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p - INTEGRABLE SELECTORS OF MULTIMEASURES

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ABSTRACT. The present work is a study of multivalued functions and measures which have many applications to mathematical economics and control problems. Both have received considerable attention in recent years. We study the set of selectors of the space of all p - integrable, X - valued additive set functions. This space contains an isometrically imbedded copy of $L^P(X)$.

KEY WORDS AND PHRASES. p-*integrable selectors, Multimeasures, Banach space, Radon-Nikodym property, Finite* p-*variation,* L^p *spaces.*

AMS (MOS) SUBJECT CLASSIFICATION (1970) Codes. *28A45, 28A48.*

1. INTRODUCTION.

In his studies of classical function spaces such as L^p spaces, S. Leader [7] found it convenient to consider the L^p spaces of finitely additive set functions which he denoted by V^p. All the set functions are real valued and L^p can be isometrically embedded in V^p.

In [9] the work of [7] is generalized in two ways. First the set functions take values in a Banach space X. Instead of studying p-summable functions, Orlicz spaces of finitely additive set functions are considered. This space is denoted by $V^\phi(X)$. Of course, if we let $p(x) = |x|^p$, then $V^\phi(X)$ becomes the space of all p-integrable, X valued additive set functions. It is shown that $L^p(X)$ is isometrically embedded in $V^p(X)$.

In recent years the study of multivalued functions and measures has received considerable attention as have their applications to mathematical economics and control problems. We refer the reader to [1], [2], [3], [4], [6], [8] for a small sample of the work done along these lines.

A multimeasure is defined to be a function from a σ-algebra into the set of non-empty, bounded closed convex subsets of a Banach space X. Countable additivity is defined through the use of the Hausdorff metric. Let $\hat{m}$ denote a multimeasure. The set of selectors of $\hat{m}$ plays a very important role in the study of $\hat{m}$. By a selector, we mean an X-valued measure m such that $m(A) \in \hat{m}(A)$ for all A. For example, Godet-Thobie (see [6]) shows that if $\hat{m}(A)$ is weakly compact for all A, then $\hat{m}$ has at least one selector and in fact a sequence of selectors $\{m_n\}$ such that $\{m_n(A)\}$ is dense in $\hat{m}(A)$ for all A. Conditions are also given on the space X that insure the existence of the sequence $\{m_n\}$.

The main purpose of the present paper is to study the set of selectors of $\hat{m}$ that belong to $V^p(X)$. Denote that set by $M^p_{\hat{m}}$. We assume here that $p > 1$.

Now $M^P_{\hat{m}}$ is an important set to consider because it will be shown that it is a convex closed subset of $V^P(X)$ that characterizes $\hat{m}$, that is $M^P_{\hat{m}_1} = M^P_{\hat{m}_2}$ if and only if $\hat{m}_1 = \hat{m}_2$. The first result deals with the case of a σ-finite space where the values of $\hat{m}$ are weakly compact subsets of X. With certain additional assumptions and using a Radon-Nikodym theorem of Tolstonogov [8], we show that if $M^P_{\hat{m}} \neq \phi$, then a sequence $\{m_n\}$ of selectors in $V^P(X)$ exists with $\{m_n(A)\}$ dense in $\hat{m}(A)$ for all A. This remains true if the values of $\hat{m}$ are assumed to be strongly convex and closed instead of weakly compact.

Next the case where the underlying space is finite is considered. Under certain assumptions using a Radon-Nikodym theorem of Costé [3], the same result is arrived at. These two results are thus along the line of results shown in [6]. Next a new space is naturally introduced, the space $AM^P_{\hat{m}}$ which is the space of all measures of the form $\sum_{A_i \in \pi} \frac{m_i'(A_i)}{r(A_i)} r_{A_i}$ where m'_i are selectors of $\hat{m}$ and where π denotes a partition (finite), r is the underlying measure and r_{A_i} is the contraction of r to A_i. A result establishes the relation between $AM^P_{\hat{m}}$, $AM^P_{\hat{m}_1}$, and $AM^P_{\hat{m}_2}$ where $\hat{m} = \hat{m}_1 + \hat{m}_2$. Also it is shown that $M^P_{\hat{m}} \subset \mathrm{cl} AM^P_{\hat{m}}$. Finally the case where m has absolutely closed convex values is considered.

