COMPLETE RESIDUE SYSTEMS IN THE RING OF MATRICES OF RATIONAL INTEGERS

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(Received April 1, 1977 and in revised form August 29, 1977)

ABSTRACT. This paper deals with the characterizations of the complete residue system mod. G, where G is any n x n matrix, in the ring of n x n matrices.

KEY WORDS AND PHRASES. Complete residue system, ring of Gaussian integers, representations for the complete residue system.

AMS(MOS)SUBJECT CLASSIFICATION (1970) CODES. 12F05, 12B35.

1. INTRODUCTION.

Let Z denote the ring of rational integers and Z(i) be the ring of
Gaussian integers. Jordan and Potratz [1] have exhibited several representations for the complete residue system (in short, C.R.S.) mod. \( r \) in the ring of Gaussian integers. Also it is well known that the ring of Gaussian integers is isomorphic to the ring of \( 2 \times 2 \) matrices of the form \[
\begin{pmatrix}
a & b \\
-b & a
\end{pmatrix}
\]
a, b in \( \mathbb{Z} \). This raises the question of characterizing the C.R.S. mod. \( G \), where \( G \) is any \( n \times n \) matrix, in the ring of \( n \times n \) matrices of which we denote by \( \text{Mat}_n(\mathbb{Z}) \).

2. THE COMPLETE RESIDUE SYSTEM IN \( \text{Mat}_n(\mathbb{Z}) \).

First of all, we define \( A \equiv B \mod. U \) mean there is a matrix \( C \) such that \( B = CA \), and \( A \equiv B \mod. U \) means that \( U|A - B \). Now we can give a definition of the C.R.S. mod. \( U \) in the ring of \( \text{Mat}_n(\mathbb{Z}) \).

DEFINITION. Let \( U \) be in \( \text{Mat}_n(\mathbb{Z}) \) with \( \det U \neq 0 \). Then a subset \( J \) of \( \text{Mat}_n(\mathbb{Z}) \) is called a C.R.S. mod. \( U \) if and only if for any \( A \) in \( \text{Mat}_n(\mathbb{Z}) \) there exists uniquely a matrix \( B \) in \( J \) such that \( A \equiv B \mod. U \).

LEMMA 1. Let \( G = \text{diag}(g_1, g_2, \ldots, g_n) \) with \( g_i \neq 0 \), \( i = 1, 2, \ldots, n \). Let \( E_{ij} \) be the matrix units, then

\[
I_{ik} = \{ a \in \mathbb{Z} : G \mid \sum_{m=1}^{n} \sum_{j=1}^{n} a_{mj} E_{mj} \text{ with } a_{mj} \in \mathbb{Z}, a_{il} = a_{i2} = \ldots = a_{ik-1} = 0, a_{ik} = a \}
\]

are the principal ideals generated by a positive integer \( g_k \), where \( i, k = 1, 2, \ldots, n \).

PROOF. It is clear the \( I_{ik} \) are ideals in \( \mathbb{Z} \). But \( \mathbb{Z} \) is a P.I.D., therefore \( I_{ik} \) are principal ideals generated by a positive integer \( d_{ik} \). Since \( g_k E_{ik} = E_{ik} G \), then \( g_k \) is in \( I_{ik} \), i.e., \( d_{ik} \mid g_k \). On the other hand, for \( d_{ik} \) in \( I_{ik} \) we have \( \sum_{m=1}^{n} \sum_{j=1}^{n} a_{mj} E_{mj} = (t_{ik}) G \) for some \( (t_{ik}) \), where \( a_{mj} \) is in \( \mathbb{Z} \), \( a_{il} = a_{i2} = \ldots = a_{ik-1} = 0 \), \( a_{ik} = d_{ik} \). It follows that \( d_{ik} = t_{ik} g_k \), i.e., \( d_{ik} = |g_k| \).

