle a_{m,n} \rangle_{m+n=k}^{\infty}$ is said to have limit a as $m+n \rightarrow \infty$, written $\lim_{m+n \rightarrow \infty} a_{m,n} = a$, if and only if for any

$\epsilon > 0$, there exists an $N = N(\epsilon) \geq 0$ such that $|a_{m,n} - a| < \epsilon$ if $m+n > N$.

Lemma 1.2. If $f(z,w) = \sum_{m,n=0}^{\infty} a_{m,n} z^m w^n \in \Gamma^2$, then for each (z,w) , the sequence $\langle a_{m,n} z^m w^n \rangle_{m+n=0}^{\infty}$ is such that $\lim_{m+n \rightarrow \infty} a_{m,n} z^m w^n = 0$.

Proof. Given (z,w) , let $S_N = \sum_{j+k=0}^N |a_{j,k} z^j w^k|$. Since $\lim_{N \rightarrow \infty} S_N$ exists, $0 = \lim_{N \rightarrow \infty} S_N - \lim_{N \rightarrow \infty} S_{N-1} = \lim_{N \rightarrow \infty} (S_N - S_{N-1}) = \lim_{N \rightarrow \infty} \sum_{j+k=N} |a_{j,k} z^j w^k|$.

Hence given $\epsilon > 0$, there exists an $M = M(\epsilon)$ such that if $N > M$,

$|a_{N-j,j} z^{N-j} w^j| < \epsilon$ for each j , $0 \leq j \leq N$. Let $m+n = N$. Then

$0 \leq n \leq N$ and $|a_{m,n} z^m w^n| < \epsilon$. Therefore given $\epsilon > 0$, there exists

an $M = M(\epsilon)$ such that $m+n > M \Rightarrow |a_{m,n} z^m w^n| < \epsilon$. Hence

$$\lim_{m+n \rightarrow \infty} a_{m,n} z^m w^n = 0.$$

Lemma 1.3. A necessary and sufficient condition that $\sum_{m,n=0}^{\infty} a_{m,n} z^m w^n$

is an entire function is that for the sequence $\langle |a_{m,n}|^{1/m+n} \rangle_{m+n=1}^{\infty}$, one has $\lim_{m+n \rightarrow \infty} |a_{m,n}|^{1/m+n} = 0$.

Proof. Let $\sum_{m,n=0}^{\infty} a_{m,n} z^m w^n \in \Gamma^2$ and $T = \overline{\lim}_{m+n \rightarrow \infty} |a_{m,n}|^{1/m+n}$.

If $T > 0$, choose (z, w) such that $|z| \geq |w| > 1/T$. ($1/T = 0$ if $T = \infty$). Then choose p such that $|w| > p > 1/T$. Then

$\left| \frac{z}{w} \right| \leq \left| \frac{1}{w} \right| < \frac{1}{p} < T$. By definition of T , there exists a sequence

$\langle (m_k, n_k) \rangle_{k=1}^{\infty}$ such that $\langle m_k + n_k \rangle_{k=1}^{\infty}$ increases monotonically to

∞ and $|a_{m_k, n_k}|^{1/(m_k + n_k)} > 1/p$ for all k . Hence $|a_{m_k, n_k} z^{m_k} w^{n_k}| >$

$\left(\frac{z}{p} \right)^{m_k} \left(\frac{w}{p} \right)^{n_k} > 1$. This contradicts Lemma 1.2. Therefore $T = 0$.

Hence $\lim_{m+n \rightarrow \infty} |a_{m,n}|^{1/m+n} = 0$.

Conversely, let $\sum_{m,n=0}^{\infty} a_{m,n} z^m w^n$ be a series such that for the

sequence $\langle |a_{m,n}|^{1/m+n} \rangle_{m+n=1}^{\infty}$, one has $\lim_{m+n \rightarrow \infty} |a_{m,n}|^{1/m+n} = 0$. To

show this series is an entire function, it suffices to show

[2] the series converges for each (z, w) . Consider (z, w) fixed.

Choose p such that $|z| < p$ and $|w| < p$. Let N be such that $m+n$

$> N \Rightarrow |a_{m,n}|^{1/m+n} < 1/p$. Then for $m+n > N$, $|a_{m,n} z^m w^n| <$

$\left(\frac{|z|}{p} \right)^m \left(\frac{|w|}{p} \right)^n$, $\sum_{m+n=N+1}^{\infty} |a_{m,n} z^m w^n| \leq \sum_{m+n=N+1}^{\infty} \left(\frac{|z|}{p} \right)^m \left(\frac{|w|}{p} \right)^n < \infty$.