2. NOTATIONS AND PERTINENT RESULTS

The purpose of this section is to present some concepts and state some results that are needed to show our results.

Let X denote a Banach space. It is said that X has the Radon-Nikodym property if every X-valued measure with finite variation has a density relative to its variation. It is known that reflexive spaces and separable dual spaces have that property. Throughout this paper we assume $p > 1$.

Let Σ denote a σ-algebra of subsets of a set Ω. Let r denote a finitely additive positive set function defined on Σ. Let F be a finitely additive set function from Σ into X with the property r(A) = 0 implies F(A) = 0, and

$$I_p(F,B) = \sup_{\pi} \sum_{E_n \varepsilon \pi} \frac{||F(E_n)||^p}{r(E_n)^{p-1}}$$

where π denotes a finite collection of disjoint subsets of B where B ε Σ and $r(E_n) < \infty$.

Now F is said to have <u>finite p-variation</u> if $I_p(F) = I_p(F,\Omega) < \infty$.

Let $F_\pi(\cdot) = \sum_{E_n \varepsilon \pi} \frac{F(E_n)}{r(E_n)} r[E_n \cap (\cdot)]$, and $A^p(X)$ denote the set of finitely additive set functions F mapping Σ into X with finite p-variation and with the property r(A) = 0 implies F(A) = 0.

Let $V^p(X)$ be the submanifold of $A^p(X)$ consisting of r-continuous set functions, that is $\lim_{\mu(E)\to 0} ||F(E)|| = 0$.

It is shown in [9] that when $p > 1$, $A^p(X) = V^p(X)$.

We denote by $N_p(F) = I_p(F)^{1/p}$. It is also shown that $V^p(X)$ is a Banach space under the norm N_p. Moreover the set functions of the form F_π form a dense subset of $V^p(X)$ provided that X has the Radon-Nikodym property and $p > 1$.

Let F be a function from Ω into the non-empty closed bounded convex subsets of X. If F has weakly compact values, we define, following Tolstonogov [8],

$$\int F dr = \{\int f dr \mid f \varepsilon L'(r) \text{ and } f(w) \varepsilon F(w) \text{ a.e.}\}$$

Otherwise,

$$Fdr = cl\{\int f dr \mid f \varepsilon L'(r) \text{ and } f(w) \varepsilon F(w) \text{ a.e.}\}$$

Let $\hat{X}_c$ denote the non-empty bounded closed convex subsets of X. Following [6] we define

$$A + B = cl\{A+B\} \text{ for } A \varepsilon \hat{X}_c,\ B \varepsilon \hat{X}_c.$$

If $\hat{m}: \Sigma \to \hat{X}_c$, then $\hat{m}$ is called a <u>multi-measure</u> if

$$\hat{m}(\cup A_i) = \dot{\Sigma}\hat{m}(A_i) \equiv \lim_{n\to\infty} [\hat{m}(A_1) \dot{+} \hat{m}(A_2) \dot{+} \dots \dot{+} \hat{m}(A_n)]$$

For every sequence of disjoint sets $A_i \in \Sigma$ where the limit is with respect to the Hausdorff metric (It is shown in [6] that $\dot{+}$ is an abelian and associative operation on $\hat{X}_c$).

The reader is referred to [3] and [8] for Radon-Nikodym theorems pertaining to multimeasures. If $A \in \hat{X}_c$, $|A|$ will denote the Hausdorff distance between $\{0\}$ and A, that is

$$|A| = \sup_{x \in A} ||x||$$

Also acoS will refer to the <u>absolute convex envelope</u> of a subset S of X, that is acoS is the set of all finite sums of the form

$$\Sigma\alpha_i s_i \text{ with } s_i \in S \text{ and } \Sigma|\alpha_i| \leq 1$$

Of course, $\overline{\text{acoS}}$ denotes the closure of acoS. Finally X is said to <u>have property</u> P if its dual space X' has a sequence $\{x_n^*\}$ that separates points of X.

