

THE STIELTJES TRANSFORM OF DISTRIBUTIONS

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ABSTRACT. In the present work, two complex inversion formulas of Byrne and Love for generalized Stieltjes transformation are shown to be valid for a class of distributions. This is accomplished by transferring the complex inversion formulas on the testing function space of a class of distributions and then showing that the limiting process in the resulting formula converges in the topology of the testing function space.

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1. INTRODUCTION.

Let p be any complex number except zero and the negative integers. Then

for all s in the "cut plane", that is all complex numbers except those which are negative real or zero, the Stieltjes Transform in its general form is defined by:

$$F(s) = \int_0^{\infty} \frac{f(t)dt}{(s+t)^p} \quad (1.1)$$

The following inversion theorems for particular values of p and s are well known.

THEOREM A (Widder). If $f(t)$ belongs to $L(0,R)$ for every positive R and is such that the integral

$$F(x) = \int_0^{\infty} \frac{f(t)dt}{t+x}$$

converges for $x > 0$, then $F(s)$ exists for complex s in the cut plane and

$$\lim_{\eta \rightarrow 0+} \frac{f(-\xi - i\eta) - F(-\xi + i\eta)}{2\pi i} = \frac{f(\xi+) + f(\xi-)}{2}$$

for any positive ξ at which $f(\xi+)$ and $f(\xi-)$ both exist.

THEOREM B (Sumner). If $p > 0$, $f(t)$ is locally integrable in $[0, \infty]$, the improper Lebesgue integral.

$$F(s) = \int_0^{\infty} \frac{f(t)dt}{(s+t)^p}$$

converges (for a certain value of s in the cut plane and so for all), $t > 0$ and the limits $f(t+0)$ exist, then:

$$\frac{1}{2} [f(t+0) + f(t-0)] = \lim_{\eta \rightarrow 0+} \frac{-1}{2\pi i} \int_0^t dx \int_{C_{\eta x}} (x+z)^{p-1} F'(z) dz$$

where $C_{\eta x}$ is a contour in the cut plane from $-x-i\eta$ to $-x+i\eta$.

THEOREM 1.3 (Byrne and Love). If $\text{Re } p > 1$, f is locally integrable in $[0, \infty)$, improper Lebesgue integral (1.1) converges, and $\lambda > 0$; then, for each positive x for which the Lebesgue limits $f(x \pm 0)$ exists,

$$\frac{1}{2} \{f(x+0) + f(x-0)\} = \lim_{\eta \rightarrow 0^+} \frac{p-1}{2\pi i} \int_{-x}^{\lambda} (x+t)^{p-2} \{F(t-i\eta) - F(t+i\eta)\} dt. \quad [2, p. 349]$$

THEOREM 1.4 (Byrne and Love). If $\text{Re } p > 1$, $\frac{f(t)}{1+t} \in L(0, \infty)$ and the improper Lebesgue integral

$$F(s) = \int_0^{\infty} \frac{f(t)}{(s+t)^p} dt$$

converges, then for each positive x for which the Lebesgue limit $f(x \pm 0)$ exists,

$$\frac{1}{2} \{f(x+0) + f(x-0)\} = \lim_{\eta \rightarrow 0^+} \int_{-x}^{\infty} (x+t)^{p-2} \{F(t-i\eta) - F(t+i\eta)\} dt \quad [2, p. 352]$$

THEOREM 1.1 has been extended to distributions by Pandey and Zemanian [13] and Pandey [14]. **Theorem 1.2** was extended to distribution by Pathak [15]. Our object is to extend theorems 1.3 and 1.4 of Byrne and Love to generalized functions (distributions).

2. THE TESTING FUNCTION SPACE, $S_{\alpha}(I)$ AND ITS DUAL.

An infinitely differentiable complex valued function $\phi(x)$ defined over $I = (0, \infty)$ belongs to the testing function spaces $S_{\alpha}(I)$ if,

$$\gamma_k(\phi) = \sup_{0 < x < \infty} \left| (1+x)^{\alpha} x^k \left(\frac{d}{dx} \right)^k \phi(x) \right| < \infty$$