Therefore $\sum_{m+n=0}^{\infty} |a_{m,n} z^m w^n| < \infty$.

Let $s_{p,q} = \sum_{m=0}^p \sum_{n=0}^q a_{m,n} z^m w^n$. To show the series converges,

it suffices to show [1] that given $\epsilon > 0$, there exists an

$N = N(\epsilon)$ such that $|s_{p,q} - s_{m,n}| < \epsilon$ if $p > m > N$ and $q > n > N$.

Since $\sum_{m+n=0}^{\infty} |a_{m,n} z^m w^n| < \infty$, given $\epsilon > 0$, there exists an $M = M(\epsilon)$

such that $N > \max\{M, 1\} \Rightarrow \sum_{j+k=N+1}^{\infty} a_{j,k} z^j w^k < \epsilon$. Choose such an N .

Then $N = N(\epsilon)$. For $p > m > N$ and $q > n > N$, $|s_{p,q} - s_{m,n}|$

$$= \left| \sum_{j=0}^p \sum_{k=0}^q a_{j,k} z^j w^k - \sum_{j=0}^m \sum_{k=0}^n a_{j,k} z^j w^k \right| \leq \sum_{j+k=m+n}^{\infty} \left| a_{j,k} z^j w^k \right|$$

$$\leq \sum_{j+k=N+1}^{\infty} |a_{j,k} z^j w^k| < \epsilon.$$

Definition 1.4. Given $f(z,w) = \sum_{m,n=0}^{\infty} a_{m,n} z^m w^n \in \Gamma^2$ and

$g(z,w) = \sum_{m,n=0}^{\infty} b_{m,n} z^m w^n \in \Gamma^2$, define $d(f,g) =$

$\sup\{|a_{0,0} - b_{0,0}|, |a_{m,n} - b_{m,n}|^{1/m+n} : m+n \geq 1\}$.

Theorem 1.5. The space (Γ^2, d) is a metric space.

Proof. Given f, g as in Definition 1.4, the set

$\{|a_{m,n} - b_{m,n}|^{1/m+n} : m+n \geq 1\}$ is a bounded set by Lemma 1.3, so d is

well defined. It is clear that $d(f,g) = 0$ if and only if $f = g$

and that $d(f,g) = d(g,f)$. Let $h(z,w) = \sum_{m,n=0}^{\infty} c_{m,n} z^m w^n \in \Gamma^2$.

Then $d(f,h) = \sup\{|a_{0,0} - c_{0,0}|, |a_{m,n} - c_{m,n}|^{1/m+n} : m+n \geq 1\} =$

$\sup\{|(a_{0,0} - b_{0,0}) + (b_{0,0} - c_{0,0})|, |(a_{m,n} - b_{m,n}) + (b_{m,n} - c_{m,n})|^{1/m+n} : m+n \geq 1\}$

$\leq \sup\{|a_{0,0} - b_{0,0}| + |b_{0,0} - c_{0,0}|, |a_{m,n} - b_{m,n}|^{1/m+n} + |b_{m,n} - c_{m,n}|^{1/m+n} :$

$m+n \geq 1\} \leq \sup\{|a_{0,0} - b_{0,0}|, |a_{m,n} - b_{m,n}|^{1/m+n} : m+n \geq 1\} + \sup$

$\{|b_{0,0}^{-c_{0,0}}|, |b_{m,n}^{-c_{m,n}}|^{1/m+n} : m+n \geq 1\} = d(f,g) + d(g,h)$. Hence d is a metric on Γ^2 .

Section 2. The class of continuous linear functionals on Γ^2 .

Definition 2.1. A function F from Γ^2 to $\mathcal{C}$ (complex plane) is a linear functional if and only if for all $f, g \in \Gamma^2, \alpha \in \mathcal{C}$, $F(f+g) = F(f)+F(g)$ and $F(\alpha f) = \alpha F(f)$.

Definition 2.2. A function F from Γ^2 to $\mathcal{C}$ is said to be continuous at $f \in \Gamma^2$ if and only if for any $\epsilon > 0$ there exists a $\delta > 0$ such that if $g \in \Gamma^2$ and $d(f,g) < \delta$, then $|F(f)-F(g)| < \epsilon$.