This completes the proof.
LEMMA 2. Let $G = \text{diag}(g_1, g_2, \ldots, g_n)$ with $g_k \neq 0$, $k = 1, 2, \ldots, n$. Then $J = \{(r_{ik}) : 0 \leq r_{ik} < |g_k|, i, k = 1, 2, \ldots, n\}$ forms a complete residue system mod. $G$.

PROOF. (1) For any $A = (a_{ik})$ in $\text{Mat}_n(Z)$, there exist $p_{ik}$, $r_{ik}$ in $Z$ such that $a_{ik} = p_{ik} |g_k| + r_{ik}$, where $0 \leq r_{ik} < |g_k|$. Therefore

$$A - (p_{ik} |g_k|) (r_{ik}) = (r_{ik}).$$

But $|g_k| \cdot E_{ik} = |g_k| \cdot g_{ik}^{-1} E_{ik} G$, and therefore

$$G \mid A - (r_{ik}).$$

This shows that $A \equiv (r_{ik}) \mod. G$.

(2) If $(r_{ik}) \equiv (s_{ik}) \mod. G$, where $0 \leq r_{ik}, s_{ik} < |g_k|$, then $r_{ik} - s_{ik}$ is in $I_{11}$ (by Lemma 1). This implies that $g_1 | (r_{11} - s_{11})$, and so $r_{11} = s_{11}$, for $0 \leq |r_{11} - s_{11}| < |g_1|$. It follows that $r_{12} - s_{12}$ is in $I_{12}$. Therefore $g_2 \mid (r_{12} - s_{12})$ and $r_{12} = s_{12}$, for $0 \leq |r_{12} - s_{12}| < |g_2|$. Continuing in this way, we must have $r_{ik} = s_{ik}$, for all $i, k = 1, 2, \ldots, n$.

THEOREM 1. If $G$ is a $n \times n$ matrix with $\det G \neq 0$, and if $U$ and $V$ are unimodular $n \times n$ matrices such that $UGV = \text{diag}(g_1, g_2, \ldots, g_n)$, then $J = \{(r_{ik})V^{-1} : 0 \leq r_{ik} < |g_k|, i, k = 1, 2, \ldots, n\}$ forms a complete residue system mod. $G$.

PROOF. (1) By Lemma 2, for any $n \times n$ matrix $A$, there exists a matrix $(r_{ik})$ with $0 \leq r_{ik} < |g_k|$ such that $AV \equiv (r_{ik}) \mod. UGV$, i.e., $A \equiv (r_{ik})V^{-1} \mod. G$.

(2) Let $(r_{ik})V^{-1} \equiv (s_{ik})V^{-1} \mod. G$, where $0 \leq r_{ik}, s_{ik} < |g_k|$. It follows that $(r_{ik}) \equiv (s_{ik}) \mod. UGV$. Therefore $(r_{ik}) = (s_{ik})$.

COROLLARY 1. If $J$ forms a C.R.S. mod. $G$, and $U$ and $V$ are unimodular $n \times n$ matrices, then $\{URV : R \in J\}$ forms a C.R.S. mod. $GV$.

COROLLARY 2. If $G$ is a $n \times n$ matrix with $\det G \neq 0$, then the cardinality of the C.R.S. mod. $G$ is $|\det G|^{n}$. 
3. THE COMPLETE RESIDUE SYSTEM IN $\text{Mat}_2(\mathbb{Z})$.

By restricting the order of the matrix we may relax the condition on the diagonable matrix.

**LEMMA 3.** Let $U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \in \text{Mat}_2(\mathbb{Z})$ with $\det U \neq 0$, then

1. $I_0 = \{a \in \mathbb{Z} : U \bigg| \begin{pmatrix} a & \alpha \\ \beta & r \end{pmatrix} \text{ for some } \alpha, \beta, r \in \mathbb{Z} \}$ and $I_0' = \{a \in \mathbb{Z} : U \bigg| \begin{pmatrix} 0 & 0 \\ a & \delta \end{pmatrix} \text{ for some } \delta \in \mathbb{Z} \}$ are nonzero principal ideals of $\mathbb{Z}$ generated by a positive integer $d = \gcd(u_{11}, u_{21})$. Moreover $I_0 = I_0'$.

