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ON THE OVERCONVERGENCE OF CERTAIN SERIES

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ABSTRACT. In this work, we consider certain class of exponential series with polynomial coefficients and study the properties of convergence of such series. Then we consider a subclass of this class and prove certain theorems on the overconvergence of such a series, which allow us to determine the conditions under which the boundary of the region of convergence of this series is a natural boundary for the function f defined by this series.

KEY WORDS AND PHRASES. LC-Dirichletian element, L-Dirichletian element, Convergence, Overconvergence.
AMS (MOS) SUBJECT CLASSIFICATION (1970) CODES. 30A16, 30A64.

1. INTRODUCTION.

Let us consider the following LC-dirichletian element

$$\{f\} : \sum_{1}^{\infty} P_n(x) \exp -\lambda_n s \ , \tag{1.1}$$

where $P_n(s) = \sum_{j=0}^{m_n} a_{nj} s^j$, a_{nj} are complex constants with $a_{n,m_n} \neq 0$, $s = \sigma + i\tau$,

$(\sigma,\tau) \in \mathbb{R}^2$ and (λ_n) is a sequence of complex numbers such that $(|\lambda_n|)$ is a D-sequence. That is to say $(|\lambda_n|)$ is a sequence of positive real numbers satisfying

$$0 < |\lambda_1| < |\lambda_2| < \dots , \lim_{n\to\infty} |\lambda_n| = \infty . \tag{1.2}$$

Let

$$L = \limsup \left\{ \frac{\log n}{|\lambda_n|} / n \in \mathbb{N}-\{0\} \right\} \tag{1.3}$$

$$A_n = \operatorname{Max} \{|a_{nj}| / j \in (0,1,\dots,m_n)\} \tag{1.4}$$

and

$$\beta^* = \limsup \left\{ \frac{m_n}{|\lambda_n|} / n \in \mathbb{N}-\{0\} \right\} . \tag{1.5}$$

Let $\mathcal{E}_n$ be the set of points of $\mathbb{C}$ which are zeros of $P_n(s)$ and $\mathcal{E} = \bigcup_n \mathcal{E}_n$. Let us denote by $\mathcal{E}^d$ the derived set of $\mathcal{E}$ and $\mathcal{E}_\infty = \{s \in \mathbb{C} \mid \exists_{(n_j)} P_{n_j}(s) = 0\}$ where (n_j) is an infinite subsequence of $\mathbb{N}-\{0\}$ depending on s ; let $\mathcal{E}^* = \mathcal{E}^d \cup \mathcal{E}_\infty$. $\mathcal{E}^*$ is a closed set. Let us suppose that $\mathbb{C}-\mathcal{E}^*$ is non empty. We put

$$\forall_{s\in\mathbb{C}-\mathcal{E}^*} \quad \delta(n,s) = -\frac{\log|P_n(s)\exp(-\lambda_n s)|}{|\lambda_n|} , \text{ for sufficiently large } n ,$$

$$\delta_*(s) = \liminf \{\delta_n(s) / n \in \mathbb{N}-\{0\}\}$$

$$\forall_{\alpha\in\mathbb{R}} \quad \mathcal{D}_{*\alpha} = \{s \in \mathbb{C}-\mathcal{E}^* / \delta_*(s) > \alpha\} .$$

In this paper, using a technique similar to that used by M. Blambert and J. Simeon [2] , we prove two lemmas for a LC-dirichletian element which enable us to discuss the properties of absolute convergence and uniform convergence for (1.1) in $\mathbb{C}-\mathcal{E}^*$ exclusively. Then we prove Jentzsch's theorem for a L-dirichletian element that is for element of the type (1.1) where λ_n are positive real numbers satisfying (1.2) $\big((\lambda_n)$ is a D-sequence$\big)$ and a theorem on the overconvergence for a L-dirichletian element.

2. MAIN RESULTS.

DEFINITION. - It is said that a function is sub-lipschitzian on an open set, if it is lipschitzian on each compact subset of that open set.

LEMMA 1. - Let $\mathcal{K}$ be any compact subset of $\mathbb{C}$. Then the following assertions are true.

(1) $\forall_{\mathcal{K}\subset\mathbb{C}-\mathcal{E}^*} \ \exists_{n'} \ \forall_{n\geq n'}$ the function $\mathcal{K}\ni s \to \delta(n,s)$ is lipschitzian.

(2) If $\beta^* < \infty$, and if there exists a $s_o \in \mathbb{C}-\mathcal{E}^*$ such that $|\delta_*(s_o)| < \infty$ then the function δ_* is sub-lipschitzian on $\mathbb{C}-\mathcal{E}^*$.

PROOF. Let $\forall_{\mathcal{K}\subset\mathbb{C}-\mathcal{E}^*} \ \epsilon_{\mathcal{K}} = \operatorname{dist}(\mathcal{K},\mathcal{E}^*)$. Then it is easy to see that

$$\forall_{s\in\mathcal{K}} \ \forall_{\epsilon\in]0,\epsilon_{\mathcal{K}}[} \ \exists_{n'} \ \forall_{n\geq n'} \ \{j\in(1,2,\dots,m_n) \Rightarrow \alpha_{nj} \notin d_{s,\epsilon}\} ,$$

where $d_{s,\epsilon}$ is the open disc centred at s and of radius ϵ and (α_{nj}), $j\in\{1,2,\dots,m_n\}$, is the sequence of zeros of $P_n(s)$ (with its order of mulplicity is taken into account). More precisely let us show that,

$$\forall_{\epsilon\in]0,\epsilon_{\mathcal{K}}[} \ \exists_{n'} \ \forall_{n\geq n'} \ \forall_{s\in\mathcal{K}} \ \{j\in(1,2,\dots,m_n) \Rightarrow \alpha_{nj} \notin d_{s,\epsilon}\} .$$

Let $G_\epsilon = \bigcup_{s\in\mathcal{K}} d_{s,\epsilon}$. It is evident that $\overline{G}_\epsilon$ the closure of G_ϵ is a compact subset of $\mathbb{C}-\mathcal{E}^*$. Let $\epsilon'\ni]0,\epsilon_{\mathcal{K}}-\epsilon[$ where $\epsilon\in]0,\epsilon_{\mathcal{K}}[$. The set of discs $d_{s,\epsilon'}$ indexed by s on $\overline{G}_\epsilon$ is an open covering of $\overline{G}_\epsilon$. Hence we have a finite subcovering ;

$$\overline{G}_\epsilon \supset \exists_{(s_1,\dots,s_k)} \bigcup_{j=1}^{k} d_{s_j,\epsilon'} \supset \overline{G}_\epsilon .$$