Definition 2.3. A function F from Γ^2 to $\mathcal{C}$ is said to be continuous if and only if it is continuous at each $f \in \Gamma^2$.

Lemma 2.4 The series $\sum_{m+n=0}^{\infty} a_{m,n} b_{m,n}$ converges for all sequences

$\langle a_{m,n} \rangle_{m+n=0}^{\infty}$ such that $\lim_{m+n \rightarrow \infty} |a_{m,n}|^{1/m+n} = 0$ if and only if

$\langle |b_{m,n}|^{1/m+n} \rangle_{m+n=1}^{\infty}$ is a bounded sequence.

Proof. Let $\langle |b_{m,n}|^{1/m+n} \rangle_{m+n=1}^{\infty}$ be a bounded sequence and $\langle a_{m,n} \rangle_{m+n=0}^{\infty}$

be such that $\lim_{m+n \rightarrow \infty} |a_{m,n}|^{1/m+n} = 0$. Choose $M > 0$ such that

$|b_{m,n}|^{1/m+n} \leq M$ if $m+n \geq 1$ and then $N \geq 0$ such that $m+n > N \Rightarrow$

$|a_{m,n}|^{1/m+n} \leq \frac{1}{2M}$. Then if $m+n > N$, $|a_{m,n} b_{m,n}| \leq \frac{1}{(2M)^{m+n}} \cdot M^{m+n} = \frac{1}{2^{m+n}}$.

Therefore $\sum_{m+n=N+1}^{\infty} |a_{m,n} b_{m,n}| \leq \sum_{m+n=N+1}^{\infty} \frac{1}{2^{m+n}} < \infty$.

Hence the series $\sum_{m+n=0}^{\infty} a_{m,n} b_{m,n}$ converges absolutely, hence it converges.

Conversely suppose for any sequence $\langle a_{m,n} \rangle_{m+n=0}^{\infty}$ such that $\lim_{m+n \rightarrow \infty} |a_{m,n}|^{1/m+n} = 0$, the series $\sum_{m+n=0}^{\infty} a_{m,n} b_{m,n}$ converges. If

$\langle |b_{m,n}|^{1/m+n} \rangle_{m+n=1}^{\infty}$ is not bounded, for each $k \in \mathbb{Z}^+$ there exists

an (m_k, n_k) such that $|b_{m_k, n_k}|^{\frac{1}{m_k+n_k}} > k$ and $\langle m_k+n_k \rangle_{k=1}^{\infty}$ is

strictly increasing. Choose $a_{m,n} = 0$ if $(m,n) \neq (m_k, n_k)$,

$a_{m_k, n_k} = k^{\frac{1}{m_k+n_k}}$. Then $\lim_{m+n \rightarrow \infty} |a_{m,n}|^{1/m+n} = \lim_{m_k+n_k \rightarrow \infty} |a_{m_k, n_k}|^{1/m_k+n_k}$

$\frac{1}{m_k+n_k} = \lim_{k \rightarrow \infty} \frac{1}{k} = 0$. But $|a_{m_k, n_k} b_{m_k, n_k}| > 1$ for each k so

$\sum_{m+n=0}^{\infty} a_{m,n} b_{m,n}$ does not converge. Therefore $\langle |b_{m,n}|^{1/m+n} \rangle_{m+n=1}^{\infty}$

is bounded. The series $\sum a_{m,n} b_{m,n}$ does not converge since the only $\neq 0$ terms are > 1 and there are an infinite number of them.

We now characterize the class of continuous linear functionals on Γ^2 .

Theorem 2.5. Let F be a function from Γ^2 to the complex plane.

Then F is a continuous linear functional on Γ^2 if and only if there

is a unique sequence $\langle b_{m,n} \rangle_{m+n=0}^{\infty}$ such that $\langle |b_{0,0}|, |b_{m,n}|^{1/m+n} \rangle_{m+n=1}^{\infty}$

is bounded and such that for all $f(z,w) = \sum_{m+n=0}^{\infty} a_{m,n} z^m w^n \in \Gamma^2$,

$$F(f) = \sum_{m+n=0}^{\infty} a_{m,n} b_{m,n}.$$

Proof. Let $\langle |b_{0,0}|, |b_{m,n}|^{1/m+n} \rangle_{m+n=1}^{\infty}$ be a bounded sequence,

$M > 0$ be such that $|b_{0,0}| < M$, $|b_{m,n}|^{1/m+n} < M$, $m+n \geq 1$ and

$f(z,w) = \sum_{m+n=0}^{\infty} a_{m,n} z^m w^n \in \Gamma^2$. Then $\lim_{m+n \rightarrow \infty} |a_{m,n}|^{1/m+n} = 0$ so

$\sum_{m+n=0}^{\infty} a_{m,n} b_{m,n}$ converges by Lemma 2.4. Hence we may define

a function F from Γ^2 to the complex plane by $F(f) =$

$\sum_{m+n=0}^{\infty} a_{m,n} b_{m,n}$. It is clear that F is a linear functional.