for all $k = 0, 1, 2, \dots$, where α is a fixed real number. Clearly, $S_\alpha(I)$ is a vector space with respect to the field of complex numbers. The zero element of the vector space $S_\alpha(I)$ is the function defined over I which is identically zero. The topology over $S_\alpha(I)$ is generated by the collection of seminorms $\{\gamma_k\}_{k=0}^\infty$ [24; p. 8]. We say that a sequence $\{\phi_\nu\}$ where ϕ_ν belongs to $S_\alpha(I)$ converges in $S_\alpha(I)$ to $\phi(x)$ if for each fixed k , $\gamma_k(\phi_\nu - \phi)$ tends to zero as ν tends to ∞ . The space $S_\alpha(I)$ is a locally convex Hausdorff topological vector space. The space $D(I)$ is a vector subspace of $S_\alpha(I)$ and the topology of $D(I)$ is stronger than the topology induced on $D(I)$ by $S_\alpha(I)$ and as such the restriction of any member of $S'_\alpha(I)$ to $D(I)$ is in $D'(I)$, where $S'_\alpha(I)$ and $D'(I)$ denote the dual spaces of $S_\alpha(I)$ and $D(I)$ respectively. We say that a sequence $\{\phi_\nu\}_{\nu=1}^\infty$ where $\phi_\nu(x)$ belongs to $S_\alpha(I)$ is a Cauchy sequence in $S_\alpha(I)$ if $\gamma_k(\phi_\nu - \phi_\mu)$ goes to zero for any non-negative integer k as μ and ν both tend to infinity independently of each other. It can be readily seen that $S_\alpha(I)$ is sequentially complete.

3. THE DISTRIBUTIONAL STIELTJES TRANSFORMATION

For a complex s not negative or zero, $\frac{1}{(s+x)^p}$ belongs to S'_α where $a < \text{Re } p$. Therefore, the distributional Stieltjes transformation $F(s)$ of an arbitrary element $f \in S'_\alpha$, $a < \text{Re } p$, is defined by

$$F(s) \triangleq \langle f(x), \frac{1}{(s+x)^p} \rangle \tag{3.1}$$

where s belongs to the complex plane cut along the negative real axis including the origin.

THEOREM 3.1. If m and k both assume non-negative integral values and Ω is a

compact set of the complex plane not meeting the negative real axis, then for fixed non-negative integers m and k , there exists a constant B_Ω satisfying

$$\gamma_m \left[\frac{1}{(s+x)^{p+k}} \right] \leq B_\Omega < \infty$$

uniformly for all s lying in the compact set Ω of the complex plane not meeting the negative real axis on the origin.

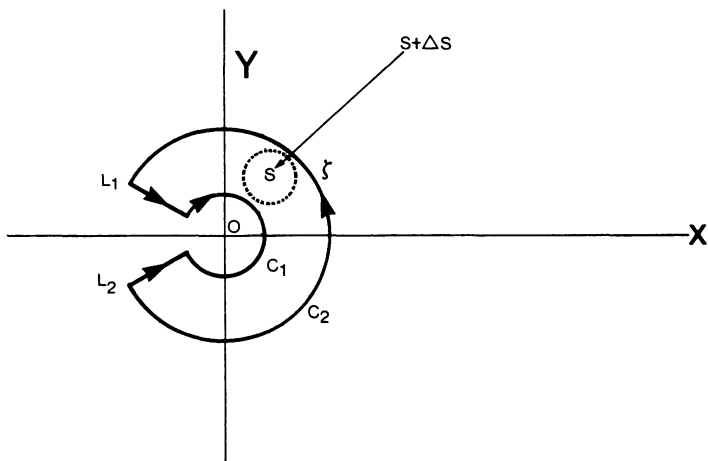
PROOF. Using the compactness of the set Ω and the fact that $\frac{1}{(s+x)^{p+k}} \in S'_\alpha$; $\alpha < \operatorname{Re} p$, the theorem is immediate.

THEOREM 3.2. For an arbitrary $f \in S'_\alpha$ and $a < \operatorname{Re} p$ let $F(s)$ be defined by the equation (3.1). Then, for $m = 1, 2, \dots$

$$\left(\frac{d}{ds} \right)^m F(s) = \langle f(x), \frac{(-1)^m (p)_m}{(s+x)^{p+m}} \rangle \quad (3.2)$$

where $(p)_m = p(p+1)(p+2) \dots (p+m-1)$.

PROOF. If p is such that s^p does not have a branch cut in the complex plane, the proof can be given in a way similar to that given in [3, Lemma 2a]. If p is such that s^p has a branch cut (along the negative real axis for the sake of definiteness), then choose the contour of integration as shown in the complex-plane cut along the negative real axis below.



Here, C_1 and C_2 are arcs of two concentric circles with centre at origin and C is the contour of integration as shown. The radii of C_1 and C_2 and paths L_1 and L_2 are so chosen that the point s is contained in the region bounded by the contour.

