Internat. J. Math. & Math. Sci.
Vol. 2 (1979) 1-13

CONVEXITY, BOUNDEDNESS, AND ALMOST PERIODICITY FOR DIFFERENTIAL EQUATIONS IN HILBERT SPACE

JEROME A. GOLDSTEIN
Department of Mathematics
Tulane University
New Orleans, Louisiana 70118 U.S.A.

(Received October 27, 1978)

ABSTRACT. There are three kinds of results. First we extend and sharpen a convexity inequality of Agmon and Nirenberg for certain differential inequalities in Hilbert space. Next we characterize the bounded solutions of a differential equation in Hilbert space involving and arbitrary unbounded normal operator. Finally, we give a general sufficient condition for a bounded solution of a differential equation in Hilbert space to be almost periodic.

KEY WORDS AND PHRASES. *Differential equations in Hilbert space, Convexity inequality, Self-adjoint operators, Bounded solutions, Almost periodic solutions.*

AMS(MOS) SUBJECT CLASSIFICATION (1970) CODES. *Primary* 34G05, 47A50 *Secondary* 34C25, 47B15.

1. INTRODUCTION. Let S_1, S_2 be two commuting self-adjoint operators on a complex Hilbert space H. Let $u : [a,b] \to H$ satisfy the inequality

$$\|du(t)/dt - (S_1 + iS_2)\, u(t)\| \le \phi(t)\, \|u(t)\| , \quad a \le t \le b , \tag{1.1}$$

where $\int_a^b \phi(t)dt \le c < 1/2$. We shall show that this implies the convexity inequality

$$\|u(t)\| \le K_c\, \|u(a)\|^{\frac{b-t}{b-a}}\, \|u(b)\|^{\frac{t-a}{b-a}} ,$$

which holds for some constant K_c and all $t \in [a,b]$. S. Agmon and L. Nirenberg [1] first proved this assuming $c = 2^{-3/2}$; recently S. Zaidman [7] extended it to weak solutions of (1.1). Our results apply to weak solutions and to the range of values $0 < c < 1/2$; moreover, we obtain a smaller constant K_c than did these previous authors. This result is presented in Section 2.

Section 3 is devoted to obtaining the structure of the set of all bounded solutions of

$$du(t)/dt = (S_1 + iS_2)u(t) \qquad (-\infty < t < \infty).$$

The results generalize and improve a recent result of Zaidman [8].

In Section 4 we study almost periodic solutions of the inhomogeneous equation

$$du(t)/dt = Au(t) + f(t) \qquad (-\infty < t < \infty) ;$$

here A is a closed linear operator on H and f is an H-valued function. Under a finite dimensionality assumption we show that bounded solutions are almost periodic. This generalizes the results obtained by Zaidman in [6].

2. A CONVEXITY THEOREM. Let u map the real interval $[a,b]$ into a complex Hilbert space H with inner product $\langle\cdot,\cdot\rangle$. Let $B : \mathcal{D}(B) \subset H \to H$ be a closed, densely defined linear operator. u is a *strong solution* of

$$\|du(t)/dt - Bu(t)\| \le \phi(t)\ \|u(t)\| \tag{2.1}$$

if u is continuously differentiable on $[a,b]$, takes values in $\mathcal{D}(B)$, and $f(t) \equiv du/dt - Bu$ satisfies $\|f(t)\| \le \phi(t)\ \|u(t)\|$, $a \le t \le b$. u is a *weak solution* of (2.1) if u is continuous and for continuously differentiable functions ψ with compact support in $]a,b[$ and with values in $\mathcal{D}(B^*)$, we have

$$-\int_a^b \langle u(t),\psi'(t)\rangle\, dt = \int_a^b \{\langle u(t),B^*\psi(t)\rangle + \langle f(t),\psi(t)\rangle\}dt\ ,$$

$$\|f(t)\| \le \phi(t)\ \|u(t)\|\ ,\quad a \le t \le b.$$

That a strong solution of (2.1) is a weak solution follows from an integration by parts.