Let $\epsilon > 0$ and $f(z,w) = \sum_{m+n=0}^{\infty} a_{m,n} z^m w^n \in \Gamma^2$ be given. We show there exists a $\delta > 0$ such that if $g \in \Gamma^2$ and $d(f,g) < \delta$, then

$|F(f) - F(g)| < \epsilon$. Choose $\delta > 0$ such that $\delta M < 1$ and

$\delta M + \left(\frac{\delta M}{1-\delta M}\right)^2 < \epsilon$. Then if $g(z,w) = \sum_{m+n=0}^{\infty} c_{m,n} z^m w^n \in \Gamma^2$ and

$d(f,g) < \epsilon$, $|F(f) - F(g)| = |F(f-g)| = \left| \sum_{m+n=0}^{\infty} (a_{m,n} - c_{m,n}) b_{m,n} \right|$

$$\leq |a_{0,0} - c_{0,0}| M + \sum_{m+n=1}^{\infty} |a_{m,n} - c_{m,n}| M^{m+n}$$

$$\leq \delta M + \sum_{m+n=1}^{\infty} (\delta M)^{m+n}$$

$$= \delta M + \sum_{m=1}^{\infty} (\delta M)^m \sum_{n=1}^{\infty} (\delta M)^n$$

$$= \delta M + \left(\frac{\delta M}{1-\delta M}\right)^2 < \epsilon.$$

Conversely, let F be a continuous linear functional on Γ^2 .

Let $F(z^m w^n) = b_{m,n}$ for all $m+n \geq 0$. Given $f(z,w) = \sum_{m+n=0}^{\infty} a_{m,n} z^m w^n$,

let $f_N(z,w) = \sum_{m+n=0}^N a_{m,n} z^m w^n$. Then $d(f_N, f) = \sup\{|a_{m,n}|^{1/m+n} : m+n$

$> N\} \rightarrow 0$ as $N \rightarrow \infty$ so by the continuity of F , $F(f_N) \rightarrow F(f)$ as $N \rightarrow \infty$.

But $F(f_N) = \sum_{m+n=0}^{\infty} a_{m,n} b_{m,n}$. Therefore $\lim_{N \rightarrow \infty} \sum_{m+n=0}^N a_{m,n} b_{m,n} = F(f)$.

Hence $\sum_{m+n=0}^{\infty} a_{m,n} b_{m,n}$ converges and $F(f) = \sum_{m+n=0}^{\infty} a_{m,n} b_{m,n}$. By

Lemma 2.4, the sequence $\langle |b_{0,0}|^{1/m+n} \rangle_{m+n=1}^{\infty}$ is bounded. Suppose

$\langle C_{m,n} \rangle_{m+n=0}^{\infty}$ is a sequence such that for all

$f(z,w) = \sum_{m+n=0}^{\infty} a_{m,n} z^m w^n \in \Gamma^2$, $F(f) = \sum_{m+n=0}^{\infty} a_{m,n} C_{m,n}$, then for

$j, k \in \mathbb{Z}_+$, $F(z^j w^k) = C_{j,k}$. But $F(z^j w^k) = b_{j,k}$.

Hence $C_{j,k} = b_{j,k}$ and the sequence is unique.

Section 3. The class of continuous scalar homomorphisms on Γ^2 .

Let $f, g \in \Gamma^2$, $\alpha \in \mathcal{C}$ (complex field). Define

$(f+g)(z,w) = f(z,w) + g(z,w)$, $(f \circ g)(z,w) = f(z,w)g(z,w)$, (αf)

$(z,w) = \alpha f(z,w)$. Then Γ^2 becomes a commutative algebra with

a unit. In this section we characterize the continuous linear functionals on Γ^2 that preserve multiplication. That is the continuous scalar homomorphisms on Γ^2 .