3. MAIN RESULTS

Let $\hat{m}$ denote a multi-measure from Σ into $\hat{X}_c$ and let $M^p_{\hat{m}}$ denote all measures $m: \Sigma \to X$ such that $m(A) \in \hat{m}(A)$ for all $A \in \Sigma$ with $m \in V^p(X)$ that is $M^p_{\hat{m}}$ is the set of all "selectors of $\hat{m}$" that are in $V^p(X)$. $M^p_{\hat{m}}$ <u>is a closed subset of</u> $V^p(X)$.

PROOF. Let $A \in \Sigma$ with $r(A) < \infty$. Let $m_i \in M^p_{\hat{m}}$ with $N_p(m-m_i) \to 0$ and $m \in V^p(X)$. Now for $\varepsilon > 0$ and i large enough:

$$\frac{||m(A)-m_i(A)||^p}{r(A)^{p-1}} < \varepsilon$$

Since $\hat{m}(A)$ is closed it follows that $m(A) \in \hat{m}(A)$. Thus $m \in M^p_{\hat{m}}$. Let $(\Omega, \Sigma, \lambda)$ denote a σ-finite space. Let $\hat{X}_w$ denote all weakly compact convex subsets of X. We assume that the convex hull of the average range of m is <u>relatively weakly compact</u>, that is $\left\{\cup \frac{\hat{m}(K)}{\lambda(K)} \,|\, K \in \Sigma, K \subset E, 0 < \lambda(K) < \infty\right\}$ is relatively compact for every $E \in \Sigma$.

THEOREM 2. Let $\acute{m}$ be a multi-measure from Σ into X_w. Assume that

(1) The convex hull of the average range of m is relatively compact.

(2) The space X' is separable (in particular X' satisfies P).

(3) For every $\varepsilon>0$ there exists $\delta>0$ such that $\lambda(A)<\delta$ implies $|m(A)|<\varepsilon$.

(4) X has the Radon-Nikodym property.

Then if $M^p_{\acute{m}} \neq \phi$ for $p > 1$, then there exists $\{m_n\}$ a sequence in $M^p_{\acute{m}}$ such that $\acute{m}(A) = \text{cl}\{m_n(A)\}$ for every $A \in \Sigma$.

PROOF. From (2) and [6] it follows that there exists a sequence $\{\sigma_n\}$ where σ_n are selectors of $\hat{m}$ with $\acute{m}(A) = \text{cl}\{\sigma_n(A)\}$ for $A \in \Sigma$, σ_n are X-valued measures. Now $\lambda(A) = 0$ implies $\acute{m}(A) = \{0\}$ by (3). Thus $\sigma_n \ll \lambda$. Now (4) implies

$$v_n(A) = \int_A f_n d\lambda \text{ for some } f_n \in L'_X(\lambda) \text{ and all } A \in \Sigma.$$

By a theorem of [8], (1) implies that there exists a function F from Ω into $\acute{X}_w$ such that $\acute{m}(A) = \int_A F d\lambda$. Thus $\int_A f_n d\lambda \in \int_A F d\lambda$ for all $A \in \Sigma$. We now show that this implies $f_n(w) \in F(w)$ a.e.

Let $\int_A h d\lambda \in \int_A F d\lambda$ for all $A \in \Sigma$. Let $\{x_n^*\}$ be a dense sequence in X'. Then

$$x \in F(w) \text{ if and only if } \langle x, x_n^* \rangle \leqq \sup_{y \in F(w)} \langle y, x_n^* \rangle \text{ for all } n.$$

This is because F(w) is convex, closed and bounded. If $h(w) \notin F(w)$ on a set of positive measure then there exists $A \in \Sigma$ and n such that $\lambda(A) > 0$ and

$$\langle h(w), x_n^* \rangle > \sup_{y \in F(w)} \langle y, x_n^* \rangle \text{ over } A$$

Consequently

$$\langle \int_A h d\lambda, x_n^* \rangle = \int_A \langle h, x_n^* \rangle d\lambda > \int_A \sup_{y \in F(w)} \langle y, x_n^* \rangle d\lambda$$

$$\geq \sup_{g(w) \in F(w)} \int_A \langle g(w), x_n^* \rangle d\lambda \geq \sup_{g(w) \in F(w)} \langle \int_A g d\lambda, x_n^* \rangle$$

Thus $\int_A g d\lambda \notin \text{cl}\{\int_A g d\lambda \mid g(w) \in F(w)\}$. This contradicts $\int_A h d\lambda \in \int_A F d\lambda$ for all $A \in \Sigma$. Thus $f_n(w) \in F(w)$ a.e. for all n.