Let $d = \inf_{\xi \in C} |\xi - s|$ and choose $|\Delta s| < \frac{d}{2}$. Now

$$\begin{aligned} \frac{F(s+\Delta s) - F(s)}{\Delta s} &= \langle f(t), \frac{-p}{(s+t)^{p+1}} \rangle \\ &= \langle f(t), \frac{1}{\Delta s} \cdot \frac{1}{2\pi i} \cdot \int_C \frac{1}{(z+t)^p} \left[\frac{1}{z-s-\Delta s} - \frac{1}{z-s} - \frac{\Delta s}{(z-s)^2} \right] dz \rangle \end{aligned}$$

where C is the contour shown in the diagram

$$= \langle f(t), \theta_{\Delta s} \rangle \tag{3.3}$$

where

$$\theta_{\Delta s} = \frac{\Delta s}{2\pi i} \int_C \frac{1}{(z+t)^p} \frac{1}{(z-s)^p(z-s-\Delta s)} dz$$

We now wish to show that $\theta_{\Delta s} \rightarrow 0$ in $S_\alpha(I)$ as $\Delta s \rightarrow 0$.

Using Theorem 3.1, we have

$$\gamma_k(\theta_{\Delta s}) \leq B_c \frac{|\Delta s|}{2\pi} \frac{L}{d^3}, \tag{3.4}$$

where L is the length of the contour C and B_c is the uniform bound of

$$\frac{(p)_k (1+t)^{\alpha} t^k}{(z+t)^{p+k}} \quad \text{for all } z \text{ lying on the closed contour } C \text{ and } t > 0.$$

Letting $\Delta s \rightarrow 0$ in (3.3) and using (3.4), we get

$$F'(s) = \langle f(t), \frac{-p}{(s+t)^{p+1}} \rangle$$

Now; the theorem follows from induction on the order of the derivative of $F(s)$.

THEOREM 3.3. The function $F^{(m)}(x)$ for real x where $F(s)$ is the Stieltjes transform of $f \in S'_{\alpha}$, satisfies the following relation:

$$F^{(m)}(x) = \begin{cases} O[x^{-m}] \text{ as } x \rightarrow \infty \text{ if } \alpha < \text{Re } p \\ O[x^{-k}] \text{ as } x \rightarrow \infty \text{ if } \alpha = \text{Re } p \\ O[x^{-k-\text{Re } p}] \text{ as } x \rightarrow 0 \text{ if } \alpha \leq \text{Re } p \end{cases}$$

The proof is immediate from the boundedness property of distributions [9, p. 18].

4. COMPLEX INVERSION THEOREMS

We are now ready to prove our first inversion theorem.

THEOREM 4.1. For a fixed $\alpha < 1$ and $\text{Re } p > 1$, let $f \in S'_{\alpha}(I)$ and let $F(s)$ be the Stieltjes transform of $f(t)$ as defined by (3.1). Then,

$$\lim_{\eta \rightarrow 0} \langle \frac{p-1}{2\pi i} \int_{-x}^{\infty} (x+t)^{p-2} [F(t-i\eta) - F(t+i\eta)] dt, \phi(x) \rangle = \langle f, \phi \rangle \text{ for all } \phi \in D(I)$$

PROOF. First consider

$$\int_{-x}^{\lambda} (x+t)^{p-2} F(t-i\eta) dt = \int_{-x}^{\lambda} (x+t)^{p-2} \langle f(y), \frac{1}{(y+t-i\eta)^p} \rangle dt \tag{4.1}$$

For fixed x and t ,

$$\frac{(x+t)^{p-2}}{(y+t-i\eta)^p} \in S_\alpha(I).$$

Since, in view of Theorem 3.2, $F(s)$ is analytic in the cut plane, the left-hand side integral in (4.1) is meaningful. By using the technique of Riemann sums it can be shown that for $\epsilon > 0$,

$$\begin{aligned} & \int_{-x+\epsilon}^\lambda (x+t)^{p-2} F(t-i\eta) dt \\ &= \langle f(y), \int_{-x+\epsilon}^\lambda \frac{(x+t)^{p-2}}{(y+t-i\eta)^p} dt \rangle \\ &= \langle f(y), \frac{1}{p-1} \cdot \frac{1}{(x-y+i\eta)} \left[\frac{(x+t)^\lambda}{(y+\lambda-i\eta)^{p-1}} - \frac{\epsilon^{p-1}}{(y-x+\epsilon-i\eta)^{p-1}} \right] \rangle \\ &= I \text{ (say).} \end{aligned}$$

[by Lemma 5,1, p. 333]

One can easily check that as $\epsilon \rightarrow 0+$, $\frac{\epsilon^{p-1}}{(y-x+\epsilon-i\eta)^{p-1}} \rightarrow 0$ in $S_\alpha(I)$ for $\text{Re } p > 1 > \alpha$

and for fixed λ, η and x . Therefore, letting $\epsilon \rightarrow 0$, we get:

$$\int_{-x}^\lambda (x+t)^{p-2} F(t-i\eta) dt = \langle f(y), \frac{(\lambda+x)^{p-1}}{(p-1)(y-x-i\eta)(y+\lambda-i\eta)^{p-1}} \rangle \tag{4.2}$$

In view of Lemma 3.5* [7, p. 12], it follows that: as $\lambda \rightarrow \infty$

$$\frac{1}{(y-x-i\eta)} \left(\frac{\lambda+x}{y+\lambda-i\eta} \right)^{p-1} \rightarrow \frac{1}{y-x-i\eta} \text{ in } S_\alpha(I) \tag{4.3}$$

for fixed x and η .