2. $I_1 = \{a \in \mathbb{Z} : U \bigg| \begin{pmatrix} 0 & a \\ \beta & r \end{pmatrix} \text{ for some } \beta, r \in \mathbb{Z} \}$ and $I_1' = \{a \in \mathbb{Z} : U \bigg| \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \}$ are nonzero principal ideals of $\mathbb{Z}$ generated by a positive integer $\frac{|\det U|}{d}$. Moreover, $I_1 = I_1'$.

**PROOF.** (1) $a \in I_0$ implies $U \bigg| \begin{pmatrix} a & \alpha \\ \beta & r \end{pmatrix}$ for some $\alpha, \beta, r \in \mathbb{Z}$, and then $U \bigg| \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & \alpha \\ \beta & r \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a & \alpha \end{pmatrix}$, i.e., $a \in I_0'$. This shows that $I_0 \subseteq I_0'$.

On the other hand, $b \in I_0'$ implies $U \bigg| \begin{pmatrix} 0 & 0 \\ b & \delta \end{pmatrix}$ for some $\delta \in \mathbb{Z}$ and then $U \bigg| \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ b & \delta \end{pmatrix} = \begin{pmatrix} b & \delta \\ 0 & 0 \end{pmatrix}$, i.e., $b \in I_0$. Therefore $I_0 = I_0'$. It is clear that $I_0$ is an ideal of $\mathbb{Z}$. Now $\det U \in I_0$, for $U \bigg| \begin{pmatrix} \det U & 0 \\ 0 & \det U \end{pmatrix}$.

Thus $I_0$ is a nonzero ideal of $\mathbb{Z}$. But $\mathbb{Z}$ is a P.I.D., therefore $I_0$ is an ideal generated by a positive integer $d$. Since $U \big| U$ implies $U \bigg| \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} U = \begin{pmatrix} U_{21} & U_{22} \\ 0 & 0 \end{pmatrix}$, we have $U_{11}, U_{12} \in I_0$, and then $d | U_{11}, d | U_{21}$. By $d \in I_0$, we have $U^{r} \bigg| \begin{pmatrix} 0 & 0 \\ d & \delta \end{pmatrix}$, i.e., $U^{r} \bigg| \begin{pmatrix} 0 & 0 \\ d & \delta \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} U$ for some $\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \in \text{Mat}_2(\mathbb{Z})$.

Therefore $d = t_{21}U_{11} + t_{22}U_{21}$. If $x \mid U_{11}$ and $x \mid U_{21}$, then $x \mid d$. Thus $d = \gcd(U_{11}, U_{21})$. 
(2) \( a \in I_1 \) implies \( U \mid \begin{pmatrix} 0 & a \\ \beta & r \end{pmatrix} \) for some \( \beta, r \in \mathbb{Z} \) and then
\[
U \mid \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & a \\ \beta & r \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ i.e., \ a \in I_1'. \] Thus \( I_1 \subseteq I_1' \). Conversely, if \( b \in I_1' \), then \( U \mid \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \) and so \( U \mid \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \), i.e., \( b \in I_1 \). It is also clear that \( I_1 \) is an ideal of \( \mathbb{Z} \). Now \( \frac{\text{det} U}{d} \in I_1 \) for all \( U \) such that \( \begin{pmatrix} 0 & 0 \\ 0 & \frac{\text{det} U}{d} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -u_{21} & u_{12} \end{pmatrix} U \), and then \( I_1 \) is a nonzero ideal of \( \mathbb{Z} \). But \( \mathbb{Z} \) is a P.I.D., and then \( I_1 \) is an ideal generated by a positive integer \( g \). Now \( \frac{\text{det} U}{d} \in I_1 \) implies \( \frac{\text{det} U}{d} \in I_1 \), i.e., \( g \mid \frac{|\text{det} U|}{d} \). By \( g \in I_1 \), we have
\[
U \mid \begin{pmatrix} 0 & 0 \\ 0 & g \end{pmatrix}, \ i.e., \ \text{det} U \mid \begin{pmatrix} 0 & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} u_{22} & -u_{12} \\ -u_{21} & u_{11} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -g u_{21} & g u_{11} \end{pmatrix}, \ \text{and then}
\]
\[
\text{det} U \mid g u_{21}, \ \text{det} U \mid g u_{11}.
\]
By the proof of (1), we have \( d = t_{21} u_{11} + t_{22} u_{21} \), and then
\[
g d = t_{21} (g u_{11}) + t_{22} (g u_{21}) \text{ or } \frac{|\text{det} U|}{d} \mid g. \ \text{Therefore} \ g = \frac{|\text{det} U|}{d}. \ \text{This completes the proof of (2)}.
\]