Now let $m_o \varepsilon M^p_m$ then $m_o(A) = \int_A f_o d\lambda$ and by [9], $f_o \varepsilon L^p_X$.

Let $A_{j,l} = \{w \mid l-1 < ||f_j|| \leq l\}$ where l is a positive integer. Let $\{A_k\}$ be a partition of Ω such that $\lambda(A_k) < \infty$. Let

$$B_{j,l,k} = A_{j,l} \cap A_k \qquad s_{j,l,k} = \chi_{B_{j,l,k}} f_j + \chi_{B'_{j,l,k}} f_o$$

clearly $s_{j,l,k} \varepsilon L^p_X$. Define $m_{j,l,k}(A) = \int_A s_{j,l,k} d\lambda$. By [9]

$m_{j,l,k} \varepsilon V^p(X)$, also $m_{j\ l,k} \varepsilon\ M^p_{\hat{m}}$ since

$$m_{j,l,k}(A) = \sigma_j(A \cap A_k \cap A_{j,l}) + m_o(A \cap B'_{j,l,k}) \varepsilon \hat{m}(A).$$

All finite sums of the form $\sum_k \hat{m}(A \cap A_k \cap A_{j,l}) + \sum_k \hat{m}(A \cap A_k \cap A'_{j,l})$ are dense in $\hat{m}(A)$ (for j and l fixed) and since

$\lambda(A'_{j,l}) \to 0$ as $l \to \infty$ (for j fixed) it is easy to see that (3) implies that all finite sums of the form

$\sum_k m_{j,l,k}(A)$ are dense in $\hat{m}(A)$ as j and l range over the positive integers.

These finite sums form the desired sequence $\{m_n\}$.

This completes the proof.

Let $\hat{X}_s$ denote all bounded closed non-empty, strictly convex subsets of X, that is $A \varepsilon \hat{X}_s$ implies that for all $x' \varepsilon X'$ there exists a unique $x \varepsilon X$ such that

$$x'(x) = \sup \{x'(y) \mid y \varepsilon A\}.$$

COROLLARY 3. Let $(\Omega, \Sigma, \lambda)$ be a σ-finite measure space. Let $\hat{m}$ be a multimeasure from Σ into $\hat{X}_s$ assume

(1) $\hat{m}$ is of the form $\hat{m}(A) = \int_A F d\lambda$ where $F(w) \varepsilon \hat{X}_c$.

(2) X' is separable.

(3) X has the Radon-Nikodym property.

Then if $M^p_{\hat{m}} \neq \phi$ for $p > 1$, then there exists a sequence $\{m_n\}$ in $M^p_{\hat{m}}$ such that

$\hat{m}(A) = cl\{m_n(A)\}$ for all $A \in \Sigma$.

PROOF. From a result in [6] and since $\hat{m}$ has values in $\hat{X}_s$ the existence of the sequence $\{\sigma_n\}$ as defined in the proof of theorem 2 follows. The rest of the proof follows as in Theorem 2.

We now consider a measure $\hat{m}$ from Σ into $\hat{X}_c$. We define the variation of $\hat{m}$ by

$|\hat{m}| = \sup \Sigma|\hat{m}(A_i)|$ where the sup is over finite collections of disjoint sets A_i.

We consider the case where (Ω,Σ,μ) is a finite measure space.

THEOREM 4. Let (Ω,Σ,μ) be a finite measure space. Let $\hat{m}$ be a multimeasure from Σ into $\hat{X}_c$. Assume:

(1) $\hat{m}$ has σ-finite variation.

(2) $\mu(a) = 0$ implies $\hat{m}(A) = \{0\}$.

(3) X' is separable.

(4) X has the Radon-Nikodym property.