Let $s\in\mathcal{K}$ and $s'\in d_{s,\epsilon}$; hence $s'\in G_\epsilon$. Then $s'\in \bigcup_{j=1}^{k} d_{s_j,\epsilon'}$ which implies that $\exists_{j^*\in(1,\dots,k)} \ s'\in d_{s_{j^*},\epsilon'}$. Now

$$\forall_{j\in(1,\dots,k)} \ \exists_{n'(=n_j)} \ \forall_{n\geq n'} \ \forall_{s\in d_{s_j,\epsilon'}} \ P_n(s)\neq 0$$

and hence

$$\forall_{n\geq \operatorname{Max}\{n_j/j\in(1,\dots,k)\}} \ \forall_{j\in(1,2,\dots,k)} \ \forall_{s\in d_{s_j,\epsilon'}} \ P_n(s)\neq 0 ,$$

which gives $\forall_{n\geq \operatorname{Max}\{n_j/j\in(1,\dots,k)\}} \ P_n(s')\neq 0$. As s is arbitrary on $\mathcal{K}$ and s' is arbitrary on $d_{s,\epsilon}$ we have

$$\underset{\epsilon\in]0,\epsilon_{\mathcal{K}}[}{\forall}\ \underset{n'(=n_\epsilon)}{\exists}\ \underset{n\geq n'}{\forall}\ \underset{s\in\mathcal{K}}{\forall}\ \{j\in(1,\dots,k) \Rightarrow \alpha_{nj}\notin d_{s,\epsilon}\}\ .$$

From which we have

$$\underset{n\geq n'(=n_\epsilon)}{\forall}\ \underset{(s,s')\in\mathcal{K}\times\mathcal{K}}{\forall}\ \log|P_n(s)|-\log|P_n(s')| \leq \sum_{j=1}^{m_n}\log\left\{1+\frac{|s-s'|}{|s'-\alpha_{nj}|}\right\} \leq \sum_{j=1}^{m_n}\log\left\{1+\frac{|s-s'|}{\epsilon}\right\}.$$

Under the above conditions related to n , s and s' with $s\neq s'$,

$$\begin{aligned}|\delta(n,s)-\delta(n,s')| &\leq |s-s'| + \frac{1}{|\lambda_n|}\sum_{j=1}^{m_n}\log\left\{1+\frac{|s-s'|}{\epsilon}\right\}\\ &\leq |s-s'| + \frac{|s-s'|}{\epsilon|\lambda_n|}\sum_{j=1}^{m_n}\left\{\log(1+\frac{|s-s'|}{\epsilon})\Big/\frac{|s-s'|}{\epsilon}\right\}\\ &\leq |s-s'| + \frac{m_n|s-s'|}{\epsilon|\lambda_n|}\sup\left\{\frac{\log(1+x)}{x}/x>0\right\}\ ;\end{aligned}$$

as $\sup\left\{\frac{\log(1+x)}{x}/x>0\right\} = 1$, $|\delta(n,s)-\delta(n,s')| \leq |s-s'|\left\{1+\frac{m_n}{\epsilon|\lambda_n|}\right\}$. Putting $\mu_{\epsilon,n} = 1+\frac{m_n}{\epsilon|\lambda_n|}$,

$$\underset{\epsilon\in]0,\epsilon_{\mathcal{K}}[}{\forall}\ \underset{n'}{\exists}\ \underset{n\geq n'}{\forall}\ \underset{(s,s')\in\mathcal{K}\times\mathcal{K}}{\forall}\ |\delta(n,s)-\delta(n,s')| \leq \mu_{\epsilon,n}|s-s'|\ ,$$

which proves the first part of the lemma.

Now let $\mu^*_\epsilon = \limsup_{n\to\infty}\mu_{\epsilon,n} = 1+\beta^*/\epsilon$ with $\epsilon\in]0,\epsilon_{\mathcal{K}}[$; as $\underset{s_0\in\mathbb{C}-\mathcal{E}^*}{\exists}\ \delta_*(s_0)<\infty$

$$\underset{\epsilon\in]0,\epsilon_{\mathcal{K}}[}{\forall}\ \underset{(s,s')\in\mathcal{K}\times\mathcal{K}}{\forall}\ |\delta_*(s)-\delta_*(s')| \leq \mu^*_\epsilon|s-s'|$$

and

$$\underset{(s,s')\in\mathcal{K}\times\mathcal{K}}{\forall}\ |\delta_*(s)-\delta_*(s')| \leq \mu^*_{\epsilon_{\mathcal{K}}}|s-s'|$$

where

$$\mu^*_{\epsilon_{\mathcal{K}}} = \operatorname{Inf}\{\mu^*_\epsilon/\epsilon\in]0,\epsilon_{\mathcal{K}}[\} = 1+\frac{\beta^*}{\epsilon_{\mathcal{K}}}\ .$$

Hence

$$\underset{\mathcal{K}\supset\mathbb{C}-\mathcal{E}^*}{\forall}\ \underset{(s,s')\in\mathcal{K}\times\mathcal{K}}{\forall}\ |\delta_*(s)-\delta_*(s')| \leq \mu^*_{\epsilon_{\mathcal{K}}}|s-s'|\ ,$$

which completes the proof of the lemma.

Under the condition (2) of Lemma 1 , δ_* is continuous on $\mathbb{C}-\mathcal{E}^*$

which implies that $\mathcal{D}_{*\alpha}$ is an open subset of $\mathbb{C}-\mathcal{E}^*$; but $\mathcal{D}_{*\alpha}$ can have several connected components.