Therefore, letting $\lambda \rightarrow \infty$ in (4.2), we obtain

$$\int_{-x}^\infty (x+t)^{p-2} F(t-i\eta) dt = \langle f(y), \frac{1}{p-1} \cdot \frac{1}{y-x-i\eta} \rangle \tag{4.4}$$

* The proof was provided by Professor E.R. Love.

Using a similar argument, we can show that

$$\int_{-x}^{\infty} (x+t)^{p-2} F(t+i\eta) dt = \langle f(y), \frac{1}{p-1} \cdot \frac{1}{y-x+i\eta} \rangle \tag{4.5}$$

Combining Equations (4.4) and (4.5), we get

$$\begin{aligned} & \frac{p-1}{2\pi i} \int_{-x}^{\infty} (x+t)^{p-2} [F(t-i\eta) - F(t+i\eta)] dt \\ &= \langle f(y), \frac{\eta}{\pi[(y-x)^2 + \eta^2]} \rangle \end{aligned} \tag{4.6}$$

Now using the technique of Riemann sums, we obtain

$$\begin{aligned} & \langle \frac{p-1}{2\pi i} \int_{-x}^{\infty} (x+t)^{p-2} [F(t-i\eta) - F(t+i\eta)] dt, \varphi(x) \rangle \\ &= \langle f(y), \int_a^b \frac{\eta \varphi(x) dx}{\pi[(y-x)^2 + \eta^2]} \rangle \end{aligned} \tag{4.7}$$

where the support of $\varphi(x) \in D(I)$ is contained in (a,b) , $b > a > 0$. Using the same techniques as followed in proving Theorem 2 of (3) one can show that

$$\frac{1}{\pi} \int_a^b \frac{\eta \varphi(x) dx}{(y-x)^2 + \eta^2} \rightarrow \varphi(y) \tag{4.8}$$

in the topology of $S_{\alpha}(I)$ as $\eta \rightarrow 0+$. Therefore, letting $\eta \rightarrow 0+$ in (4.7), we have

$$\lim_{\eta \rightarrow 0+} \langle \frac{p-1}{2\pi i} \int_{-x}^{\infty} (x+t)^{p-2} [F(t-i\eta) - F(t+i\eta)] dt, \varphi(x) \rangle = \langle f, \varphi \rangle$$

This completes the proof of the theorem.

To prove our other inversion theorems, we require a couple of Lemmas.

LEMMA 4.2. Let $t, s, \eta > 0$. Then, for finite $b > a > 0$ and $\alpha < 1$,

$$\lim_{\eta \rightarrow 0+} (1+x)^{\alpha} \eta \int_a^b \frac{|t-x|}{(t-x)^2 + \eta^2} dt = 0$$

uniformly for all $x > 0$.

PROOF. Let

$$I = (1+x)^{\alpha\eta} \int_a^b \frac{|t-x| dt}{(t-x)^2 + \eta^2}$$

Since $\sup_{\substack{x > N > b \\ a \leq t \leq b}} \left| (1+x)^{\alpha} \frac{(t-x)}{(t-x)^2 + \eta^2} \right|$ is bounded, for $\epsilon > 0$, there exists a

positive $N > b$ and $0 < q < 1$ such that

$$|I| < \epsilon \quad (4.9)$$

uniformly for all $\eta \in (0, q)$ and $x > N$.

Now assume that δ is a positive number $\leq \min(1, \frac{a}{2})$ and for $\delta \leq x \leq N$

let us write.

$$I = (1+x)^{\alpha\eta} \left(\int_a^{a+x-\delta} + \int_{a+x-\delta}^{a+x+\delta} + \int_{a+x+\delta}^b \right) \frac{|t-x|}{(t-x)^2 + \eta^2} dt. \quad (4.10)$$

Denote the three expressions on the right-hand side of Eqn. (4.10) by I_1 , I_2 and I_3 respectively.