THEOREM 2.1. *Let* $u : [a,b] \to H$ *be a weak solution of* (2.1) *where* B *is symmetric. If*

$$\int_a^b \phi(t)dt \le c < 1/2\ , \tag{2.2}$$

then the convexity inequality

$$\|u(t)\| \le K_c\ \|u(a)\|^{\alpha}\ \|u(b)\|^{1-\alpha}\ , \tag{2.3}$$

holds, where

$$\alpha = \frac{b-t}{b-a}\ ,\quad K_c = \left(\frac{2}{1-2c}\right)^{1/2}\ .$$

In particular, when $c = 1/2\sqrt{2}$, we get $K_c = (4 + 2\sqrt{2})^{1/2}$. Agmon and Nirenberg [1] proved this result for strong solutions, taking $c = 1/2\sqrt{2}$ and obtaining the constant $K_c = 2\sqrt{2}$ ($> (4 + 2\sqrt{2})^{1/2}$). This result also appears in Friedman's book [3, p.219]. Zaidman [7] extended the Agmon-Nirenberg result to weak solutions. The new features of Theorem 2.1 are (i) the result is extended to cover the case $\frac{1}{2\sqrt{2}} < c < \frac{1}{2}$, (ii) the constant K_c is sharpened for each value of c (including $c \leq 1/2\sqrt{2}$).

By enlarging the Hilbert space H, we can extend B to be a self-adjoint operator (cf. Sz.-Nagy [5]). Also, for S_1 and S_2 commuting self-adjoint operators (i.e., e^{itS_1} and e^{isS_2} commute for all real t and s), we may extend the theorem to the case where B is replaced by the (unbounded) normal operator $S_1 + iS_2$ according to the observation made in [1, p.138].

PROOF OF THEOREM 2.1. The proof follows Zaidman [7, pp. 236-244] with the following changes on pp. 242-244. We use Zaidman's notation. From

$$\|u(t)\|^2 \leq \|u_1(b)\|^2 + \|u_2(a)\|^2 + 2M \int_a^b \|f(s)\|\, ds$$

(cf. [7, p.242, line 3]) we get

$$\|u(t)\|^2 \leq \|u_1(b)\|^2 + \|u_1(a)\|^2 + \varepsilon M^2 + \varepsilon^{-1}\left(\int_a^b \|f(s)\|\, ds\right)^2$$

for each $\varepsilon > 0$; here $M = \sup \{\|u(s)\| : a \leq s \leq b\}$. This implies

$$M^2 \leq \beta + \varepsilon M^2 + \varepsilon^{-1} N^2$$

where $\beta = \|u(a)\|^2 + \|u(b)\|^2$, $N = \int_a^b \|f(s)\|\, ds$. Consequently

$$M^2 \leq (\beta + \varepsilon^{-1} N^2)(1 - \varepsilon)^{-1} \tag{2.4}$$

for $0 < \varepsilon < 1$. (This becomes [7, p.242, eqn. (*)] when $\varepsilon = 1/2$.) Since u is a weak solution of $u' - Bu = f$ (where $\|f(t)\| \le \phi(t)\,\|u(t)\|$), it follows that $\omega_\sigma(t) \equiv e^{\sigma t}\,u(t)$ is a weak solution of $\omega' - B_\sigma \omega_\sigma = e^{\sigma t}\,f(t)$ where $B_\sigma = B - \sigma I$ (cf. [7, Lemma 4, p.242]). Letting

$$M_\sigma = \sup\{\|e^{\sigma t}\,u(t)\|^2 : a \le t \le b\}\ ,$$

$$B_\sigma = \|e^{\sigma a}\,u(a)\|^2 + \|e^{\sigma b}\,u(b)\|^2\ ,$$

$$N_\sigma = \int_a^b \|e^{\sigma t}\,f(t)\|\,dt\ ,$$

we have that (2.4) (applied to ω_σ rather than u) yields

$$M_\sigma^{\,2} \le (\beta_\sigma + \varepsilon^{-1}\,N_\sigma^{\,2})(1 - \varepsilon)^{-1} \tag{2.5}$$