LEMMA 2. - When $\beta^*<\infty$, then

$$\forall_{\alpha\in\mathbf{R}}\left\{\mathcal{D}_{*\alpha}\neq\phi\Rightarrow\forall_{\mathcal{K}\subset\mathcal{D}_{*\alpha}}\ \forall_{\beta'>\beta^*}\ \exists_{n'}\ \forall_{n\geq n'}\ \forall_{s\in\mathcal{K}}\ |P_n(s)\exp(-\lambda_n s)|<\exp(-|\lambda_n|(\alpha-\beta'))\right\} .$$

PROOF. Let $\alpha\in\mathbf{R}$ such that $\mathcal{D}_{*\alpha}\neq\phi$ (otherwise the lemma is trivial) and let $\mathcal{K}$ be a compact subset of $\mathcal{D}_{*\alpha}$. We can easily see that

$$\forall_{s\in\mathbb{C}-\mathcal{E}^*}\ \forall_{\varepsilon\in]0,\mathrm{dist}(s,\mathcal{E}^*)[}\ \exists_{n'(=n_{s,\varepsilon})}\ \forall_{n\geq n'}\ \forall_{s'\in\overline{d_{s,\varepsilon}}}\ P_n(s')\neq 0$$

where $\overline{d_{s,\varepsilon}}$ is the closed disc centred at s and of radius ε . Hence

$$\forall_{s\in\mathbb{C}-\mathcal{E}^*}\ \forall_{\varepsilon\in]0,\varepsilon_{\mathcal{K}}[}\ \exists_{n'(=n_{s,\varepsilon})}\ \forall_{n\geq n'}\ \forall_{s'\in\overline{d_{s,\varepsilon}}}\ P_n(s')\neq 0 .$$

Let us consider the compact subset $\mathcal{K}_\varepsilon=\overline{\bigcup_{s\in\mathcal{K}} d_{s,\varepsilon}}$, of $\mathbb{C}-\mathcal{E}^*$. As $\mathrm{dist}(\mathcal{K}_\varepsilon,\mathcal{E}^*)>0$, we have from lemma 1,

$$\forall_{\varepsilon'\in]0,\mathrm{dist}(\mathcal{K}_\varepsilon,\mathcal{E}^*)[}\ \exists_{n'(=n_{\varepsilon'})}\ \forall_{n\geq n'}\ \forall_{(s,s')\in\mathcal{K}_\varepsilon\times\mathcal{K}_\varepsilon}\ |\delta(n,s)-\delta(n,s')|\leq\mu_{\varepsilon',n}|s-s'|$$

where $\mu_{\varepsilon',n}=1+\dfrac{m_n}{\varepsilon'|\lambda_n|}$. In particular,

$$\forall_{n\geq n_{\varepsilon'}}\ \forall_{s\in\mathcal{K}}\ \forall_{s'\in d_{s,\varepsilon}}\ |\delta(n,s)-\delta(n,s')|\leq\mu_{\varepsilon',n}|s-s'| ,$$

and hence

$$\delta(n,s')\geq\delta(n,s)-\mu_{\varepsilon',n}|s-s'| .$$

Further $\forall_{\beta'>\beta^*}\ \exists_{n'(=n_{\beta'})}\ \forall_{n\geq n'}\ \dfrac{m_n}{|\lambda_n|}<\beta'$ and

$$\forall_{n\geq\max\{n_{\varepsilon'},n_{\beta'}\}=n_1}\ \forall_{s\in\mathcal{K}}\ \forall_{s'\in d_{s,\varepsilon}}\ \delta(n,s')\geq\delta(n,s)-|s-s'|(1+\frac{\beta'}{\varepsilon'}) .$$

Since $\mathcal{K}$ is a compact subset of $\mathcal{D}_{*\alpha}$, $\forall_{s\in\mathcal{K}}\ \exists_{n'(=n_s)}\ \forall_{n\geq n'}\ \delta(n,s')>\alpha$; finally we have

$$\forall_{s\in\mathcal{K}}\ \forall_{\beta'>\beta^*}\ \forall_{\varepsilon'\in]0,\mathrm{dist}(\mathcal{E}^*,\mathcal{K}_\varepsilon)[}\ \exists_{n'}\ \forall_{n\geq n'}\ \forall_{s'\in d_{s,\varepsilon}}\ \delta(n,s')>\alpha-|s-s'|(1+\frac{\beta'}{\varepsilon'}) \tag{2.1}$$

where ε is arbitrary in $]0,\varepsilon_{\mathcal{K}}[$. The set of discs $d_{s,\varepsilon}$ indexed by s on $\mathcal{K}$ is an open covering for $\mathcal{K}$ and hence $\underset{\mathcal{K}\supset(s_1\ldots s_k)}{\exists} \bigcup_{j=1}^{k} d_{s_j,\varepsilon} \supset \mathcal{K}$. Further we have $\underset{s\in\mathcal{K}}{\forall} \underset{j'\in(1\ldots k)}{\exists} s \in d_{s_{j'},\varepsilon}$. Using (2.1) for the particular pair $(s_{j'},s)$, we have

$$\underset{\beta'>\beta^*}{\forall} \underset{\varepsilon'\in]0,\mathrm{dist}(\mathcal{E}^*,\mathcal{K}_\varepsilon)[}{\forall} \underset{n'(=n_{s_{j'},\beta',\varepsilon'})}{\exists} \underset{n\geq n'}{\forall} \delta(n,s) > \alpha - |s-s_{j'}|(1+\frac{\beta'}{\varepsilon'}) .$$

Let $n'' = \mathrm{Max}\{n_{s_j,\beta',\varepsilon'} \,|\, j\in(1\ldots k)\}$ and as $|s-s_j| < \varepsilon$, we have

$$\underset{\beta'>\beta^*}{\forall} \underset{\varepsilon'\in]0,\mathrm{dist}(\mathcal{E}^*,\mathcal{K}_\varepsilon)[}{\forall} \underset{n''}{\exists} \underset{n\geq n''}{\forall} \delta(n,s) > \alpha - \varepsilon(1+\frac{\beta'}{\varepsilon'}) .$$

Choosing $\varepsilon = \varepsilon' < \frac{\varepsilon_{\mathcal{K}}}{2}$ we have $\frac{\mathrm{dist}(\mathcal{K},\mathcal{E}^*)}{2} < \mathrm{dist}(\mathcal{E}^*,\mathcal{K}_\varepsilon)$ and

$$\underset{\beta'>\beta^*}{\forall} \underset{\varepsilon\in]0,\frac{\varepsilon_{\mathcal{K}}}{2}[}{\forall} \underset{n''}{\exists} \underset{n\geq n''}{\forall} \delta(n,s) > \alpha - \varepsilon - \beta'$$

where s is any arbitrary point of $\mathcal{K}$ and n'' does not depend on s . Hence

$$\underset{\beta'>\beta^*}{\forall} \underset{\varepsilon\in]0,\frac{\varepsilon_{\mathcal{K}}}{2}[}{\forall} \underset{n''}{\exists} \underset{n\geq n''}{\forall} \underset{s\in\mathcal{K}}{\forall} \delta(n,s) > \alpha-\varepsilon-\beta' .$$