Then if $M^p_{\hat{m}} \neq \phi$ for $p > 1$ then there exists a sequence $\{m_n\}$ in $M^p_{\hat{m}}$ such that

$\hat{m}(A) = cl\{m_n(A)\}$ for all $A \in \Sigma$.

PROOF. From a result in [2] and since $\hat{m}$ takes values in $\hat{X}_c$ and since (2) holds, $\hat{m}$ is uniformly continuous with respect to μ. Since μ is finite by a result of [3] it follows that there exists a function F from Ω into $\hat{X}_c$ such that $\hat{m}(A) = \int_A F d\mu$ for all $A \in \Sigma$.

The rest of the proof proceeds as in Theorem 2.

We now show that $M^p_{\hat{m}}$ characterizes $\hat{m}$.

COROLLARY 4. Let $\hat{m}_1$ and $\hat{m}_2$ be two multimeasures. Under the hypothesis of theorem 2 or theorem 4

$M^p_{\hat{m}_1} = M^p_{\hat{m}_2}$ if and only if $\hat{m}_1 = \hat{m}_2$.

PROOF. The proof follows immediately from the statement $\hat{m}(A) = \mathrm{cl}\{m_n(A)\}$.

PROPOSITION 5. Let $\hat{m}$ be a multimeasure from Σ into $\hat{X}_c$ of finite variation and having no atoms. Assume X has the Radon-Nikodym property then $M^P_{\hat{m}}$ is a convex set.

PROOF. From a result in [2] and since $\hat{m}$ has no atoms, $\hat{m}(A)$ is convex for all $A \in \Sigma$. It is then straight-forward to check that $M^P_{\hat{m}}$ is convex.

From now on γ will denote λ when the hypothesis of theorem 2 are verified and μ when the hypothesis of theorem 4 are verified. Recall that we use the word "partition" as in [9] that is a finite collection of disjoint subsets of finite measure.

THEOREM 6. Assume the hypothesis of theorem 2 or of theorem 4 and assume $M^P_{\hat{m}} \neq \phi$. Then there exists a sequence $\{m_i\}$ in $M^P_{\hat{m}}$ such that for $\varepsilon > 0$ and $m \varepsilon M^P_{\hat{m}}$ there exists a partition π_0 such that if π refines π_0 then there is a subset $\{m_{i_n}\}$ of $\{m_j\}$ such that

$$N_p \left[m - \sum_{E_n \varepsilon \pi} \frac{m_{i_n}(E_n)}{\gamma(E_n)} \gamma_{E_n} \right] < \varepsilon$$

where γ_{E_n} denotes the contraction of γ to E_n.

PROOF. Let $\varepsilon > 0$, $m \varepsilon M^P_{\hat{m}}$. By [9] there exists a partition π_0 such that if π refines π_0 then

$$N_p \left[m - \sum_{E_n \varepsilon \pi} \frac{m(E_n) \gamma_{E_n}}{\gamma(E_n)} \right] < \varepsilon/2.$$

By theorem 2 or theorem 4 there exists a sequence $\{m_i\}$ in $M^P_{\hat{m}}$ such that

$$m(E) = \mathrm{cl}\{m_i(E)\} \text{ for all } E \varepsilon \Sigma.$$

Let $\varepsilon' > 0$ then there exists an integer i_n such that

$$||m_{i_n}(E_n) - m(E_n)|| < \varepsilon'.$$

Now

$$N_p\left[\sum_{E_n\varepsilon\pi}\frac{m(E_n)}{\gamma(E_n)}\gamma_{E_n}-\sum_{E_n\varepsilon\pi}\frac{m_{i_n}(E_n)}{\gamma(E_n)}\gamma_{E_n}\right]$$

$$\leq \sum N_p\left[\frac{m(E_n)-m_{i_n}(E_n)}{\gamma(E_n)}\gamma_{E_n}\right] \text{ and}$$

$$I_p\left[\frac{m(E_n)-m_{i_n}(E_n)}{\gamma(E_n)}\gamma_{E_n}\right] = \sup_{\hat{\pi}}\sum_{A_j\varepsilon\hat{\pi}}\frac{||m(E_n)-m_{i_n}(E_n)||^p}{\gamma(E\ )^p}\frac{\gamma_{E_n}(A_j)^p}{\gamma(A_j)^{p-1}}$$