Now

$$\begin{aligned} I_2 &= \eta \int_{a+x-\delta}^{a+x+\delta} \frac{|t-x| (1+x)^{\alpha}}{(t-x)^2 + \eta^2} dt \\ &\leq \eta 2\delta \frac{1}{2\eta} (1+N)^{\alpha} = \delta (1+N)^{\alpha} \end{aligned}$$

Now choose δ such that $\delta(1+N)^{\alpha} < \epsilon$ and fix δ this way. Therefore

$$|I_2| < \epsilon \quad (4.11)$$

uniformly for all $x \in [\delta, N]$ and $\eta \in [0, q]$.

$$I_3 = \eta \int_{a+x+\delta}^b \frac{(1+x)^{\alpha} |t-x|}{(t-x)^2 + \eta^2} dt = (1+x)^{\alpha\eta} \left[\int_{a+x+\delta}^x + \int_x^b \right] \frac{|t-x|}{(t-x)^2 + \eta^2} dt$$

Therefore,

$$I_3 = \begin{cases} = \frac{\eta}{2} \ln \left(\frac{(b-x)^2 + \eta^2}{(a+\delta)^2 + \eta^2} \right) (1+x)^\alpha, & \text{if } \delta < x < b \\ = (1+x)^\alpha \frac{\eta}{2} \ln \left(\frac{\eta^4}{[(a+\delta)^2 + \eta^2][(b-x)^2 + \eta^2]} \right) & \text{if } x \geq b. \end{cases}$$

Therefore, $I_3 \rightarrow 0$ as $\eta \rightarrow 0+$ (4.12)

uniformly for all $x \in [\delta, N]$.

Next,

$$I_1 = (1+x)^\alpha \eta \int_a^{a+x-\delta} \frac{|t-x| dt}{(t-x)^2 + \eta^2}$$

Now for $\delta \leq x \leq a$

$$I_1 = (1+x)^\alpha \frac{\eta}{2} \ln \left(\frac{(a-\delta)^2 + \eta^2}{(a-x)^2 + \eta^2} \right) \rightarrow 0 \text{ as } \eta \rightarrow 0+ \tag{4.13}$$

uniformly for all x lying in $[\delta, a]$.

For $a \leq x \leq N$,

$$I_1 = \left[-\frac{\eta}{2} \ln \left(\frac{\eta^2}{(a-x)^2 + \eta^2} \right) + \frac{\eta}{2} \ln \left(\frac{(a-\delta)^2 + \eta^2}{\eta^2} \right) \right] (1+x)^\alpha$$

$\rightarrow 0$ as $\eta \rightarrow 0+$ uniformly for all $x \in [a, N]$ (4.14)

Combining results (4.11) through (4.14) we have

$$\overline{\lim}_{\eta \rightarrow 0+} |I| \leq \epsilon \text{ uniformly for all } x \geq 0 \tag{4.15}$$

Now, for $0 < x \leq \delta$, we can see that

$$|I| \leq (1+\delta)^\alpha \eta \ln \left[\frac{(b-x)^2 + \eta^2}{(a-x)^2 + \eta^2} \right] \rightarrow 0 \tag{4.16}$$

As $\eta \rightarrow 0+$ uniformly for all $x \in [0, \delta]$.

Combining results (4.15) and (4.16), we get

$$\overline{\lim}_{\eta \rightarrow 0^+} |I| \leq \epsilon \quad \text{uniformly for all } x > 0.$$

Thus, the proof of the lemma is complete.

LEMMA 4.3. Let $\text{Re } p > 1 > \alpha$. Assume that t, x, λ and η are all positive numbers and $\bar{\varphi}(t) \in D(I)$. Then,

$$I(x, t) \triangleq \frac{1}{2\pi i} \int_a^b \left\{ \frac{1}{x-t+i\eta} \left[\left(\frac{t+\lambda}{x+\lambda+i\eta} \right)^{p-1} - 1 \right] - \frac{1}{x-t-i\eta} \left[\left(\frac{t+\lambda}{x+\lambda-i\eta} \right)^{p-1} - 1 \right] \right\} \bar{\varphi}(t) dt \rightarrow 0$$

as $\eta \rightarrow 0^+$ in the topology of S_α where the support of $\bar{\varphi}(t) \in D(I)$ is contained in (a, b) ; $b > a > 0$.

PROOF. We have to show that for each $m = 0, 1, 2, \dots$ $(1+x)^\alpha x^m D_x^m I(x, \eta) \rightarrow 0$ as $\eta \rightarrow 0$ uniformly.

It can be easily shown that

$$D_x^m I(x, \eta) = O\left(\frac{1}{x^{m+1}}\right) \quad \text{as } x \rightarrow \infty$$

uniformly for all η satisfying $0 < \eta < 1$. So,

$$(1+x)^\alpha x^m D_x^m I(x, \eta) \rightarrow 0 \quad \text{as } x \rightarrow \infty$$

uniformly for all $\eta \in (0, 1)$. Therefore, for $\epsilon > 0$ there exists $N > 0$ such that

$$|(1+x)^\alpha x^m D_x^m I(x, \eta)| < \epsilon \tag{4.17}$$

uniformly for all $x \geq N$, and $0 < \eta < 1$.