for all real σ and all ε, $0 < \varepsilon < 1$. But by (2.1) and (2.2),

$$\begin{aligned} N_\sigma &\le \int_a^b e^{\sigma t}\,\phi(t)\,\|u(t)\|\,dt \\ &\le \sup\{\|e^{\sigma s}\,u(s)\| : a \le s \le b\}\int_a^b \phi(t)\,dt \\ &\le M_\sigma\,c. \end{aligned}$$

Squaring this gives

$$N_\sigma^{\,2} \le M_\sigma^{\,2}\,c^2.$$

Plugging into (2.5) yields

$$M_\sigma^{\,2} \le (\beta_\sigma + \varepsilon^{-1}\,c^2\,M_\sigma^{\,2})(1 - \varepsilon)^{-1}$$

or

$$M_\sigma^{\,2} \le \frac{\varepsilon\,\beta_\sigma}{\varepsilon(1-\varepsilon) - c^2} \tag{2.6}$$

provided $0 < \varepsilon < 1$ and $\varepsilon(1 - \varepsilon) > c^2$, i.e., $0 < c < 1/2$ and $|2\varepsilon - 1| < (1 - 4c^2)^{1/2}$. As in [7, pp. 243, 244], $u(a) = 0$ or $u(b) = 0$ implies $u \equiv 0$, so to prove the theorem we may suppose $u(a) \neq 0$, $u(b) \neq 0$. Choosing $\sigma = (b - a)^{-1} \log(\|u(a)\| / \|u(b)\|)$ makes

$e^{\sigma t} = (\|u(a)\| / \|u(b)\|)^{\frac{t}{b-a}}$ and $\|e^{\sigma a} u(a)\| = \|e^{\sigma b} u(b)\|$. Thus (2.6) becomes, for all $t \in [a,b]$,

$$\left(\frac{\|u(a)\|}{\|u(b)\|}\right)^{\frac{2t}{b-a}} \|u(t)\|^2 \le L \left\{ \|u(a)\|^2 \left(\frac{\|u(a)\|}{\|u(b)\|}\right)^{\frac{2a}{b-a}} + \|u(b)\|^2 \left(\frac{\|u(a)\|}{\|u(b)\|}\right)^{\frac{2b}{b-a}} \right\}$$

$$= 2L \left(\frac{\|u(a)\|^{2b}}{\|u(b)\|^{2a}}\right)^{\frac{1}{b-a}}$$

where $L = \varepsilon(\varepsilon(1-\varepsilon) - c^2)^{-1}$. Consequently

$$\|u(t)\| \le (2L)^{1/2} \|u(a)\|^{\frac{b-t}{b-a}} \|u(b)\|^{\frac{t-a}{b-a}}$$

holds for $a \le t \le b$. Regard $g(\varepsilon) \equiv (2L)^{1/2} = \left(\dfrac{2\varepsilon}{\varepsilon(1-\varepsilon) - c^2}\right)^{1/2}$ as a function of ε. It is minimized when $\varepsilon = c$, in which case $(2L)^{1/2} = \left(\dfrac{2}{1-2c}\right)^{1/2}$. This is a legitimate choice of ε since $|2\varepsilon - 1| < (1 - 4c^2)^{1/2}$ holds in this case. The proof of the theorem is now complete.

3. BOUNDED SOLUTIONS. Let S_1, S_2 be commuting self-adjoint operators on H. We study functions $u \in C^1(\mathbb{R}, H)$ ($\mathbb{R} =]-\infty,\infty[$) which are bounded (strong) solutions of

$$du(t)/dt = (S_1 + iS_2)\,u(t)\ ,\quad t \in \mathbb{R}\,. \tag{3.1}$$

LEMMA 3.1. *Let* u *be a bounded solution of* (3.1). *Then* $u(t) = e^{itS_2}h$ *for all* $t \in \mathbb{R}$ *and some* $h \in \mathrm{Ker}(S_1) = \{f \in \mathcal{H} : S_1 f = 0\}$.