Lemma 3.1. Given $\epsilon > 0$ and $(b,c) \in \mathcal{C} \times \mathcal{C}$, there exists a $\delta > 0$ such that if $f, g \in \Gamma^2$ and $d(f,g) < \delta$, then $|f(b,c) - g(b,c)| < \epsilon$.

Proof. Given $\epsilon > 0$ and $(b,c) \in \mathcal{C} \times \mathcal{C}$, let $R = \max\{|b|, |c|\}$ choose $\delta > 0$ such that $\delta R < 1$ and $\delta + \left(\frac{\delta R}{1 - \delta R}\right)^2 < \epsilon$. Then if $f(z,w) = \sum_{m,n=0}^{\infty} a_{m,n} z^m w^n$ and $g(z,w) = \sum_{m,n=0}^{\infty} b_{m,n} z^m w^n$ are in Γ^2 and $d(f,g) < \delta$, $|f(b,c) - g(b,c)| = \left| \sum_{m+n=0}^{\infty} (a_{m,n} - b_{m,n}) b^m c^n \right|$
 $\leq |a_{0,0} - b_{0,0}| + \sum_{m+n=1}^{\infty} |a_{m,n} - b_{m,n}| R^{m+n} < \delta + \sum_{m+n=1}^{\infty} (\delta R)^{m+n} =$

$$\delta + \sum_{m=1}^{\infty} (\delta R)^m \sum_{n=1}^{\infty} (\delta R)^n = \delta + \left(\frac{\delta R}{1-\delta R} \right)^2 < \epsilon.$$

Theorem 3.2. Let F be a function from Γ^2 to $\mathcal{C}$, $F \not\equiv 0$.

Then F is a continuous scalar homomorphism on Γ^2 if and only if there exists a unique $(b, c) \in \mathcal{C} \times \mathcal{C}$ such that for all $f(z, w) =$

$$\sum_{m,n=0}^{\infty} a_{m,n} z^m w^n \in \Gamma^2,$$

$$F(f) = f(b, c).$$

Proof. Let F be a $\not\equiv 0$ continuous scalar homomorphism on Γ^2 .

By Theorem 2.5, there is a unique sequence $\langle b_{m,n} \rangle_{m+n=0}^{\infty}$ such that

$$\text{for all } f(z, w) = \sum_{m,n=0}^{\infty} a_{m,n} z^m w^n \in \Gamma^2, \quad F(f) = \sum_{m+n=0}^{\infty} a_{m,n} b_{m,n}.$$

For each m and n , $b_{m,n} = F(z^m w^n) = F(z)^m F(w)^n = b_{1,0}^m b_{0,1}^n$. Therefore

$$F(f) = \sum_{m+n=0}^{\infty} a_{m,n} b_{1,0}^m b_{0,1}^n = f(b_{1,0}, b_{0,1}).$$

Conversely, given $(b, c) \in \mathcal{C} \times \mathcal{C}$, define a function F from Γ^2 to $\mathcal{C}$ by $F(f) = f(b, c)$. Then F is clearly a $\not\equiv 0$ scalar homomorphism. Given $\epsilon > 0$, let $\delta > 0$ be such that if $f, g \in \Gamma^2$ and $d(f, g) < \delta$, then $|f(b, c) - g(b, c)| < \epsilon$. Then $|F(f) - F(g)| = |F(f - g)| = |(f - g)(b, c)| = |f(b, c) - g(b, c)| < \epsilon$. Hence F is continuous.

Section 4. The class of bounded linear functionals on Γ^2 .

Definition 4.1 . Let F be a linear functional on Γ^2 . Then F

is said to be bounded if and only if there exists an $M \geq 0$ such

that for all $f \in \Gamma^2$, $|F(f)| \leq Md(f, 0)$. Here, 0 denotes the function identically zero on $\mathcal{C} \times \mathcal{C}$.

Lemma 4.2. Let F be a linear functional on Γ^2 . If F is bounded, F is continuous but not conversely.

Proof. Let F be a bounded linear functional on Γ^2 . Let $f_0 \in \Gamma^2$ $\epsilon > 0$ be given and let $M \geq 0$ be such that for all $f \in \Gamma^2$, $|F(f)| \leq Md(f, 0)$. Choose $\delta = \epsilon/M+1$. Then if $g \in \Gamma^2$ and $d(f_0, g) < \delta$, $|F(f_0) - F(g)| = |F(f_0 - g)| \leq Md(f_0 - g, 0) \leq (m+1)d(f_0, g) < (M+1)\delta < \epsilon$. Therefore F is continuous at f_0 . Hence F is continuous.