$$\leq \frac{\varepsilon'^p}{\gamma(E_n)^p}\sup_{\hat{\pi}}\sum_{A_j\varepsilon\hat{\pi}}\left[\frac{\gamma_{E_n}(A_j)}{\gamma(A_j)}\right]^{p-1}\gamma_{E_n}(A_j)\leq\frac{\varepsilon'^p}{\gamma(E_n)^{p-1}}$$

clearly ε' may be picked so that

$$N_p\left[\sum_{E_n\varepsilon\pi}\frac{m(E_n)-m_{i_n}(E_n)}{\gamma(E_n)}\gamma_{E_n}\right]<\varepsilon/2.$$

This finishes the proof.

Motivated by the above theorem we define $AMP_{\hat{m}}$ to be the set of all finite sums of the form $\sum_{A_i\varepsilon\pi}\frac{m'_i(A_i)}{\gamma(A_i)}\gamma_{A_i}$ where $m'_i\varepsilon MP_{\hat{m}}$.

If $\hat{m}_1$ and $\hat{m}_2$ are multimeasures from Σ into $\hat{X}_c$ we define

$$(\hat{m}_1+\hat{m}_2)(A)=\hat{m}_1(A)\dot{+}\hat{m}_2(A).$$

Also let $AMP_{\hat{m}_1}\oplus_s AMP_{\hat{m}_2}$ be the set of all finite sums of the form

$$\sum_{A_i\varepsilon\pi}\frac{m'_{1i}(A_i)+m'_{2i}(A_i)}{\gamma(A_i)}\gamma_{A_i}\quad\text{with } m'_{1i}\varepsilon MP_{\hat{m}_1},\ m'_{2i}\varepsilon MP_{\hat{m}_2}.$$

That is the partition π is kept the same for $\hat{m}_1$ and $\hat{m}_2$.

THEOREM 7. Let $\hat{m}=\hat{m}_1+\hat{m}_2$. Under the hypothesis of theorem 2 or theorem 4 $cl\{AMP_{\hat{m}}\}=cl\{AMP_{\hat{m}_1}\oplus_s AMP_{\hat{m}_2}\}$.

PROOF. By theorem 2 or theorem 4 there exists sequences $\{m_i\}$, $\{m_{1i}\}$, $\{m_{2i}\}$, m, $M^P_{\hat{m}}$, $M^P_{\hat{m}_1}$, $M^P_{\hat{m}_2}$ such that

$\hat{m}(A) = cl\{m_i(A)\}$, $\hat{m}_1(A) = cl\{m_{1i}(A)\}$, $\hat{m}_2(A) = cl\{m_{2i}(A)\}$ for $A \in \Sigma$.

Consider $\Sigma \frac{m'_i(A_i)}{\gamma(A_i)} \gamma_{A_i}$, then as in theorem 6 one may pick $\varepsilon' > 0$ small enough so that

$$||m'_i(A_i)-m_i(A_i)||<\varepsilon' \text{ implies } N_p\left[\Sigma \frac{m'_i(A_i)}{\gamma(A_i)} \gamma_{A_i} - \Sigma \frac{m_i(A_i)}{\gamma(A_i)} \gamma_{A_i}\right] < \varepsilon$$

Since $\hat{m}(A) = cl\{m_{1i}(A)+m_{2_j}(A)\}$ again let

$||m_i(A_i)-(m_{1ki}+m_{21_i})(A_i)||<\varepsilon$ "where ε" is small enough so that

$$N_p\left[\Sigma \frac{m_i(A_i)}{\gamma(A_i)} \gamma_{A_i} - \Sigma \frac{(m_{1ki}+m_{21i})(A_i)}{\gamma(A_i)} \gamma_{A_i}\right] < \varepsilon. \text{ This shows that}$$

$cl AM^P_{\hat{m}} \subset cl\left[AM^P_{\hat{m}1} \dot{+}_s AM^P_{\hat{m}2}\right]$. Since $m'_{1i}(A_i)+m'_{2j}(A_i) \in \hat{m}_1(A_i)\dot{+}\hat{m}_2(A_i)$, the converse is also clear.