Now consider the case $0 < x \leq \eta$. We will first give the proof for $m = 0$ and complete the proof for $m = 1, 2, 3, \dots$ by using the result for the case $m = 0$.

I. For $m = 0$. Write

$$\begin{aligned} (1+x)^\alpha I(x,\eta) &= \frac{(1+x)^\alpha}{2\pi i} \int_a^b \frac{x-t}{(x-t)^2 + \eta^2} \left[\left(\frac{t+\lambda}{x+\lambda+i\eta} \right)^{p-1} - \left(\frac{\lambda+t}{\lambda+x-i\eta} \right)^{p-1} \right] \bar{\varphi}(t) dt \\ &- (1+x)^\alpha \frac{\eta}{2\pi} \int_a^b \frac{\bar{\varphi}(t)}{(x-t)^2 + \eta^2} \left[\left(\frac{t+\lambda}{x+\lambda+i\eta} \right)^{p-1} - 1 + \left(\frac{t+\lambda}{x+\lambda-i\eta} \right)^{p-1} - 1 \right] \bar{\varphi}(t) dt \\ &= I_1(x,\eta) - I_2(x,\eta) \qquad \text{(say)} \end{aligned}$$

First, consider $I_1(x,\eta)$. Since,

$$\begin{aligned} \left| \frac{1}{(x+\lambda+i\eta)^{p-1}} - \frac{1}{(x+\lambda-i\eta)^{p-1}} \right| &= \left| (p-1) \int_{x+\lambda-i\eta}^{x+\lambda+i\eta} z^{-p} dz \right| \\ &\leq 2|p-1| \eta \frac{e^{\pi |\operatorname{Im} p|}}{(x+\lambda)^{\operatorname{Re} p}} \\ &\leq 2|p-1| \eta e^{\pi |\operatorname{Im} p|} / \lambda^{\operatorname{Re} p} \end{aligned}$$

Therefore we can find a constant $B(p)$ independent of x and η such that

$$|I_1(x,\eta)| \leq B (1+x)^\alpha \eta \int_a^b \frac{|x-t| dt}{(x-t)^2 + \eta^2}$$

In view of Lemma 4.2, the right-hand side converges to 0 uniformly for all $x > 0$ as $\eta \rightarrow 0+$.

We now consider $I_2(x,\eta)$. For $0 < \eta < b$, using Lemma 3.3 (7, p. 9) we get

(i) For $\operatorname{Re} p \geq 2$, $t \leq x$

$$\begin{aligned} \left| \left(\frac{t+\lambda}{x+\lambda+i\eta} \right)^{p-1} - 1 \right| &\leq |p-1| e^{2\pi |\operatorname{Im} p|} \frac{|t-x|+\eta}{\sqrt{(x+\lambda)^2 + \eta^2}}, \quad t \leq x \\ &\leq |p-1| e^{2\pi |\operatorname{Im} p|} \frac{|t-x|+\eta}{\lambda}, \end{aligned} \tag{4.18}$$

(ii) For $\text{Re } p \geq 2, t \geq x$

$$\begin{aligned} \left| \left(\frac{t+\lambda}{x+\lambda+i\eta} \right)^{p-1} - 1 \right| &\leq |p-1| e^{2\pi |\text{Im } p|} \left| \frac{t+\lambda+i\eta}{x+\lambda+i\eta} \right|^{\text{Re } p-2} \cdot \frac{|t-x|+\eta}{\sqrt{(\lambda+x)^2+\eta^2}} \\ &\leq \frac{|p-1| e^{2\pi |\text{Im } p|}}{\lambda^{\text{Re } p-1}} (b+\lambda+1)^{\text{Re } p-2} (|t-x|+\eta); 0 < \eta < 1 \end{aligned} \tag{4.19}$$

(iii) For $\text{Re } p \leq 2, t \leq x$ and $t \in [a,b], b > a > 0, 0 < \eta < 1$.

$$\left| \left(\frac{t+\lambda}{x+\lambda+i\eta} \right)^{p-1} - 1 \right| \leq \frac{|p-1| e^{2\pi |\text{Im } p|} (a+\lambda)^{\text{Re } p-2}}{\lambda^{\text{Re } p-1} \left[1 + \frac{1}{\lambda^2} \right]^{\text{Re}(p-2)^2}} \cdot \frac{(|t-x|+\eta)}{\left(1 + \frac{b}{\lambda} \right)^{\text{Re } p-2}} \tag{4.20}$$

