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ON HAUSDORFF COMPACTIFICATIONS OF NON-LOCALLY COMPACT SPACES

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ABSTRACT. Let X be a completely regular, Hausdorff space and let R be the set of points in X which do not possess compact neighborhoods. Assume R is compact. If X has a compactification with a countable remainder, then so does the quotient X/R, and a countable compactification of X/R implies one for X-R. A characterization of when X/R has a compactification with a countable remainder is obtained. Examples show that the above implications cannot be reversed.

KEY WORDS AND PHRASES. Countable remainders, compactifications, non-locally compact spaces, components of $\beta X - X$.

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1. INTRODUCTION.

Let X be a completely regular, Hausdorff topological space. The question of characterizing when X has a Hausdorff compactification αX, where $\alpha X - X$ is countably infinite, has been answered for the locally compact case by Magill [2] and for the case when $\alpha X = \beta X$ by Okuyama [4] (where βX is the Stone-Cech compactification of X). In case X is an arbitrary completely regular space, no such characterization has been given. The purpose of this paper is to contribute results toward such a characterization.

Let R be the set of points in X which do not possess compact neighborhoods. Then for all compactifications αX of X, $R = Cl_{\alpha X}(\alpha X - X) \cap X$. (See [5].) Herein we observe that for compact R, a necessary condition for X to have a countable compactification is that X/R have one. The main theorem of this paper characterizes when X/R has a countable compactification.

2. CHARACTERIZATION OF $\alpha(X/R)$.

Throughout this paper all compactifications are Hausdorff compactifications. Let N denote the natural numbers. If R is a compact, non-empty subset of a completely regular space X and if X has a coutable compactification γX, then a countable compactification of X/R can be obtained from γX by identifying R to a single point. It is readily verified that the resulting space is Hausdorff.

If $\alpha(X/R)$ is a countable compactification of X/R, then $\alpha(X/R)$ is also a countable compactification of X - R. Thus, we have the following:

THEOREM 1. If X is completely regular and R is compact, then each of the following conditions implies the next:

(A) X has a countable compactification;

(B) X/R has a countable compactification;

(C) $X - R$ has a countable compactification.

Examples will be provided to show that none of these implications can be reversed.

If R is non-compact, then (A) no longer implies (C) as in Theorem 1. Let X be the unit disc in the standard plane with a countable dense subset removed from the boundary. The remaining boundary points constitute R. Then, clearly, X has a countable compactification but $X - R$, the open disc, has no countable compactification.

Let $Y = (\beta X - X) \cup R$.

THEOREM 2. Let X be a completely regular Hausdorff space with R compact and non-empty. Then the following are equivalent:

(A) X/R has a countable compactification.

(B) R is a G_δ-set in Y and components of R are components of Y.

PROOF. (A) implies (B). Take $\{p_n | n \in N\} = \gamma(X/R) - X/R$, where $\gamma(X/R)$ is a countable compactification of X/R, and let t_0 be the canonical mapping of X into $\gamma(X/R)$. Then t_0 has an extension t which maps βX onto $\gamma(X/R)$. We first show that t carries $\beta X - X$ onto $\gamma(X/R) - X/R$. Since the restriction of t to $X - R$ is a homeomorphism and $X - R$ is dense in βX and in $\gamma(X/R)$, t carries Y onto $[\gamma(X/R) - X/R] \cup \{r\}$, where $r = t[R]$ (cf. Lemma 6.11 [1]). If $x \in R$ and $y \in \beta X - X$, then since R is compact there exists a compact neighborhood N_R of R in βX such that $y \notin N_R$. Set $N = N_R \cap X$. Since $R \subseteq N$, $t_0[N]$ is a neighborhood of $t(x) = r$ in X/R. Thus, there is a neighborhood G in $\gamma(X/R)$ for which $t_0[N] = G \cap X/R$. If N_y is any neighborhood of y in βX, choose $z \in N_y \cap (X - N)$. Then $t(z) \notin G$ and it follows from the continuity of t that $t(x) \neq t(y)$. Hence $t[\beta X - X] = \gamma(X/R) - X/R$.

Next, let $K_n = t^{-1}(p_n)$, for each $n \in N$. Evidently, $\beta X - X = \bigcup\{K_n | n \in N\}$. Since each K_n is compact, the sets $Y - K_n$ are open in Y and $R = \bigcap\{Y - K_n | n \in N\}$. Thus R is a G_δ-set in Y.

Let C be a component of R and let C_1 be a component of Y, where $C \_ C_1$. If $C \neq C_1$, choose $x \in C_1 - C$. Now there exists a continuous injection f of $\{p_n \in N\} \quad \{r\}$ into the real numbers. (See [3]). But $f \circ t \; C_1$ must be connected and not a singleton, since $t[R] \neq t(x)$. This contradicts the fact that the image of f is countable. Thus, $C = C_1$, so that components of R are components of Y.