As β' is arbitrary and strictly greater than β^* , we have

$$\underset{\mathcal{K}\subset\mathcal{D}_{*\alpha}}{\forall} \underset{\beta'>\beta^*}{\forall} \underset{n'}{\exists} \underset{n\geq n'}{\forall} \underset{s\in\mathcal{K}}{\forall} \delta(n,s) > \alpha - \beta'$$

and hence

$$\underset{\mathcal{K}\subset\mathcal{D}_{*\alpha}}{\forall} \underset{\beta'>\beta^*}{\forall} \underset{n'}{\exists} \underset{n\geq n'}{\forall} \underset{s\in\mathcal{K}}{\forall} |P_n(s)\exp -\lambda_n s| < \exp(-|\lambda_n|(\alpha-\beta')) .$$

THEOREM 1. - When $\beta^* < \infty$, $L < \infty$, the LC-dirichletian element $\{f\}$ converges absolutely on $\mathcal{D}_{*,L+\beta^*}$ and uniformly on any compact subset of $\mathcal{D}_{*,L+\beta^*}$.

PROOF. Let us suppose that $\mathcal{D}_{*,L+\beta^*}$ is non empty. Let $\mathcal{K}_o$ be a compact subset of $\mathcal{D}_{*,L+\beta^*}$. We know that $\underset{\alpha>L+\beta^*}{\exists} \mathcal{K}_o \subset \mathcal{D}_{*\alpha}$. Let

$\beta' \in]\beta^*, \alpha - L[$. From Lemma 2 we have,

$$\exists_{n'} \ \forall_{n \geq n'} \ \forall_{s \in \mathcal{K}_o} \ |P_n(s)\exp(-\lambda_n s)| < \exp\{-|\lambda_n|(\alpha-\beta')\} ,$$

where $\alpha - \beta' > L$. Hence

$$\sum_{n=n'}^{\infty} |P_n(s)\exp(-\lambda_n s)| < \sum_{n=n'}^{\infty} \exp\{-|\lambda_n|(\alpha-\beta')\}$$

and the series on the right hand side is convergent which proves that $\{f\}$ converges absolutely and uniformly on $\mathcal{K}_o$. Since $\mathcal{K}_o$ is any arbitrary compact subset of $\mathcal{D}_{*,L+\beta^*}$, $\{f\}$ converges uniformly on any compact subset of $\mathcal{D}_{*,L+\beta^*}$ and absolutely on $\mathcal{D}_{*,L+\beta^*}$.

REMARK 1. By the following method, we obtain a bigger set of absolute convergence for $\{f\}$. Let $\mathcal{D}_{*L}$ be supposed to be non-empty and $L < \infty$. Then $\forall_{s \in \mathcal{D}_{*L}} \ \exists_{\epsilon_s > 0} \ \delta_*(s) > L + \epsilon_s$; $\exists_{n'_s} \ \forall_{n \geq n'_s} \ \delta(n,s) > L + \epsilon_s$ and $\forall_{n \geq n'_s} \ -\mathrm{Log}\,|P_n(s)\exp(-\lambda_n s)| > (L+\epsilon_s)|\lambda_n|$. Hence

$$\sum_{n=n'_s}^{\infty} |P_n(s)\exp -\lambda_n s| < \sum_{n=n'_s}^{\infty} \exp\{-(L+\epsilon_s)|\lambda_n|\}$$

and as the series on the right hand side converges, the series (1.1) converges absolutely on $\mathcal{D}_{*L}$. In this result, we have no restriction on β^* .

REMARK 2. $\{f\}$ diverges on $\mathbb{C}-\mathcal{E}^*-\overline{\mathcal{D}}_{*o}$. If $s \in \mathbb{C}-\mathcal{E}^*-\overline{\mathcal{D}}_{*o}$, then $\delta_*(s) < 0$ and $\exists_{\alpha \in \mathbb{R}_o^+} \ \delta_*(s) < -\alpha$. Hence $\forall_{s \in \mathbb{C}-\mathcal{E}^*-\overline{\mathcal{D}}_{*o}} \ \exists_{\alpha > 0} \ \exists_{(n_j)} \ \delta(n_j, s) < -\alpha$ where (n_j) is an infinite subsequence of $\mathbb{N}-\{0\}$. Therefore

$$|P_{n_j}(s)\exp(-\lambda_{n_j} s)| > \exp(\alpha|\lambda_{n_j}|) > 1$$

and which shows that $\{f\}$ diverges on $\mathbb{C}-\mathcal{E}^*-\overline{\mathcal{D}}_{*o}$. When $L = 0$, we have convergence of the series (1.1) in $\mathcal{D}_{*o} \subset \mathbb{C}-\mathcal{E}^*$ and divergence in $\mathbb{C}-\mathcal{E}^*-\overline{\mathcal{D}}_{*o}$. We do not discuss the property of convergence of the series in $\mathcal{E}^*$.

From here onwards we consider a L-dirichletian element,

$$\{f\} : \sum_{1}^{\infty} P_n(s)\exp(-\lambda_n s) \tag{2.2}$$

where (λ_n) is a D-sequence (here λ_n are positive real numbers).

DEFINITION. It is said that a D-sequence (λ_n) is of the type (Λ) if the following conditions are satisfied :

i) the Dirichlet series $\sum_{j=1}^{\infty} \exp(-\lambda_n s)$ converges on $P_o = \{s \in \mathbb{C} \mid \sigma > 0\}$. (this gives that $\forall_{n\in\mathbb{N}-\{0\}} \sum_{j=n}^{\infty} \exp(-s(\lambda_j-\lambda_n))$ converges on P_o . Let $\theta_n(s)$ be its sum at the point s);

ii) $\forall\eta > 0$ the sequence of functions (θ_n) where $\theta_n : P_o \ni s \to \theta_n(s)$ is bounded on $\overline{P}_\eta = \{s \in \mathbb{C} / \sigma \geq \eta\}$;

iii) $\forall\eta > 0$ the sequence of functions (θ_n^*) where $\theta_n^* : P_o \ni s \to \sum_{j=1}^{n} \exp(-s(\lambda_n-\lambda_j))$ is bounded on $\overline{P}_\eta$.