PROOF. Let $h = u(0)$. Then

$$u(t) = e^{tS_1}(e^{itS_2}h) = e^{itS_2}(e^{tS_1}h).$$

(Recall that e^{tS_1}, e^{itS_2} are defined by the operational calculus associated with the spectral theorem.) Since e^{itS_2} is unitary, $\|u(t)\| = \|e^{tS_1}h\|$ follows. But $\|e^{tS_1}h\|$ is bounded for $t \in \mathbb{R}$ if and only if $h \in \mathrm{Ker}(S_1)$, in which case $e^{tS_1}h = h$, and so $u(t) = e^{itS_2}h$, as advertised.

A special case occurs when

$$\mathrm{Ker}(S_1) = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_n\ ,$$

where S_2 restricted to $\mathcal{M}_j$ is a real constant λ_j times the identity on $\mathcal{M}_j$ for $1 \le j \le n$. Then any bounded solution of (3.1) is of the form

$$u(t) = \sum_{j=1}^{n} e^{it\lambda_j} h_j \tag{3.2}$$

where h_j is the orthogonal projection of $u(0)$ onto $\mathcal{M}_j$, $1 \le j \le n$. This covers the result obtained by Zaidman in [8]. More precisely, let $\{E(\theta) : \theta \in \mathbb{R}\}$ be a resolution of the identity and let

$$S_1 = \int_{-\infty}^{\infty} x(\theta)\,dE(\theta)\ ,\quad S_2 = \int_{-\infty}^{\infty} y(\theta)\,dE(\theta)$$

be associated commuting self-adjoint operators, where x and y are continuous real functions on $\mathbb{R}$. If the zero set of x is the finite set $\{\theta_1,\ldots,\theta_n\}$ then S_2 is $\lambda_j = y(\theta_j)$ times the identity on $M_j = (E(\theta_j^+) - E(\theta_j^-))(H)$, $1 \le j \le n$, and so any bounded solution of (3.1) is of the form (3.2) with $h_j \in M_j$, $1 \le j \le n$. This is Zaidman's result [8].

4. ALMOST PERIODIC SOLUTIONS.

THEOREM 4.1. *Let* $A : H \to H$ *be a bounded linear operator and let* $f : \mathbb{R} \to H$ *be almost periodic. Let* $u : \mathbb{R} \to H$ *be a bounded (i.e.* $\sup\{\|u(t)\| : t \in \mathbb{R}\} < \infty$*) strong solution of*

$$du(t)/dt = Au(t) + f(t) \qquad (t \in \mathbb{R}). \tag{4.1}$$

$$\left.\begin{array}{l}\textit{Suppose there is a finite dimensional subspace } H_1 \textit{ of } H\\ \textit{such that } H_1 \supset \{Af(s) : s \in \mathbb{R}\} \cup \{Au(0)\} \textit{ and}\\ e^{tA}(H_1) \subset H_1 \textit{ for all } t \in \mathbb{R}.\end{array}\right\} \tag{4.2}$$

Then u *is almost periodic.*

When H is finite dimensional, this is the classical Bohr-Neugebauer-Bochner theorem (cf. Amerio-Prouse [2, p.85]). When A is a finite rank operator we can take H_1 to be the range of A, and Theorem 4.1 becomes the theorem of Zaidman [6] in this case.