For an example of a continuous linear functional that is not bounded, let $b_{m,n} = n$. Then $\langle |n|^{1/m+n} \rangle_{m+n=1}^{\infty}$ is a bounded sequence.

Define a function F from Γ^2 to $\mathcal{C}$ by $F(\sum_{m,n=0}^{\infty} a_{m,n} z^m w^n) = \sum_{m+n=1}^{\infty} na_{m,n}$.

By Theorem 2.5, F is a continuous linear functional on Γ^2 . If F

is bounded, there exists an $M \geq 0$ such that $|\sum_{m+n=1}^{\infty} na_{m,n}| \leq M$
 $\sup \{|a_{0,0}|, |a_{m,n}|^{1/m+n} : m+n \geq 1\}$ for all $\langle a_{m,n} \rangle_{m+n=0}^{\infty}$ such that
 $|a_{m,n}|^{1/m+n} \rightarrow 0$ as $m+n \rightarrow \infty$. Let $k \in \mathbb{Z}^+$, $k > \max\{M, 2\}$. Let
 $a_{0,k} = k$, $a_{m,n} = 0$ if $(m,n) \neq (0,k)$. Then $|a_{m,n}|^{1/m+n} \rightarrow 0$ as

$m+n \rightarrow \infty$ since the sequence has only one non-zero term. But

$\left| \sum_{m+n=1}^{\infty} na_{m,n} \right| = k^2$, $M \sup\{|a_{0,0}|, |a_{m,n}|^{1/m,n} : m+n \geq 1\} = M \cdot k^{1/k} < k \cdot k^{1/k} < k \cdot k^{1/k} < k^2$, a contradiction. Hence F is not bounded.

Definition 4.3. Let B denote the class of bounded linear functionals on Γ^2 . For $F, G \in B$, $\alpha \in \mathbb{C}$, $f \in \Gamma^2$, define $(F+G)(f) = F(f) + G(f)$, $(\alpha F)(f) = \alpha \cdot F(f)$, $\|F\| = \inf\{M \geq 0 \mid \text{for all } f \in \Gamma^2, |F(f)| \leq M d(f, 0)\}$.

Theorem 4.4. With respect to Definition 4.3, B is a normed linear space.

Proof. Let $F \in B$. Then $|F(f)| \leq \|F\| d(f, 0)$ for all $F \in \Gamma^2$. If not, for some $f_0 \in \Gamma^2$, $|F(f_0)| > \|F\| d(f_0, 0)$. Then $d(f_0, 0) \neq 0$ so choose $\epsilon > 0$ such that $|F(f_0)| = \|F\| d(f_0, 0) + \epsilon d(f_0, 0)$. By Definition of $\|F\|$, there exists an $M \geq 0$ such that $|F(f)| \leq M d(f, 0)$ for all $f \in \Gamma^2$ and $\|F\| + \epsilon > M$. Then $d(f_0, 0)(\|F\| + \epsilon) = |F(f_0)| \leq M d(f_0, 0)$. Hence $\|F\| + \epsilon \leq M$, a contradiction.

Therefore $|F(f)| \leq \|F\| d(f, 0)$ for all $f \in \Gamma^2$ and $\|F\|$ is the smallest number to satisfy this inequality for all $f \in \Gamma^2$.

For $F, G \in B$, $\alpha \in \mathbb{C}$, $F+G$ and αF are clearly linear functionals on Γ^2 . For $f \in \Gamma^2$, $|(F+G)(f)| = |F(f) + G(f)| \leq |F(f)| + |G(f)| \leq \|F\| d(f, 0) + \|G\| d(f, 0) = (\|F\| + \|G\|) d(f, 0)$. Hence $F+G \in B$ and $\|F+G\| \leq \|F\| + \|G\|$. Also $|(\alpha F)(f)| = |\alpha \cdot F(f)| = |\alpha| |F(f)| \leq |\alpha| \|F\| d(f, 0)$. Hence $\alpha F \in B$ and $\|\alpha F\| \leq |\alpha| \|F\|$. Suppose it is possible to have $\|\alpha F\| < |\alpha| \|F\|$. Choose

$\epsilon > 0$ such that $||\alpha F|| + \epsilon ||\alpha|| ||F|| = ||\alpha|| ||F||$. Then for all $f \in \Gamma^2$, $||\alpha|| ||F(f)|| = ||(\alpha F)(f)|| \leq ||\alpha F|| d(f, 0) = (1 - \epsilon) ||\alpha|| ||F|| d(f, 0)$. Therefore $||F(f)|| \leq (1 - \epsilon) ||F|| d(f, 0)$, a contradiction. Hence $||\alpha F|| = ||\alpha|| ||F||$.