EXAMPLE.- If (λ_n) is a D-sequence and $\exists_{\mu>0} \operatorname{Inf}(\lambda_{n+1}-\lambda_n) = \mu$, then it is easy to see that (λ_n) is of the type (Λ) .

If the D-sequence (λ_n) is of the type (Λ) , then we can easily show that $L = 0$.

Now let us prove Jentzsch's theorem for L-dirichletian element. This theorem for Dirichlet series with complex exponents was proved by T.M. Gallie [3] . First let us consider the associated Dirichlet series of $\{f\}$.

$$\{f_A\} : \sum_{1}^{\infty} A_n \exp(-\lambda_n s)$$

where A_n is defined by (1.4). Let

$$\sigma_p^{f_A} = \operatorname{Inf}\left\{\sigma\in\mathbb{R} / \lim |A_n \exp(-\lambda_n s)| = 0 \ , \ n \to \infty\right\}$$

be the abscisse of pseudo convergence of $\{f_A\}$. Then we know that

$$\sigma_p^{f_A} = \limsup_{n\to\infty} \left\{\frac{\log A_n}{\lambda_n}\right\} ;$$

when $L = 0$, $\sigma_p^{f_A}$ is the same as $\sigma_c^{f_A}$, the abscisse of convergence of $\{f_A\}$.

Let n and n' be two natural numbers such that $n' \geq n$. Let $E_{n,n'}$ denote the set, indexed by (n,n'), of points of $\mathbb{C}$ which are zeros of the LC-dirichletian polynomial

$$S_{n,n'}(S) = \sum_{j=n}^{n'} P_j(s)\exp(-s\lambda_j) \ ;$$

let E denote the union of all sets $E_{n,n'}$ corresponding to all pairs (n,n') and E_∞ be the set formed by the points which are zeros for an infinity of polynomials $S_{n,n'}(s)$. Let us put $E^* = E^d \cup E_\infty$ where E^d is the derived set of E. E^* is a closed subset of $\mathbb{C}$. It is evident that $E \supset \mathcal{E}$ and $E_\infty \supset \mathcal{E}_\infty$ and hence $E^* \supset \mathcal{E}^*$. We suppose in what follows that $\mathbb{C}-E^* \neq \phi$ (which implies $\mathbb{C}-\mathcal{E}^* \neq \phi$). Then we have

THEOREM 2. - When the D-sequence (λ_n) is of the type (N), $\sigma_c^{f_A} < \infty$ and $\beta^* < \infty$, then we have $(\mathrm{Fr}(\mathcal{D}_{*o}) \cap \mathbb{C}-\mathcal{E}^*) \subset E^*$.

PROOF. Let us suppose that the theorem is not true. Then there exists a point $b \in (\mathrm{Fr}(\mathcal{D}_{*o}) \cap \mathbb{C}-\mathcal{E}^*)$ and a disc $d(b,\rho)$ centred at b of radius $\rho > 0$, included in $\mathbb{C}-\mathcal{E}^*$ such that

$$\underset{n_o}{\exists} \ \underset{n' \geq n \geq n_o}{\forall} \ \underset{s \in d(b,\rho)}{\forall} \ S_{n,n'}(s) \neq 0 \ .$$

We have $|P_n(s)\exp(-\lambda_n s)| \leq A_n(1+|s|)^{m_n}|\exp(-\lambda_n s)|$ and

$\underset{\beta'>\beta^*}{\forall} \ \underset{n'_o(=n_{\beta'})}{\exists} \ \underset{n \geq n'_o}{\forall} \ (m_n/\lambda_n) < \beta'$. Let us take a certain $\beta' > \beta^*$ and put $\omega = \beta' \operatorname{Log}[1+\sup\{|s|/s \in d(b,\rho)\}] - \operatorname{Inf}\{\sigma/s \in d(b,\rho)\}$ and hence $\underset{n \geq n'_o}{\forall} \ \underset{s \in d(b,\rho)}{\forall} \ |P_n(s)\exp(-\lambda_n s)| < A_n \exp(\omega\lambda_n)$. From the definition of $\sigma_c^{f_A}$ we have $\underset{\sigma'>\sigma_c^{f_A}}{\forall} \ \underset{n''_o(=n_{\sigma'})}{\exists} \ \underset{n \geq n''_o}{\forall} \ A_n < \exp(\sigma'\lambda_n)$. Hence putting $n_1 = \operatorname{Max}(n_o, n'_o, n''_o)$, we get $\underset{n \geq n_1}{\forall} \ \underset{s \in d(b,\rho)}{\forall} \ |P_n(s)\exp(-\lambda_n s)| < \exp((\omega+\sigma')\lambda_n)$.

Let $S_{n_1,n}(s) = \sum_{j=n_1}^{n} P_j(s)\exp(-\lambda_j s)$ and $\underset{s \in d(b,\rho)}{\forall} \ T_{n_1,n}(s) = (S_{n_1,n}(s))^{1/\lambda_n}$; $[S_{n_1,n}(s)]^{1/\lambda_n}$ is defined to be equal to $\exp((1/\lambda_n)\operatorname{Log} S_{n_1,n}(s))$ where

$\operatorname{Im} \operatorname{Log} S_{n_1,n}(s) \in]-\pi,\pi]$. For each integer $n \geq n_1$ the function $T_{n_1,n} : d(b,\rho) \ni s \to T_{n_1,n}(s)$ is holomorphic on $d(b,\rho)$. We have

$$\forall_{s\in d(b,\rho)} \; |T_{n_1,n}(s)| = |(\sum_{j=n_1}^{n} P_j(s)\exp(-\lambda_j s))^{1/\lambda_n}| \leq \{\exp(\lambda_n(\omega+\sigma')+\log n)\}^{1/\lambda_n}$$
$$= \exp(\omega+\sigma')\exp(\frac{\log n}{\lambda_n}) \; .$$

Since (λ_n) is of the type (Λ) which implies $L = 0$, we have $\lim_{n\to\infty}\exp(\frac{\log n}{\lambda_n}) = 1$. Hence the sequence of functions $(T_{n_1,n})$, $n \geq n_1$, is bounded and hence normal on $d(b,\rho)$.

Let $\mathcal{K}$ be a compact subset of $d(b,\rho)$ such that $\operatorname{Int} \mathcal{K} \cap \mathcal{B}_{*o} \neq \phi$. From any extracted subsequence of $(T_{n_1,n})$ we can extract a subsequence which converges uniformly on $\mathcal{K}$ and the limit function is holomorphic on the $\operatorname{Int} \mathcal{K}$.