(iv) For $\text{re } p \leq 2, t \geq x, t \in [a,b], b > a > 0, 0 < \eta < 1$

$$\begin{aligned} \left| \left(\frac{t+\lambda}{x+\lambda+i\eta} \right)^{p-1} - 1 \right| &\leq |p-1| e^{2\pi |\text{Im } p|} \left| \frac{x+\lambda}{x+\lambda+i\eta} \right|^{\text{Re } p-2} \frac{|t-x|+\eta}{\sqrt{(x+y)^2+\eta^2}} \\ &\leq \frac{|p-1| e^{2\pi |\text{Im } p|}}{\lambda} \left[1 + \frac{1}{\lambda^2} \right]^{2-\text{Re } p} (|t-x|+\eta) \end{aligned} \tag{4.21}$$

Therefore, from inequalities (4.19) through (4.21), it is evident that for $\text{Re } p > 1$ and $0 < \eta < 1$, there exists a positive constant $K(\lambda, p)$ independent of t and x satisfying

$$\left| \left(\frac{t+\lambda}{x+\lambda+i\eta} \right)^{p-1} - 1 \right| \leq K(\lambda, p) [(t-x) + \eta] \quad t \in (a,b)$$

Similarly, under the same set of conditions

$$\left| \left(\frac{t+\lambda}{x+\lambda-i\eta} \right)^{p-1} - 1 \right| \leq K(\lambda, p) [(t-x) + \eta], \quad t \in (a,b)$$

In view of these inequalities we, therefore, have established that

$$|I_2(x, \eta)| \leq \frac{K(\lambda, p)M(1+x)^\alpha}{\pi} (1+x)^\alpha \int_a^b \frac{\eta(|t-x| + \eta)dt}{(t-x)^2 + \eta^2}$$

where $M = \sup |\phi(t)|$

$$a \leq t \leq b$$

That is

$$|I_2(x, \eta)| \leq \frac{K(\lambda, p)M}{\pi} (1+x)^\alpha \left\{ \eta \int_a^b \frac{|t-x|dt}{(t-x)^2 + \eta^2} + \eta^2 \int_a^b \frac{dt}{(t-x)^2 + \eta^2} \right\}$$

Since

$$(1+x)^\alpha \eta^2 \int_a^b \frac{dt}{(t-x)^2 + \eta^2} \rightarrow 0$$

as $\eta \rightarrow 0+$ uniformly for all $x > 0$. Therefore, in view of Lemma 5.2, it follows that

$$I_2(x, \eta) \rightarrow 0 \text{ as } \eta \rightarrow 0+ \text{ uniformly for all } x > 0.$$

II) The case $m = 1, 2, 3, \dots$

A careful computation along with integration by parts will show that

$$\begin{aligned} (1+x)^\alpha x^m D_x I(x, \eta) &= \frac{(1+x)^\alpha}{2\pi i} x \int_a^b \left\{ \frac{1}{x-t+i\eta} \left[\left(\frac{t+\lambda}{x+\lambda+i\eta} \right)^{p-1} - 1 \right] \right. \\ &\quad \left. - \frac{1}{x-t-i\eta} \left[\left(\frac{t+\lambda}{x+\lambda-i\eta} \right)^{p-1} - 1 \right] \right\} \phi'(t) dt \\ &\quad + \frac{(p-1)}{2\pi i} (1+x)^\alpha x \int_a^b [(x+\lambda+i\eta)^{-p} - (x+\lambda-i\eta)^{-p}] (t+\lambda)^{p-2} \phi(t) dt \end{aligned}$$

Using the technique of induction, we obtain:

$$\begin{aligned} (1+x)^\alpha x^m D_x^m I(x, \eta) &= \frac{(1+x)^\alpha x^{-m}}{2\pi i} \int_a^b \left\{ \frac{1}{x-t+i\eta} \left[\left(\frac{t+\lambda}{x+\lambda+i\eta} \right)^{p-1} - 1 \right] \right. \\ &\quad \left. - \frac{1}{x-t-i\eta} \left[\left(\frac{t+\lambda}{x+\lambda-i\eta} \right)^{p-1} - 1 \right] \right\} \phi^{(m)}(t) dt \end{aligned}$$

$$\begin{aligned}
 &+ \dots \\
 &+ \frac{(1+x)^{\alpha} x^m}{2\pi i} \int_a^b [(x+\lambda+i\eta)^{-p} - (x+\lambda-i\eta)^{-p}] (x+\lambda)^{p-2\frac{m-1}{2}} (t) dt \\
 &- \frac{p(p-1)}{2\pi i} (1+x)^{\alpha} x^m \int_a^b [(x+\lambda+i\eta)^{-p-1} - (x+\lambda-i\eta)^{-p-1}] (x+\lambda)^{p-2\frac{m-2}{2}} (t) dt \\
 &+ \dots \\
 &(-1)^{m-1} \frac{p(p-1) \dots (p+m-2)}{2\pi i} (1+x)^{\alpha} x^m \times \\
 &\int_a^b [(x+\lambda+i\eta)^{-p-m+1} - (x+\lambda-i\eta)^{-p-m+1}] (x+\lambda)^{p-2\frac{m}{2}} (t) dt \tag{4.22}
 \end{aligned}$$