**THEOREM 2.** Let \( U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \in \text{Mat}_2(\mathbb{Z}) \) with \( \text{det} U \neq 0 \), let
\[
d = \text{g.c.d.}(u_{11}, u_{21}). \ \text{Then} \ J = \{ R = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \in \text{Mat}_2(\mathbb{Z}) : 0 \leq r_{11},
\]
\[
r_{21} < d, 0 \leq r_{12}, r_{22} < \frac{|\text{det} U|}{d} \} \text{is a complete residue system (mod. } U) \text{ in } \text{Mat}_2(\mathbb{Z}).
\]

**PROOF.** (1) From \( d \in I_0, \frac{|\text{det} U|}{d} \in I_1 \), we have
\[
U \mid \begin{pmatrix} d & a \\ \beta & r \end{pmatrix}, \ U \mid \begin{pmatrix} 0 & 0 \\ d & \eta \end{pmatrix}, \ U \mid \begin{pmatrix} 0 & \frac{|\text{det} U|}{d} \\ \epsilon & \delta \end{pmatrix}, \ U \mid \begin{pmatrix} 0 & 0 \\ 0 & \frac{|\text{det} U|}{d} \end{pmatrix}, \ i.e.,
\]
there exists \( T_i \in \text{Mat}_2(\mathbb{Z}) \), \( i = 1, 2, 3, 4 \) such that
\[
\begin{pmatrix}
\frac{|\text{det}U|}{d} & 0 \\
\varepsilon & \delta
\end{pmatrix}

(\begin{pmatrix}
d & a \\
\beta & r
\end{pmatrix}) = T_1U,

(\begin{pmatrix}
0 & 0 \\
d & \eta
\end{pmatrix}) = T_2U,

(\begin{pmatrix}
0 & 0 \\
\frac{|\text{det}U|}{d} & 0
\end{pmatrix}) = T_4U.

For any matrix \( A = \begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix} \in \text{Mat}_2(\mathbb{Z}) \), there exists \( p_{11}, r_{11} \in \mathbb{Z} \) such that
\[a_{11} = p_{11}d + r_{11}\]
where \( 0 < r_{11} < d \). Thus
\[A - p_{11}T_1U = \begin{pmatrix}
r_{11} & b_{12} \\
b_{21} & b_{22}
\end{pmatrix},
\]
for some \( b_{12}, b_{21}, b_{22} \in \mathbb{Z} \). Moreover, \( b_{12} = p_{12} \frac{|\text{det}U|}{d} + r_{12} \) for some
\[p_{12}, r_{12} \in \mathbb{Z}, \quad 0 < r_{12} < \frac{|\text{det}U|}{d}.
\]
Then
\[A - p_{11}T_1U - p_{12}T_2U = \begin{pmatrix}
r_{11} & r_{12} \\
c_{21} & c_{22}
\end{pmatrix},
\]
for some \( c_{21}, c_{22} \in \mathbb{Z} \). Again \( c_{21} = p_{21} - d + r_{21} \) for some \( p_{21}, r_{21} \in \mathbb{Z} \),
\[0 < r_{21} < d.
\]
Then
\[A - p_{11}T_1U - p_{12}T_2U - p_{21}T_3U = \begin{pmatrix}
r_{11} & r_{12} \\
r_{21} & r_{22}
\end{pmatrix},
\]
for some \( d_{22} \in \mathbb{Z} \). Finally \( d_{22} = p_{22} \frac{|\text{det}U|}{d} + r_{22} \) for some \( p_{22}, r_{22} \in \mathbb{Z}, \quad 0 < r_{22} < \frac{|\text{det}U|}{d},
\)
implies
\[A - p_{11}T_1U - p_{12}T_2U - p_{21}T_3U - p_{22}T_4U = \begin{pmatrix}
r_{11} & r_{12} \\
r_{21} & r_{22}
\end{pmatrix},
\]
where \( 0 < r_{11}, r_{21} < d, \quad 0 < r_{22}, r_{12} < \frac{|\text{det}U|}{d} \).