PROOF OF THEOREM 4.1. Let $H_2 = H \ominus H_1$ be the orthogonal complement of H_1, and let P_j be the orthogonal projection onto H_j, $j = 1,2$. Let $u_j(t) = P_j u(t)$, $j = 1,2$. Note that if L is as upper bound for $\|u(s)\|$ $(s \in \mathbb{R})$, then for all real t,

$$L^2 \ge \|u(t)\|^2 = \|u_1(t)\|^2 + \|u_2(t)\|^2,$$

whence u_1 and u_2 are bounded. Also,

$$du/dt = du_1/dt + du_2/dt = Au_1 + Au_2 + P_1 f + P_2 f \ .$$

Applying P_1 to both sides gives

$$du_1/dt = P_1 Au_1 + P_1 Au_2 + P_1 f \ . \tag{4.3}$$

The function u admits the variation of parameters representation

$$\begin{aligned} u(t) &= e^{tA}u(0) + \int_0^t e^{(t-s)A} f(s)\, ds \\ &= e^{tA}u(0) + \int_0^t f(s)\, ds + \sum_{n=1}^{\infty} \int_0^t \frac{(t-s)^n}{n!} A^n f(s)\, ds \ . \end{aligned}$$

The last (summation) term belongs to H_1 by (4.2). Applying P_2 to this expression gives

$$u_2(t) = P_2\, e^{tA}u(0) + \int_0^t P_2\, f(s)\, ds \ ;$$

differentiating yields

$$du_2(t)/dt = P_2\, e^{tA}Au(0) + P_2\, f(t) = P_2\, f(t)$$

by (4.2). Since f is almost periodic and P_2 is bounded it follows that du_2/dt is almost periodic. Since u_2 is bounded, u_2 itself is almost periodic (see [2, p.55]).

Next, by (4.3),

$$du_1(t)/dt = P_1 Au_1(t) + g(t) \ , \tag{4.4}$$

where $g(t) \equiv P_1 Au_2(t) + P_1 f(t)$ is almost periodic. Since u_1 is bounded and $P_1 A : H_1 \to H_1$ is linear, (4.4) is a linear system in the finite dimensional Hilbert space H_1 (see (4.2)). It follows from the classical Bohr-Neugebauer-Bochner theorem [2] that u_1 is almost periodic. Consequently $u = u_1 + u_2$ is almost periodic, and the proof is complete.

Theorem 4.1 can be easily extended to the case when A is unbounded, as follows.

THEOREM 4.2. *Let* $A : \mathcal{D}(A) \subset H \to H$ *generate a* (C_0) *group of bounded linear operators* $\{T(t) : t \in \mathbb{R}\}$ *on* H (cf. [4]). *Let* $u : \mathbb{R} \to H$ *be a bounded solution of* (4.1) *where* f *is almost periodic. Suppose there is a finite dimensional subspace* H_1 *of* H *such that*

$$H_1 \supset \{(T(t) - I) f(s) : s \in \mathbb{R}, t \in \mathbb{R}\} \cup \{Au(0)\}$$

and $T(t)(H_1) \subset H_1$ *for all* $t \in \mathbb{R}$. *Then* u *is almost periodic.*

The proof, which differs from the proof of Theorem 4.1 only in inessential ways, is omitted.

COROLLARY 4.3. *Let* $\lambda_1, \ldots, \lambda_n$ *be eigenvalues of the linear operator* $A : \mathcal{D}(A) \subset H \to H$ *and let* $\phi_1, \ldots, \phi_n$ *be corresponding eigenvectors. Let* H_1 *be the span of* $\phi_1, \ldots, \phi_n$. *Then any bounded solution of* (4.1) *is almost periodic, provided* $f : \mathbb{R} \to H_1$ *is almost periodic and* $u(0) \in H_1$.

This follows immediately from Theorem 4.2.

COROLLARY 4.4. *In Corollary* 4.3 *one can omit the hypothesis that* $u(0) \in H_1$ *provided that one assumes that* A *is a compact normal operator.*