Denote the integrals on the right-hand side of (5.22) by J_1, J_2, \dots, J_{m+1} in that order. In view of case $m = 0, J_1 \rightarrow 0$ as $\eta \rightarrow 0+$ uniformly for all $x \in (0, N)$. To show that other integrals converge to 0 as $\eta \rightarrow 0+$ uniformly for $x \in (0, N)$, we consider the most general integral J_{m+1} . As before,

$$\begin{aligned}
 |(x+\lambda+i\eta)^{-p-m+1} - (x+\lambda-i\eta)^{-p-m+1}| &= |(-p-m+1) \int_{x+\lambda-i\eta}^{x+\lambda+i\eta} z^{-p-m} dz| \\
 &\leq 2\eta |p+m-1| \frac{e^{\pi |\text{Im } p|}}{(x+\lambda)^{\text{Re } p+m}}
 \end{aligned}$$

Therefore, we can find a positive constant C independent of x and η such that

$$|J_{m+1}| \leq \eta C \frac{(1+x)^{\alpha} x^m}{(x+\lambda)^{\text{Re } p+m}} \rightarrow 0 \text{ as } \eta \rightarrow 0+$$

uniformly for all $x \in (0, N)$ and each fixed $m = 1, 2, \dots$

Thus, we have proved that

$$(1+x)^{\alpha} x^m D^m I(x, \eta) \rightarrow 0 \text{ as } \eta \rightarrow 0+ \text{ uniformly for all } x \in (0, N) \text{ and each fixed } m = 0, 1, 2, 3, \dots$$

Combining this fact with inequality (4.17), we have

$$\lim_{\eta \rightarrow 0+} |(1+x)^{\alpha} x^m D^m I(x, \eta)| < \epsilon$$

uniformly for all $x > 0$ and each fixed $m = 0, 1, 2, \dots$; since ϵ is arbitrary our claim is established.

THEOREM 4.4. Let $\text{Re } p > 1 > \alpha$ and $f(t) \in S'_\alpha$. If $F(s)$ is the Stieltjes transform of $f(t)$ defined by (3.1) then for $\lambda > 0$ and each $\phi \in D(I)$

$$\lim_{\eta \rightarrow 0+} \langle \frac{p-1}{2\pi i} \int_{\lambda}^{\infty} [f(y-i\eta) - F(y+i\eta)] (y+t)^{p-2} dy, \phi(t) \rangle = 0$$

PROOF. By using the same technique as used in proving Theorem 4.1, it can be shown that:

$$\begin{aligned} & \langle \frac{p-1}{2\pi i} \int_{\lambda}^{\infty} [F(y-i\eta) - F(y+i\eta)] (y+t)^{p-2} dy, \phi(t) \rangle \\ &= \int_a^b \langle f(x), \frac{1}{2\pi i} \left\{ \frac{1}{x-t+i\eta} \left[\left(\frac{t+\lambda}{x+\lambda+i\eta} \right)^{p-1} - 1 \right] \right. \right. \\ & \quad \left. \left. - \frac{1}{x-t-i\eta} \left[\left(\frac{t+\lambda}{\lambda+x+i\eta} \right)^{p-1} - 1 \right] \right\} \right\rangle \phi(t) dt \end{aligned}$$

where the support of $\phi(t)$ is contained in (a,b) , $b > a > 0$,

$$\begin{aligned} &= \langle f(x), \frac{1}{2\pi i} \int_a^b \left\{ \frac{1}{x-t+i\eta} \left[\left(\frac{t+\lambda}{x+\lambda+i\eta} \right)^{p-1} - 1 \right] \right. \right. \\ & \quad \left. \left. - \frac{1}{(x-t-i\eta)} \left[\left(\frac{t+\lambda}{x+\lambda-i\eta} \right)^{p-1} - 1 \right] \right\} \right\rangle \phi(t) dt. \quad (\text{By using Riemann's sum technique}) \end{aligned}$$

Letting $\eta \rightarrow 0+$, the result follows in view of Lemma 4.3.

THEOREM 4.5. For a fixed $\alpha < 1 < \text{Re } p$, let $f(t) \in S'_\alpha(I)$ and let $F(s)$ be the Stieltjes transform of $f(t)$ defined by (3.1). Then,

$$\lim_{\eta \rightarrow 0+} \langle \frac{p-1}{2\pi i} \int_{-x}^{\lambda} (x+t)^{p-2} [F(t-i\eta) - F(t+i\eta)] dt, \phi(x) \rangle = \langle f, \phi \rangle$$

for all $\phi \in D(I)$ and $\lambda > 0$.

PROOF. The result follows quite easily in view of Theorems 4.1 and 4.4.

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