Let $\mathcal{K}_1$ be a compact subset of $d(b,\rho) \cap \mathcal{B}_{*o}$ such that $\operatorname{Int}\mathcal{K} \cap \operatorname{Int}\mathcal{K}_1 \neq \phi$. Then we have $\forall_{s\in\mathcal{K}_1} \lim_{n\to\infty} T_{n_1,n} = 1$. Now $\mathcal{K} \cup \mathcal{K}_1$ is a compact subset of $d(b,\rho)$. Then the subsequence extracted from the arbitrarily extracted subsequence of $(T_{n_1,n})$ converges uniformly on $\mathcal{K}\cup\mathcal{K}_1$ to a limit function holomorphic in $\operatorname{Int}(\mathcal{K}\cup\mathcal{K}_1)$ and continuous on the boundary of $\mathcal{K}\cup\mathcal{K}_1$ and takes the value one at each point of $\mathcal{K}_1$. Hence the limit function takes the value one at each point of $\mathcal{K}\cup\mathcal{K}_1$. This results that the sequence $(T_{n_1,n})$ converges to the same limit function on $\mathcal{K}\cup\mathcal{K}_1$.

As $\mathcal{K}$ is any arbitrary compact subset of $d(b,\rho)$ and $\mathcal{K}_1$ is any arbitrary compact subset of $d(b,\rho) \cap \mathcal{B}_{*o}$ such that $\operatorname{Int}\mathcal{K} \cap \operatorname{Int}\mathcal{K}_1 \neq \phi$, we have

$$\forall_{s\in d(b,\rho)} \lim_{n\to\infty} T_{n_1,n}(s) = 1 \; .$$

Let $s_o \in d(b,\rho) \cap (\mathbb{C}-\mathcal{E}^*-\overline{\mathcal{B}}_{*o})$. Then

$$\forall_{\epsilon>0} \; \exists_{n_1'(=n_{s_o,\epsilon})\geq n_1} \; \forall_{n\geq n_1'} \; |\sum_{j=n_1}^{n} P_j(s)\exp(-\lambda_j s)| < (1+\epsilon)^{\lambda_n}$$

and hence

$$\forall_{n>n_1'} \; |P_n(s_o)\exp(-\lambda_n s_o)| = |S_{n_1,n}(s_o) - S_{n_1,n-1}(s_o)| < 2(1+\varepsilon)^{\lambda_n}$$

which gives

$$-\frac{\operatorname{Log}|P_n(s_o)\exp(-\lambda_n s_o)|}{\lambda_n} > \frac{-\operatorname{Log} 2}{\lambda_n} - \operatorname{Log}(1+\varepsilon) \; ;$$

$\delta_*(s_o) \geq 0$ as ε is arbitrary. Hence we arrive at a contradiction that $s_o \in \overline{\mathcal{D}}_{*o} \cap \mathbb{C}-\mathcal{E}^*$ which establishes the result.

Finally, let us prove a theorem on the overconvergence of $\{f\}$ defined by (7). Before proving the theorem let us note that

REMARK 3. Let $\overline{\Delta}$ be any compact subset of $\mathbb{C}-\mathcal{E}^*$ and (λ_n) be a D-sequence of the type (Λ). We have $|P_n(s)\exp(-\lambda_n s)| \leq A_n(1+|s|)^{m_n}\exp(-\sigma\lambda_n)$. If $s \in \overline{\Delta}$, then $|P_n(s)\exp(-\lambda_n s)| \leq A_n(1+m_\Delta)^{m_n}\exp(m_\Delta\lambda_n)$ where $m_\Delta = \sup\{|s|/s \in \overline{\Delta}\}$. As $\overline{\Delta}$ is a compact set, m_Δ is finite ; for sufficiently large n we have

$$\frac{\operatorname{Log}|P_n(s)\exp(-\lambda_n s)}{\lambda_n} \leq \frac{\operatorname{Log} A_n}{\lambda_n} + \frac{m_n}{\lambda_n}\operatorname{Log}(1+m_\Delta) + m_\Delta \; ;$$

$$\delta_*(s) \geq -\sigma_c^{f_A} - \beta^* \operatorname{Log}(1+m_\Delta) - m_\Delta \; .$$

Hence $\forall_{\varepsilon>0} \; \overline{\Delta} \subset \mathcal{D}_{*,\alpha_o-\varepsilon}$ with $\alpha_o = -\sigma_c^{f_A} - \beta^* \operatorname{Log}(1+m_\Delta) - m_\Delta$. If

$\beta^* < \dfrac{-\sigma_c^{f_A} - m_\Delta}{1+\operatorname{Log}(1+m_\Delta)}$, we have $\overline{\Delta} \subset \mathcal{D}_{*\beta^*}$.

THEOREM 3. - When (λ_n) is a D-sequence of the type (Λ), $\beta^* < \infty$ and $\mathcal{D}_{*\beta^*} \neq \emptyset$ if there exist an infinite subsequence (n_ν), $\nu \in \mathbb{N}$, of $\mathbb{N}-\{0\}$ and a sequence of strictly positive numbers (θ_ν) such that

$$\lim_{\nu\to\infty} \theta_\nu = +\infty$$

and

$$\forall_{\nu\in\mathbb{N}} \; \lambda_{n_\nu+1} > (1+\theta_\nu)\lambda_{n_\nu} \tag{2.3}$$

then the sequence $\{S_{n_\nu}(s)\}$, $\nu \in \mathbb{N}$, where $S_{n_\nu}(s) = \sum_{j=1}^{n_\nu} P_j(s)\exp(-\lambda_j s)$, converges at each point s of any open simply connected subset (whose intersection with $\mathcal{D}_{*,\beta^*}$ in non empty) of an open set included in $\mathbb{C}-\mathcal{E}^*$ in which the function f defined by $\{f\}$ is holomorphic.

