

## AN EXAMPLE OF A BLOCH FUNCTION

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ABSTRACT. A Bloch function is exhibited which has radial limits of modulus one almost everywhere but fails to belong to  $H^p$ , for each  $0 < p \leq \infty$ .

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### 1. INTRODUCTION.

The purpose of this note is to give an example which seems to be useful in settling several questions about Bloch functions.

Let  $E$  be the subset of the complex plane  $\mathbb{C}$  consisting of the closed unit disc together with the Gaussian integers  $\mathbb{Z}^2$ . Let  $G$  be the complement of  $E$  in  $\mathbb{C}$ . Let  $g : D \rightarrow G$  be the analytic universal covering map of  $G$  given by the uniformization theorem ( $D$  denotes the unit disc).

PROPOSITION. The function  $g$  is an unbounded Bloch function with the properties

- (i)  $g$  has a radial limit  $g(e^{i\theta})$  at almost every point  $e^{i\theta}$  of the unit circle.

- (ii) the function  $g(e^{i\theta})$  is of modulus one almost everywhere on the unit circle,
- (iii)  $g$  is the reciprocal of a singular inner function, and so  $g$  does not belong to any  $H^p$  class.

Bloch functions on the unit disc may be defined as those analytic functions  $f$  on  $D$  for which the radii of the schlicht discs in the range of  $f$  are bounded above. The Bloch functions are somewhat analogous to functions in the disc algebra--Bloch functions can be characterized (see [1]) as those analytic functions which are uniformly continuous when  $D$  is given the hyperbolic metric and  $\mathbb{C}$  the Euclidean metric. Since Bloch functions may be characterized (see [1]) as those analytic functions  $f$  on  $D$  for which the quantity  $|f'(z)|(1 - |z|^2)$  is bounded for  $z \in D$ , it follows that the modulus of a Bloch function grows rather slowly--at most as fast as  $\log(1/(1 - |z|))$ . Because functions in the disc algebra and bounded functions have good boundary behaviour, it is natural to ask about boundary values of Bloch functions--in particular about radial boundary values. (It is shown in [4] that a Bloch function has a radial limit at a point of the unit circle if and only if it has a non-tangential limit there.)

In [5], Pommerenke gave an example of a Bloch function with radial limits almost nowhere. The example given here is constructed in a similar way, but it contrasts with Pommerenke's in that it shows that Bloch functions which have radial limits almost everywhere need not be particularly well-behaved.

The example answers a question posed by Joseph Cima (private communication). He asked whether a Bloch function which has radial limits

almost everywhere and has the additional property that the boundary function belongs to  $L^p$  need be in  $H^p$ . The function  $g$  provides a negative answer to this question since  $g(e^{i\theta}) \in L^\infty$  while  $g \notin H^p$  for any  $0 < p \leq \infty$ . In fact  $g$  does not belong to the class  $N^+$  (see [2] p. 25) which contains  $H^p$  for every  $p$ .

PROOF. It is evident that  $g$  is an unbounded Bloch function. Also, to verify properties (i), (ii) and (iii), it is clearly sufficient to verify (iii).

To establish (iii), consider the analytic function  $f = 1/g$  on  $D$ . The function  $f$  is bounded (by 1) and is the universal covering map:  $D \rightarrow D - K$ , where  $K$  is the countable set

$$\{0\} \cup \{1/(m+in) \mid m, n \in \mathbb{Z}, |m+in| > 1\}.$$

Being a bounded analytic function,  $f$  has radial limits almost everywhere on the unit circle. It is easy to see from the properties of covering maps that these radial limits are either of modulus 1 or else belong to  $K$ . To complete the proof that  $f$  is a singular inner function, it is only necessary to show that the radial limit  $f(e^{i\theta})$  belongs to  $K$  on a subset of the unit circle of measure zero.

But, for each  $k \in K$  it is true that the set of  $e^{i\theta}$  for which  $f(e^{i\theta}) = k$  has measure zero (see [2] p. 17). Since  $K$  is countable, it follows that the set of  $e^{i\theta}$  for which  $f(e^{i\theta})$  belongs to  $K$  also has measure zero. The proof is now complete.

The example may also be viewed as elucidating the almost total lack of relationships between the class  $\mathcal{B}$  of Bloch functions on  $D$  and the subclasses  $H^p$  and  $N^+$  of the Nevanlinna class  $N$  (see [2]). The only

containment which holds between  $\mathcal{B}$  and the other classes is the relation  $H^\infty \subseteq \mathcal{B}$ . It is known that  $H^p \not\subseteq \mathcal{B}$  for any  $0 < p < \infty$  and that  $\mathcal{B} \not\subseteq N$ . The example  $g$  given above belongs to  $\mathcal{B} \cap N$  but not to  $N^+$ . The fact  $\mathcal{B} \not\subseteq N$  is shown by the example of Pommerenke's [5] mentioned above.

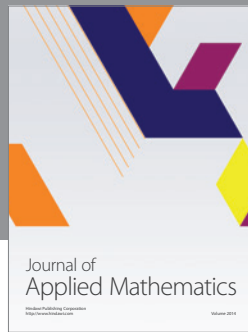
Finally, the example given here can be modified to show that there is no  $\delta > 0$  such that an analytic function  $f : D \rightarrow \mathbb{C}$  satisfying

$$f(e^{i\theta}) = \lim_{r \rightarrow 1} f(re^{i\theta}) = 1$$

almost everywhere on the unit circle must have a disc of radius  $\delta$  in its range. (Merely replace  $\mathbb{Z}^2$  by  $\delta\mathbb{Z}^2$  in the construction of  $g$ ). This answers a question raised by J.S. Hwang. By contrast, he showed (see [3]) that a singular inner function (for example) must have a (Schlicht) disc of radius at least  $2B/e$  in its range, where  $B$  denotes Bloch's constant.

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