It is clear that $||\bullet||$ evaluated at the zero linear functional on Γ^2 is 0 and if $||F|| = 0$ then $||F(f)|| = 0$ for all $f \in \Gamma^2$, hence $F \equiv 0$. Also the remaining properties required for B to be a normed linear space follow trivially. Hence B is a normed linear space with respect to Definition 4.3.

Theorem 4.5. Let F be a function from Γ^2 to $\mathcal{C}$. Then $F \in B$ if and only if there exists unique $(a, b, c) \in \mathcal{C} \times \mathcal{C} \times \mathcal{C}$ such that for all $f(z, w) = \sum_{m,n=0}^{\infty} a_{m,n} z^m w^n \in \Gamma^2$,

$$F(f) = a_{0,0}a + a_{1,0}b + a_{0,1}c.$$

Also, $||F|| = |a| + |b| + |c|$.

Proof. Let $F \in B$. Then F is continuous so there exist a unique

sequence $\langle b_{m,n} \rangle_{m+n=0}^{\infty}$ such that $F\left(\sum_{m,n=0}^{\infty} a_{m,n} z^m w^n\right) = \sum_{m+n=0}^{\infty} a_{m,n} b_{m,n}$

and $\left|\sum_{m+n=0}^{\infty} a_{m,n} b_{m,n}\right| \leq ||F|| \sup\{|a_{0,0}|, |a_{m,n}|^{1/m+n} : m+n \geq 1\}$ for

all sequences $\langle a_{m,n} \rangle_{m+n=0}^{\infty}$ such that $|a_{m,n}|^{1/m+n} \rightarrow 0$ as $m+n \rightarrow \infty$.

Suppose $b_{k,j} \neq 0$ for some (k, j) with $k+j \geq 2$. Choose $a_{m,n} = 0$ if $(m, n) \neq (k, j)$ and choose $a_{k,j}$ such that $||F|| >$

$|a_{k,j}|^{1-\frac{1}{k+j}} \cdot |b_{k,j}|$. Then $|a_{m,n}|^{1/m+n} \rightarrow 0$ as $m+n \rightarrow \infty$,

$$\left| \sum_{m+n=0}^{\infty} a_{m,n} b_{m,n} \right| = |a_{k,j} b_{k,j}| \leq \|F\| |a_{k,j}|^{\frac{1}{k+j}}. \text{ Therefore}$$

$$|a_{k,j}|^{1-\frac{1}{k+j}} \cdot |b_{k,j}| \leq \|F\|, \text{ a contradiction. Hence } b_{k,j} = 0$$

$$\text{if } k+j \geq 2. \text{ Hence } F\left(\sum_{m,n=0}^{\infty} a_{m,n} z^m w^n\right) = a_{0,0} b_{0,0} + a_{1,0} b_{1,0} +$$

$$a_{0,1} b_{0,1}. \text{ Also } |F\left(\sum_{m,n=0}^{\infty} a_{m,n} z^m w^n\right)| \leq |a_{0,0}| |b_{0,0}| +$$

$$|a_{1,0}| |b_{1,0}| + |a_{0,1}| |b_{0,1}| \leq (|b_{0,0}| + |b_{1,0}| + |b_{0,1}|) d(f, 0).$$

$$\text{Therefore } \|F\| \leq |b_{0,0}| + |b_{1,0}| + |b_{0,1}|. \text{ To show equality}$$

$$\text{here, it suffices to show there exists an } f_0 \in \Gamma^2 \text{ such that}$$

$$|F(f_0)| = (|b_{0,0}| + |b_{1,0}| + |b_{0,1}|) d(f_0, 0). \text{ If } b_{0,0} =$$

$$|b_{0,0}| e^{i\theta_1}, b_{1,0} = |b_{1,0}| e^{i\theta_2}, b_{0,1} = |b_{0,1}| e^{i\theta_3}, \text{ choose}$$

$$f_0(z, w) = e^{-i\theta_1} + e^{-i\theta_2} z + e^{-i\theta_3} w. \text{ Then } |F(f_0)| =$$