PROOF. Let us choose 3 bounded domains Δ_1, Δ_2 and Δ_3 in the following manner : $\bar{\Delta}_1 \subset \Delta_2$, $\bar{\Delta}_2 \subset \Delta_3$, $\bar{\Delta}_3 \subset \mathbb{C}-\mathcal{E}^*$, $\bar{\Delta}_1 \subset \mathcal{D}_{*\beta^*}$ and $\bar{\Delta}_3$ is included in an open subset of $\mathbb{C}-\mathcal{E}^*$ in which the function f defined by $\{f\}$ is holomorphic. Further let $Fr(\Delta_1)$, $Fr(\Delta_2)$ and $Fr(\Delta_3)$ satisfy a condition of Hadamard's type, namely

$$\exists_{b\in]0,1[} \ \mathrm{Log}\, M_2 \leq b \,\mathrm{Log}\, M_1 + (1-b)\,\mathrm{Log}\, M_3$$

where $\forall_{i\in(1,2,3)} \ M_i = \mathrm{Max}\{\,|f(s)|\,/\,s\in Fr(\Delta_i)\}$.

It is easy to see that $\exists_{\alpha>\beta^*} \ \bar{\Delta}_1 \subset \mathcal{D}_{*\alpha}$. Let us consider the set $I_1 = \{\alpha/\mathcal{D}_{*\alpha} \supset \bar{\Delta}_1\}$. I_1 is non empty and is an interval. Let $\alpha_{\Delta_1} = \sup I_1$. Then $\alpha_{\Delta_1} > \beta^*$ and $\forall_{\varepsilon>0} \ \mathcal{D}_{*,\alpha_{\Delta_1}-\varepsilon} \supset \bar{\Delta}_1$. We can easily show that, $\alpha_{\Delta_1} = \mathrm{Inf}\{\delta_*(s)\,|\,s\in\bar{\Delta}_1\}$ which implies that α_{Δ_1} is a finite number. Hence from lemma 2,

$$\forall_{\beta'\in]\beta^*,\alpha_{\Delta_1}[} \ \exists_{n_1} \ \forall_{n\geq n_1} \ \forall_{s\in Fr(\Delta_1)} \ |P_n(s)\exp(-\lambda_n s)| < \exp(-\lambda_n(\alpha_{\Delta_1}-\beta'))\ ;$$

hence for $n \geq n_1$

$$\sum_{j=n+1}^{\infty} |P_j(s)\exp(-\lambda_j s)| < \sum_{j=n+1}^{\infty} \exp\{-\lambda_j(\alpha_{\Delta_1}-\beta^*)\}$$
$$= \exp\{-\lambda_{n+1}(\alpha_{\Delta_1}-\beta')\} \sum_{j=n+1}^{\infty} \exp\{-(\alpha_{\Delta_1}-\beta')(\lambda_j-\lambda_{n+1})\}\ .$$

Since (λ_n) is a D-sequence of the type (Λ) and $\alpha_{\Delta_1}-\beta' > 0$, there exists a finite number strictly positive $B(\beta')$ such that

$$\forall_{n\in\mathbb{N}} \ \sum_{j=n+1}^{\infty} |\exp\{-(\lambda_j-\lambda_{n+1})s\,| \leq B(\beta') \quad \text{where} \quad \Re e\, s \geq \alpha_{\Delta_1}-\beta'\ ;$$

thus we have for each $n \geq n_1$

$$\sum_{j=n+1}^{\infty} |P_j(s)\exp(-\lambda_j s)| < B(\beta')\exp\{-\lambda_{n+1}(\alpha_{\Delta_1}-\beta')\} . \tag{2.4}$$

Now let $I_2 = \{\alpha \in \mathbf{R} \mid \mathcal{D}_{*\alpha} \supset \bar{\Delta}_3\}$. We have

$$\forall_{s\in\bar{\Delta}_3} \ \exists_{n'(=n_s)} \ \forall_{n\geq n'} \ \delta(n,s) \geq \frac{-\operatorname{Log} A_n}{\lambda_n} - \frac{m_n}{\lambda_n}\operatorname{Log}(1+|s|)+\sigma .$$

Let $m_{\Delta_3} = \sup\{|s| / s\in\bar{\Delta}_3\}$. Then

$$\forall_{n\geq n'} \ \delta(n,s) \geq \frac{-\operatorname{Log} A_n}{\lambda_n} - \frac{m_n}{\lambda_n}\operatorname{Log}(1+m_{\Delta_3}) - m_{\Delta_3}$$

$$\delta_*(s) \geq -\sigma_c^{f_A} - \beta^*\operatorname{Log}(1+m_{\Delta_3}) - m_{\Delta_3} ,$$

which shows that $\bar{\Delta}_3 \subset \mathcal{D}_{*\alpha}$ with $\alpha < -\sigma_c^{f_A} - \beta^*\operatorname{Log}(1+m_{\Delta_3}) - m_{\Delta_3}$, and hence $I_2 \neq \emptyset$ and is an interval in $\mathbf{R}$. Let $\alpha_{\Delta_3} = \sup I_2$. Then $\forall_{\varepsilon>0} \ \mathcal{D}_{*,\alpha_{\Delta_3}-\varepsilon} \supset \bar{\Delta}_3$. We can easily show that $\alpha_{\Delta_3} = \operatorname{Inf}\{\delta_*(s) \mid s\in\bar{\Delta}_3\}$ which implies that α_{Δ_3} is a finite number. Once again, from lemma 2, we get

$$\forall_{\beta'>\beta^*} \ \exists_{n_2} \ \forall_{n\geq n_2} \ \forall_{s\in Fr(\Delta_3)} \ |P_n(s)\exp(-\lambda_n s)| < \exp\{-\lambda_n(\alpha_{\Delta_3}-\beta')\}$$

which gives

$$\forall_{s\in Fr(\Delta_3)} \ \sum_{j=1}^{n(\geq n_2)} |P_j(s)\exp(-\lambda_j s)| = \sum_{j=1}^{n_2-1} |P_j(s)\exp(-\lambda_j s)| + \sum_{j=n_2}^{n} |P_j(s)\exp(-\lambda_j s)|$$

$$\leq \max\left\{\sum_{j=1}^{n_2-1} |P_j(s)\exp(-\lambda_j s)| / s\in Fr(\Delta_3)\right\} + \sum_{j=n_2}^{n} \exp(-\lambda_j(\alpha_{\Delta_3}-\beta')) .$$

Let us choose $\beta' > \beta^*$ such that $\alpha_{\Delta_3}-\beta' \neq 0$. Now we examine the two cases.