Let us note that if $M^P_{\hat{m}_1} \neq \phi$, $M^P_{\hat{m}_2} \neq \phi$ then whenever $\hat{m}$ satisfies the above hypothesis, $\hat{m}_1$ and $\hat{m}_2$ do.

Let π be the partition $\{A_i\}$, let $\sigma_\pi = \sum_{A_i \in \pi} \frac{m'_i(A_i)}{\gamma(A_i)} \gamma_{A_i}$ where $m'_i \in M^P_{\hat{m}}$

Now let $\hat{m}$ be a multimeasure from Σ into $\hat{X}_c$, define $(\overline{aco}\,\hat{m})(A) = \overline{aco}(\hat{m}(A))$.

PROPOSITION 8. The following are true.

(1) If $\hat{m}$ is an $\hat{X}_c$ valued multimeasure, then so is $\overline{aco}\,\hat{m}$.

(2) $M^P_{\hat{m}} \subset cl AM^P_{\hat{m}}$.

(3) Let $\pi = \{A_i\}$ be a partition and assume that the values of $\hat{m}$ are absolutely convex sets in $\hat{X}_c$. Then $\sigma_\pi(A) \in \hat{m}(A)$ whenever π is a partition of A with

$$\sum_{A_i \in \pi} \frac{\gamma(A_i \cap A)}{\gamma(A_i)} \leqslant 1$$

PROOF. We would like to show $\overline{aco}\,\hat{m}\,(\bigcup_{i=1}^{\infty} A_i) = \cdot\Sigma_{i=1}^{\infty}\ \overline{aco}\,\hat{m}(A_i)$ for any sequence of disjoint sets A_i.

Now the proof in [5] (p. 415) shows with obvious modifications that

$\text{aco}(A+B) = \text{aco } A + \text{aco } B$. Thus

$\bar{\text{aco}}(A+B) \subset \text{aco } A + \text{aco } B$ and by continuity of +

$\bar{\text{aco}}\, A + \bar{\text{aco}}\, B \subset \overline{\text{aco}A+\text{aco}B} = \bar{\text{aco}}\,(A+B)$

Thus $\bar{\text{aco}}\, A \dotplus \bar{\text{aco}}\, B = \bar{\text{aco}}\,(A+B)$ and by induction

$$\bar{\text{aco}}\left[\cdot\sum_{i=1}^{n} \hat{m}(A_i)\right] = \cdot\sum_{i=1}^{n} \bar{\text{aco}}\ \hat{m}(A_i)$$

By a result of [4], $\mathcal{J}[\bar{\text{aco}}\, A, \bar{\text{aco}}\, B] \leq \mathcal{J}[A,B]$ where A and B are in $\hat{X}_c$ and $\mathcal{J}$ denotes the Hausdorff metric. Thus letting n go to infinity

$$\bar{\text{aco}}\left[\cdot\sum_{i=1}^{\infty}\hat{m}(A_i)\right] = \cdot\sum_{i=1}^{\infty} \bar{\text{aco}}\ \hat{m}(A_i)$$

This shows the first part of the theorem.

$M^P_{\hat{m}} \subset \text{cl}AM^P_{\hat{m}}$ follows from [9] since F_π are dense in $V^P(X)$. Finally let $\sigma_\pi \in AM^P_{\hat{m}}$. Then

$$\left(\sum_{A_i\in\pi} \frac{m_i'(A_i)}{\gamma(A_i)}\gamma_{A_i}\right)(A) = \sum_{A_i\in\pi} m_i'(A_i)\ \frac{\gamma(A\cap A_i)}{\gamma(A_i)}$$

If $\sum \frac{\gamma(A\cap A_i)}{\gamma(A_i)} \leq 1$ since $m_i'(A_i)\in\hat{m}(A_i)\subset \hat{m}(A)$ (this because $0\in\hat{m}(B)$ if $\hat{m}(B)$ is absolutely convex for all $B\in\Sigma$) this implies $\sigma_\pi(A)\in\hat{m}(A)$.

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