(B) implies (A). First we show that there exist sets $\{U_n | n \in N\}$ which are clopen in Y such that $\bigcap\{U_n | n \in N\} = R$. Note that Y is compact. Let $\{V_n | n \in N\}$ be open subsets of Y satisfying $\bigcap\{V_n | n \in N\} = R$. For each $n \in N$, set $K_n = Y - V_n$. We assume that each $K_n \neq \emptyset$. Let $(x,r) \in K_n \times R$. Since x and r are in distinct quasi-components of Y, there exists a clopen neighborhood $W_n(x,r)$ of r in Y, where $x \notin W_n(x,r)$. Now $\{W_n(x,r) | r \in R\}$ is an open covering of R so that a finite subfamily $\{W_n(x,r_i) | i = 1,\ldots,p(x)\}$ covers R. Take $W_n(x) = \bigcup\{W_n(x,r_i) | i = 1,\ldots,p(x)\}$. Thus $W_n(x)$ is a clopen subset of Y. $R \subseteq W_n(x)$, and $x \notin W_n(x)$. Since $\{Y - W_n(x) | x \in K_n\}$ is an open cover of K_n, there is a finite subcover $\{Y - W_n(x_j) | j = 1,\ldots,q(n)\}$.

For each $n \in N$, let $U_n = \bigcap\{W_n(x_j) | j = 1,\ldots,q(n)\}$. Then each U_n is a clopen subset of Y, $R \subseteq U_n$ and $K_n \subseteq Y - U_n$. Hence $R = \bigcap\{U_n | n \in N\}$.

Let $C_1 = Y - U_1$, and for $n > 1$, take $C_n = [Y - \bigcap\{U_i | i = 1,\ldots,n\}] - \bigcup\{C_i | i = 1,\ldots,n-1\}$. Then each C_n is a clopen subset of Y and $\beta X - X = \bigcup\{C_n | n \in N\}$.

Let $\sim$ be the equivalence relation in βX which identifies each C_n to a point and R to a point. The projection of βX onto $\beta X/\sim$ is denoted by Π.

For each $n \in N$, consider the point $\Pi[C_n]$ in $\beta X/\sim$. Now $\{C_n, Y - C_n\}$ is a partition of Y into disjoint open sets. Thus, C_n and $Y - C_n$ can be separated by open sets U and V in βX. Evidently, $\Pi[U]$ and $\Pi[V]$ are disjoint open subsets of $\beta X/\sim$. This shows that $\Pi[C_n]$ can be separated from any other point of $\beta X/\sim$. Since points of $\beta X - Y$ have compact βX - neighborhoods in $\beta X - Y$, it follows that $\beta X/\sim$ is a compact Hausdorff space.

It remains to show that X/R can be embedded in $\beta X/\sim$ in the desired manner. Let i be the natural embedding of X in βX and let p be the projection of X onto X/R. Since i is relation preserving, a continuous mapping j of X/R into $\beta X/\sim$ is induced such that $j \circ p = \Pi \circ i$. It follows that j is also a closed mapping, hence an embedding of X/R into $\beta X/\sim$ as desired. This completes the proof.

In [2] Magill shows that a locally compact space X has a countable compactification if and only if $\beta X - X$ has infinitely many components. As an application of the proof of Theorem 2, the following is proven.

COROLLARY 3. Let X be completely regular with R compact. If X has a countable compactification, then $\beta X - X$ has infinitely many components.

PROOF. Let t be a continuous mapping of βX onto $\alpha(X/R)$ which carries $\beta X - X$ onto $\alpha(X/R) - X/R$. Since the subspace $K = (\alpha(X/R) - X/R) \cup \{t(R)\}$ is compact and countable, it contains an open countable discrete subspace. Since $\alpha(X/R) - X/R$ contains infinitely many components of K, Y must contain infinitely many components.

The converse of Corollary 3 is false when X is not locally compact. Example (A) shows that X/R can have a countable compactification, so that $\beta X - X$ has infinitely many components, but X has no countable compactification. Example (A) also shows that condition (B) of Theorem 1 is not sufficient to insure that X has a countable compactification when R is compact.

EXAMPLE (A). Let S be the closed unit square in R^2, I be the unit interval, $L_0 = I \times \{0\}$, and, for $n \in N$, $L_n = I \times \{\frac{1}{n+1}\}$. For $X = S - \bigcup_{n \in N} L_n$, it is clear that X is not rim compact, and hence does not have a countable compactification (cf. [6]). Furthermore, $R = L_0$ and S is a compactification of X. The existence of a continuous surjection from βX onto S which leaves X fixed and which carries $\beta X - X$ onto $S - X$ guarantees that condition (B) of Theorem 2 is satisfied. Hence X/R has a countable compactification.

The following example shows that for R non-empty and compact the implication of (C) by (B) of Theorem 1 cannot be reversed. It suffices to exhibit X, with R a singleton, where $X - R$ has a countable compactification but X does not.

EXAMPLE (B). In the plane R^2 take
$X = [\{(x,y) \mid -1 < x < 1;\ -1 < y < 1\} \bigcup \{(1,0)\}] - \{(\frac{-n}{n+1}, 0) \mid n \in N\}$. Then $R = \{(1,0)\}$. Since X is not rim compact, it has no countable compactification. However, a countable compactification for $X - R$ is obtained by adjoining the points $(\frac{-n}{n+1}, 0)$, for each $n \in N$, and taking the one-point compactification of the resulting space.

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