<u>Case 1</u>. - If $\alpha_{\Delta_3}-\beta' > 0$, then

$$\sum_{j=n_2}^{n} \exp(-\lambda_j(\alpha_{\Delta_3}-\beta')) = \exp(\lambda_n(\alpha_{\Delta_3}-\beta'))\sum_{j=n_2}^{n} \exp\{-(\alpha_{\Delta_3}-\beta')(\lambda_j+\lambda_n)\} < B''(\beta')\exp(\lambda_n(\alpha_{\Delta_3}-\beta'))$$

where $B''(\beta')$ is the sum of the series $\sum_{j=0}^{\infty} \exp(-2(\alpha_{\Delta_3}-\beta')\lambda_j)$.

Case 2. - If $\alpha_{\Delta_3}-\beta' < 0$, then

$$\sum_{j=n_2}^{n} \exp(-\lambda_j(\alpha_{\Delta_3}-\beta')) = \sum_{j=n_2}^{n} \exp(\lambda_j|\alpha_{\Delta_3}-\beta'|) = \exp(\lambda_n|\alpha_{\Delta_3}-\beta'|)\sum_{j=n_2}^{n} \exp(-(\lambda_n-\lambda_j)|\alpha_{\Delta_3}-\beta'|)$$

Since the D-sequence (λ_n) is of the type (Λ) there exists a finite number strictly positive $B'(\beta')$ such that

$$\forall_{n\in\mathbb{N}-\{0\}} \sum_{j=1}^{n} \exp\{-(\lambda_n-\lambda_j)|\alpha_{\Delta_3}-\beta'|\} \le B'(\beta')$$

which implies that

$$\sum_{j=n_2}^{n} \exp(-\lambda_j(\alpha_{\Delta_3}-\beta')) \le B'(\beta')\exp(\lambda_n|\alpha_{\Delta_3}-\beta'|) .$$

On putting $B'''(\beta') = \mathrm{Max}\{B'(\beta'),B''(\beta')\}$ we have

$$\sum_{j=n_2}^{n} \exp(-\lambda_j(\alpha_{\Delta_3}-\beta')) \le B'''(\beta')\exp(\lambda_n|\alpha_{\Delta_3}-\beta'|) . \tag{2.5}$$

Using the generalized form of Hadamard three circle theorem [4] we have

$$\exists_{b\in]0,1[} \quad \mathrm{Log}\, M_{2,\nu} \le b\,\mathrm{Log}\, M_{1,\nu} + (1-b)\mathrm{Log}\, M_{3,\nu} \tag{2.6}$$

where

$$M_{i,\nu} = \mathrm{Max}\{|R_{n_\nu}(s)|/s\in \mathrm{Fr}(\Delta_i)\} , \quad i = 1,2,3$$

with

$$R_{n_\nu}(s) = f(s) - \sum_{j=1}^{n_\nu} P_j(s)\exp(-\lambda_j s) .$$

From (2.4) we have for $n_\nu \ge n_1$

$$M_{1,\nu} \le B(\beta')\exp\{-(\lambda_{n_\nu+1})(\alpha_{\Delta_1}-\beta')\} < B(\beta')\exp\{-(1+\theta_\nu)\lambda_{n_\nu}(\alpha_{\Delta_1}-\beta')\} \tag{2.7}$$

because of (2.3) . On putting

$$B_o = \mathrm{Max}\{|f(s)|/s\in \mathrm{Fr}(\Delta_3)\} + \mathrm{Max}\left\{\sum_{j=1}^{n_2-1} |P_j(s)\exp(-\lambda_j s)|/s\in \mathrm{Fr}(\Delta_3)\right\}$$

we have from (2.5) for $n_\nu \ge n_2$,

$$M_{3,\nu} \le B_o + B'''(\beta')\exp(\lambda_n|\alpha_{\Delta_3}-\beta'|) .$$

Let $B'_o(\beta') = \mathrm{Max}(B_o,B'''(\beta')$. Then for $n_\nu \ge n_2$,

$$M_{3,\nu} \leq B'_o(\beta') \exp(\lambda_n |\alpha_{\Delta_3} - \beta'|) \ . \tag{2.8}$$

Then using (2.7) and (2.8) in (2.6) we get, for $n_\nu \geq \max\{n_1, n_2\}$

$$\operatorname{Log} M_{2,\nu} \leq b \operatorname{Log} B(\beta') + (1-b) \operatorname{Log} B'_o(\beta') + \{-b(1+\theta_\nu)(\alpha_{\Delta_1} - \beta') + (1-b)|\alpha_{\Delta_3} - \beta'|\}\lambda_{n_\nu}$$

Since $\lim_{\nu\to\infty} \theta_\nu = \infty$, we have $\lim -b(1+\theta_\nu)(\alpha_{\Delta_1} - \beta') + (1-b)|\alpha_{\Delta_3} - \beta'| = -\infty$, $\nu \uparrow \infty$ and hence $\lim_{\nu\to\infty} \operatorname{Log} M_{2,\nu} = -\infty$ which proves the theorem.

When the polynomial $P_n(s)$ reduces to a complex number $a_{n,o}$, we get the famous Ostrowaski's theorem [1] for Dirichlet series. Our theorem contains G.L. Lunt'z theorem [5] as a particular case when $P_n(s) = a_n s^{m_n}$.

COROLLARY. - In theorem 3 if we replace (2.3) by the condition that there exists a sequence (θ_n) of strictly positive numbers such that $\lim_{n\to\infty} \theta_n = \infty$ and $\exists_{n'} \ \forall_{n\geq n'} \ \lambda_{n+1} > (1+\theta_n)\lambda_n$, then each point of $(\operatorname{Fr} \mathcal{D}_{*o}) \cap \mathbb{C} - \mathcal{E}^*$ is a singular point for f defined by (2.2). In particular if $(\operatorname{Fr} \mathcal{D}_{*o}) \subset \mathbb{C} - \mathcal{E}^*$ then $\operatorname{Fr} \mathcal{D}_{*o}$ is a natural boundary for f .

PROOF. Let us suppose that the corollary is false. Then there exists a point $b \in (\operatorname{Fr} \mathcal{D}_{*o}) \cap \mathbb{C} - \mathcal{E}^*$ and a disc $d(b,\rho)$ centred at b and of radius $\rho > 0$ on which f is holomorphic. As a result of theorem 3 the sequence (S_n) converges on $d(b,\rho)$. From remark 2 $\{f\}$ diverges on $\mathbb{C} - \mathcal{E}^* - \overline{\mathcal{D}}_{*o}$. There exists necessarily points common to $\mathbb{C} - \mathcal{E}^* - \overline{\mathcal{D}}_{*o}$ and $d(b,\rho)$. For these points there is a contradiction which establishes the corollary.

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