$$|b_{0,0}| + |b_{1,0}| + |b_{0,1}| = (|b_{0,0}| + |b_{1,0}| + |b_{0,1}|) d(f_0, 0).$$

$$\text{Conversely, given } (a, b, c) \in \mathcal{L} \times \mathcal{L} \times \mathcal{L}, \text{ define a function } F \text{ from } \Gamma^2 \text{ to } \mathcal{L} \text{ by } F(f) = a_{0,0} a + a_{1,0} b + a_{0,1} c, f(z, w) =$$

$$\sum_{m,n=0}^{\infty} a_{m,n} z^m w^n \in \Gamma^2. \text{ By Theorem 2.5, } F \text{ is a continuous linear}$$

$$\text{functional on } \Gamma^2. \text{ Since } |F(f)| \leq |a_{0,0}| |a| + |a_{1,0}| |b| +$$

$$|a_{0,1}| |c| \leq (|a| + |b| + |c|) d(f, 0), F \in B.$$

Corollary 4.6. With respect to Definition 4.3, B is a Banach space.

Proof. Let $\langle F_n \rangle_{n=1}^{\infty}$ be a Cauchy sequence in B , F_n corresponding to (a_n, b_n, c_n) . Then for any $\epsilon > 0$, there exists an $N = N(\epsilon)$ such that $m, n > N(\epsilon) \Rightarrow ||F_n - F_m|| < \epsilon$. That is $|a_n - a_m| + |b_n - b_m| + |c_n - c_m| < \epsilon$. Hence each of $\langle a_n \rangle, \langle b_n \rangle, \langle c_n \rangle$ is a Cauchy sequence. Let $a_n \rightarrow a, b_n \rightarrow b, c_n \rightarrow c$ as $n \rightarrow \infty$. Define a function F from Γ^2 to $\mathcal{C}$ by

$$F\left(\sum_{m,n=0}^{\infty} a_{m,n} z^m w^n\right) = a_{0,0}a + a_{1,0}b + a_{0,1}c. \text{ By Theorem 4.5, } F \in B.$$

Given $\epsilon > 0$, there exists an $N = N(\epsilon)$ such that $|a_m - a| + |b_m - b| + |c_m - c| < \epsilon/2$ if $m, n > N(\epsilon)$. Let $m \rightarrow \infty$ to get $|a - a_n| + |b - b_n| + |c - c_n| \leq \epsilon/2$ if $n > N(\epsilon)$. Hence given $\epsilon > 0$, there exists an $N = N(\epsilon)$ such that if $n > N(\epsilon)$, $|a - a_n| + |b - b_n| + |c - c_n| < \epsilon$. That is $||F_n - F|| < \epsilon$. Therefore B is a Banach Space.

If (a_1, b_1, c_1) and (a_2, b_2, c_2) are in $\mathcal{C} \times \mathcal{C} \times \mathcal{C}$ and $\alpha \in \mathcal{C}$, it is clear that if addition and scalar multiplication are defined by $(a_1, b_1, c_1) + (a_2, b_2, c_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$, $\alpha(a_1, b_1, c_1) = (\alpha a_1, \alpha b_1, \alpha c_1)$, that $\mathcal{C} \times \mathcal{C} \times \mathcal{C}$ is a vector space over $\mathcal{C}$. Also if $||\bullet||$ is defined by $||(a, b, c)|| = |a| + |b| + |c|$, $||\bullet||$ is a norm on $\mathcal{C} \times \mathcal{C} \times \mathcal{C}$ making $\mathcal{C} \times \mathcal{C} \times \mathcal{C}$ into a Banach space. Define a function Ψ from B to $\mathcal{C} \times \mathcal{C} \times \mathcal{C}$ as follows: For $F \in B$, let (a, b, c) be the unique element of $\mathcal{C} \times \mathcal{C} \times \mathcal{C}$ such that $F(f) = a_{0,0}a + a_{1,0}b + a_{0,1}c$ for all

$$f(z, w) = \sum_{m,n=0}^{\infty} a_{m,n} z^m w^n \in \Gamma^2. \text{ Let } \Psi(F) = (a, b, c). \text{ The following}$$

theorem is straightforward to prove so the proof is omitted

Theorem 3.4.7. The spaces B and $\mathcal{O}_X \otimes \mathcal{O}_X$ are isometrically isomorphic Banach spaces.

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ABSTRACT. Three classes of linear functionals on the space of entire functions of two variables are characterized. Several results are proved.

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