This proves that for any matrix \( A \in \text{Mat}_2(\mathbb{Z}) \) there exists \( R \in J_2 \) such that
\[A \equiv R(\text{mod. } U).
\]

(2) Assume that
\[\begin{pmatrix}
r_{11} & r_{12} \\
r_{21} & r_{22}
\end{pmatrix} \equiv \begin{pmatrix}
s_{11} & s_{12} \\
s_{21} & s_{22}
\end{pmatrix}(\text{mod. } U)
\]
where
\[0 < r_{11}, r_{21}, s_{11}, s_{21} < d, \quad 0 < r_{12}, r_{22}, s_{12}, s_{22} < \frac{|\text{det}U|}{d}.
\]
This implies

\[
U \begin{pmatrix}
  r_{11} - s_{11} & r_{12} - s_{12} \\
  r_{21} - s_{21} & r_{22} - s_{22}
\end{pmatrix}, \text{ i.e., } r_{11} - s_{11} \in I_0 \text{ or } d \mid r_{11} - s_{11}.
\]

Now \(0 < |r_{11} - s_{11}| < d\), so \(r_{11} = s_{11}\). It follows that

\[
U \begin{pmatrix}
  0 & r_{12} - s_{12} \\
  r_{21} - s_{21} & r_{22} - s_{22}
\end{pmatrix},
\]

i.e., \(r_{12} - s_{12} \in I_1\) or \(\frac{|\det U|}{d} \mid (r_{12} - s_{12})\). But \(0 < |r_{12} - s_{12}| < \frac{|\det U|}{d}\),

so that \(r_{12} = s_{12}\).

It follows that

\[
U \begin{pmatrix}
  0 & 0 \\
  r_{21} - s_{21} & r_{22} - s_{22}
\end{pmatrix}, \text{ i.e., } r_{21} - s_{21} \in I_0 \text{ or } d \mid (r_{21} - s_{21}).
\]

Also \(0 < |r_{21} - s_{21}| < d\), so that \(r_{21} = s_{21}\). This implies that

\[
U \begin{pmatrix}
  0 & 0 \\
  0 & r_{22} - s_{22}
\end{pmatrix},
\]

i.e., \(r_{22} - s_{22} \in I_1\) or \(\frac{|\det U|}{d} \mid (r_{22} - s_{22})\). Finally \(0 < |r_{22} - s_{22}| < \frac{|\det U|}{d}\),

so that \(r_{22} = s_{22}\), i.e.,

\[
\begin{pmatrix}
  r_{11} & r_{12} \\
  r_{21} & r_{22}
\end{pmatrix} = \begin{pmatrix}
  s_{11} & s_{12} \\
  s_{21} & s_{22}
\end{pmatrix}.
\]

This proves that any two elements in \(J_2\) are incongruent.

COROLLARY 3. Let \(U \in \text{Mat}_2(\mathbb{Z})\) with \(\det U \neq 0\). Then the cardinality of the complete residue system (mod. \(U\)) is \(|\det U|^2\).