PROOF. Let P_1, P_2, u_1, u_2 be as in the proof of Theorem 4.1. Applying P_j to (4.1) and noting that A commutes with P_j in this case gives

$$du_1(t)/dt = Au_1(t) + f(t) ,$$

$$du_2(t)/dt = Au_2(t) \qquad (t \in \mathbb{R}) . \tag{4.5}$$

u_1 is almost periodic by the Bohr-Neugebauer-Bochner theorem. Thus it only remains to show that u_2 is almost periodic. Let B be the restriction of A to H_2. B is a compact normal operator, hence by the spectral theorem there is an orthonormal basis $\{\psi_m\}$ for H_2 and complex numbers $\mu_m \to 0$ such that

$$B\phi = \sum_{m=1}^{\infty} \mu_m <\phi,\psi_m> \psi_m$$

for all $\phi \in H_2$. Let Q_m be the orthogonal projection (in H_2) onto the span of $\psi_1, \ldots, \psi_m$. Let $v_m = Q_m u_2$. Then

$$dv_m/dt = Q_m\, du_2/dt = Q_m\, Au_2 = Bv_m$$

by (4.5). Also, v_m is bounded (since u_2 is) and takes values in a finite dimensional space, whence v_m is almost periodic. We *claim* that $u_2(t) = \lim_{m\to\infty} v_m(t)$, uniformly for $t \in \mathbb{R}$. It then follows that u_2 is almost periodic [2] and the proof is done. So it only remains to prove the *claim*. We have

$$\frac{d}{dt} (u_2(t) - v_m(t)) = B(u_2(t) - v_m(t)) = (B - Q_m B)(u_2(t) - v_m(t)) ,$$

therefore

$$u_2(t) - v_m(t) = \sum_{k=m+1}^{\infty} e^{t\mu_k} <(u_2 - v_m)(0), \psi_k> \psi_k .$$

Consequently

$$||u_2(t) - v_m(t)||^2 = \sum_{k=m+1}^{\infty} e^{t \,\mathrm{Re}\, \mu_k} |<(u_2 - v_m)(0), \psi_k>|^2 .$$

Since $\|u_2(t) - v_m(t)\| \le \|u_2(t)\| \le L < \infty$ for some L and all $t \in \mathbb{R}$, it follows that for every k for which $\langle (u_2 - v_m)(0), \psi_k \rangle \neq 0$ for some m, μ_k must be purely imaginary. Therefore

$$\|u_2(t) - v_m(t)\|^2 = \sum_{k=m+1}^{\infty} |\langle (u_2 - v_m)(0), \psi_k \rangle|^2$$

$$= \|(I - Q_m)u_2(0)\|^2 \to 0$$

as $n \to \infty$, uniformly for $t \in \mathbb{R}$. Q.E.D.

ACKNOWLEDGEMENT: This research is an outgrowth of discussions I had with Samuel Zaidman in April, 1978 when I had the pleasure to visit the Université de Montréal. I thank Professor Zaidman for some stimulating dicussions, and I gratefully acknowledge that my trip to Montréal was supported by his NRC grant. Finally, I gratefully acknowledge the support of an NSF grant.

REFERENCES

[1] Agmon, S. and L. Nirenberg, Properties of solutions of ordinary differential equations in Banach spaces, Comm. Pure Appl. Math. 16 (1963), 121-239.

[2] Amerio, L. and G. Prouse, Almost-Periodic Functions and Functional Equations, Van Nostrand Rheinhold, New York, 1971.

[3] Friedman, A., Partial Differential Equations, Holt, Rinehart and Winston, New York, 1969.

[4] Hille, E. and R. S. Phillips, Functional Analysis and Semi-Groups, Amer. Math. Soc. Colloq. Publ. 31, Providence, R. I., 1957.

[5] Sz.-Nagy, B., Extensions of linear transformations in Hilbert space which extend beyond the space, appendix to F. Riesz and B. Sz.-Nagy, Functional Analysis, Ungar, New York, 1960.

[6] Zaidman, S., Bohr-Neugebauer theorem for operators of finite rank in Hilbert spaces, Atti Acad. Sci. Torino 109 (1974/1975), 183-185.

[7] Zaidman, S., A convexity result for weak differential inequalities, Canad. Math. Bull. 19 (1976), 235-244.

[8] Zaidman, S., Structure of bounded solutions for a class of abstract differential equations, Ann. Univ. Ferrara 22 (1976), 43-47.