REMARK. If we consider the ring of 3x3 matrices, the corresponding results will read as follows, the proofs will be as in Lemma 3 and Theorem 2, with possible minor changes.

LEMMA 4. Let \(u = (u_{ij}) \in \text{Mat}_3(\mathbb{Z})\) with \(\det U \neq 0\). Then

\[
I_o = \{a \in \mathbb{Z} : U \begin{pmatrix}
  a & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{pmatrix} \text{ for some } a_{ij} \in \mathbb{Z}\}.
\]
\[ I' = \{a \in \mathbb{Z} : U \begin{pmatrix} 0 & 0 & 0 \\ a & \alpha_{21} & \alpha_{22} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \text{ for some } \alpha_{ij} \in \mathbb{Z} \} \]

\[ I'' = \{a \in \mathbb{Z} : U \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a & \alpha_{32} & \alpha_{33} \end{pmatrix} \text{ for some } \alpha_{32}, \alpha_{33} \in \mathbb{Z} \} \]

are nonzero principal ideals of \( \mathbb{Z} \) generated by the positive integer \( g' = \text{g.c.d.}(u_{11}, u_{21}, u_{31}) \). Moreover, \( I_o = I'_o = I''_o \).

\[ I_2 = \{a \in \mathbb{Z} : U \begin{pmatrix} 0 & 0 & a \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \text{ for some } \alpha_{ij} \in \mathbb{Z} \}, \]

\[ I'_2 = \{a \in \mathbb{Z} : U \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \text{ for some } \alpha_{ij} \in \mathbb{Z} \}, \]

\[ I''_2 = \{a \in \mathbb{Z} : U \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix} \} \]

are nonzero principal ideals of \( \mathbb{Z} \) generated by the positive integer \( g_2 = \frac{|\text{det}U|}{g'} \), where \( g' = \text{g.c.d.}(\text{cofu}_{13}, \text{cofu}_{23}, \text{cofu}_{33}) \), and

\( \text{cofu}_{ij} \) is the cofactor of the element \( u_{ij} \). Moreover, \( I_2 = I'_2 = I''_2 \).

\[ I_1 = \{a \in \mathbb{Z} : U \begin{pmatrix} 0 & a_1 & a_1 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \text{ for some } \alpha_{ij} \in \mathbb{Z} \}, \]

\[ I'_1 = \{a \in \mathbb{Z} : U \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \text{ for some } \alpha_{ij} \in \mathbb{Z} \}, \]

\[ I''_1 = \{a \in \mathbb{Z} : U \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \text{ for some } a_{33} \in \mathbb{Z} \} \]
are nonzero principal ideals of \( \mathbb{Z} \) generated by the positive integer \( g_1 = \frac{R'}{g_0} \).

Moreover, \( I_1 = I_1' = I_1'' \).

**THEOREM 3.** Let \( \text{det} \mathbf{U} \neq 0 \), let

\[
\mathbf{U} = \begin{pmatrix}
  u_{11} & u_{12} & u_{13} \\
  u_{21} & u_{22} & u_{23} \\
  u_{31} & u_{32} & u_{33}
\end{pmatrix}
\in \text{Mat}_3(\mathbb{Z})
\]

\( g_0 = \text{g.c.d.}(u_{11}, u_{21}, u_{31}), \ g' = \text{g.c.d.}(\text{cofu}_{13}, \text{cofu}_{23}, \text{cofu}_{33}) \). Then

\( J_3 = \{ \mathbf{R} = [r_{ij}] \in \text{Mat}_3(\mathbb{Z}) : 0 \leq r_{ij} < g_{j-1} \ \text{i,j} = 1,2,3 \} \) is a complete residue system (mod. \( \mathbf{U} \)) where \( g_1 = \frac{R'}{g_0}, \ g_2 = \frac{|\text{det} \mathbf{U}|}{g_0} \).

**REFERENCE**
