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ON THE HANKEL DETERMINANTS OF CLOSE-TO-CONVEX UNIVALENT FUNCTIONS

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ABSTRACT. The rate of growth of Hankel determinant for close-to-convex functions is determined. The results in this paper are best possible.

KEY WORDS AND PHRASES. Starlike and close-to-convex Functions, Hankel Determinant
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1. INTRODUCTION.

Let K and S* be the classes of close-to-convex and starlike functions in $\gamma = \{z: |z| < 1\}$. Let f be analytic in γ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. The qth Hankel determinant of f is defined for $q \geq 1$, $n \geq 1$ by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots\cdots & a_{n+q-1} \\ a_{n+1} & \cdots\cdots\cdots\cdots & & \\ \vdots & & & \\ a_{n+q-1} & \cdots\cdots\cdots & & a_{n+2q-2} \end{vmatrix}$$

For $f \in S^*$, Pommerenke [2] has solved the Hankel determinant problem completely. Following essentially the same method, we extend his results in this paper to the class K.

2. MAIN RESULTS.

THEOREM 1. Let $f \in K$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then, for $m = 0,1,\ldots,$ there are numbers γ_m and $c_{m\mu}$ ($\mu = 0, \ldots, m$) that satisfy $|c_{mo}| = |c_{mm}| = 1$ and

$$\sum_{k=0}^{\infty} \gamma_k \le 3 \;,\; 0 \le \gamma_m \le \frac{2}{m+1} \tag{2.1}$$

such that

$$\sum_{\mu=0}^{m} c_{m\mu}\, a_{n+\mu} = 0(1)n^{-1+\gamma_m} \qquad (n\to\infty).$$

The bounds (2.1) are best possible.

PROOF. Since $f \in K$, there exists $g \in S^*$ such that, for $z \in \gamma$

$$zf'(z) = g(z)h(z),\ \mathrm{Re}h(z) > 0 \tag{2.2}$$

Now g can be represented as in [1], $g(z) = z \exp\left[\int_0^{2\pi} \log \frac{1}{1-ze^{-it}}\, d\mu(t)\right]$, where $\mu(t)$ is an increasing function and $\mu(2\pi) - \mu(0) = 2$. Let $\alpha_1 \ge \alpha_2 \ge \ldots$ be the jumps of $\mu(t)$, and $t=\theta_1, \theta_2,\ldots$ be the values at which these jumps occur. We may assume that $\theta_1 = 0$. Then $\alpha_1 + \alpha_2 + \ldots. \le 2$ and $\alpha_1 + \alpha_2 + \ldots + \alpha_q = 2$ for some q if and only if g is of the form

$$g(z) = z \prod_{j=1}^{q} (1-e^{-i\theta_j} z)^{-\frac{2}{q}} \tag{2.3}$$

We define ϕ_m by

$$\phi_m(z) = \prod_{\mu=1}^{m} (1-e^{i\theta_\mu} z),$$

and

$$\beta_m = \alpha_{m+1} \quad (m = 0,1,\ldots)$$

We consider the three cases i.e.

(i) $0 \le \alpha_1 \le 1$, (ii) $1 < \alpha_1 < \frac{3}{2}$, (iii) $\frac{3}{2} \le \alpha_1 \le 2$

as in [2] and the first part, that is the bounds (2.1), follows similarly. For the rest, we need the following which is well-known [2].

LEMMA. Let $\theta_1 < \theta_2 < \ldots < \theta_q < \theta_1 + 2\pi$, let $\lambda_1, \ldots, \lambda_q$ be real, and $\lambda > 0$, $\lambda \ge \lambda_j$ $(j=1, \ldots, q)$. If

$$\psi(z) = \prod_{j=1}^{q} (1-e^{-i\theta_j}z)^{-\lambda_j} = \sum_{n=1}^{\infty} b_n z^n \tag{2.4}$$

then $b_n = 0(1)\, n^{\lambda-1}$ as $n \to \infty$.

We write

$$\phi_m(z) = \sum_{\mu=0}^{m} c_{m\mu} z^{m-\mu},$$

and

$$\phi_m(z) z f'(z) = \sum_{n=1}^{m} b_{mn} z^{n+m} + \sum_{n=1}^{\infty} (n+m) a_{mn} z^{n+m} \tag{2.5}$$

where

$$b_{mn} = \sum_{\nu=0}^{n} (n+\nu)\, c_{m-\nu}\, a_{n-\nu},$$

$$a_{mn} = \sum_{\mu=0} c_{m\mu}\, a_{n+\mu}, \quad |c_{mo}| = |c_{mm}| = 1.$$

There are two cases.

(a) Let g in (2.2) have the form (3); that is, $\alpha_1 + \alpha_2 + \ldots + \alpha_q = 2$.

With $\gamma_m = \beta_m$, it follows that $\gamma_m \le \frac{2}{m+1}$, $\gamma_0 + \gamma_1 + \ldots \le 3$ and $\lambda_m = \frac{2}{m+1}$ implies $m = q-1$, $\alpha_1 = \ldots = \alpha_q = \frac{2}{q}$.

Now from (2.2), (2.5) and the Cauchy Integral formula, we have, with

$$B_m(r) = \frac{1}{r^{m+n}} \sum_{k=1}^{m} |\, b_{mk} \,| r^{k+m},$$

$$(n+m)\, |a_{mn}| \le \frac{1}{2\pi r^{n+m}} \int_0^{2\pi} |\phi_m(z) g(z) h(z)|\, d\theta + B_m(r). \tag{2.6}$$

Applying the Schwarz inequality, we have

$$(n+m)|a_{mn}| \le \frac{1}{2\pi r^{n+m}} \left(\int_0^{2\pi} |\phi_m(z)g(z)|^2 d\theta\right)^{\frac{1}{2}}\left(\int_0^{2\pi} |h(z)|^2 d\theta\right)^{\frac{1}{2}} + B_m(r) .$$

When we write $[\phi_m(z)g(z)]^2$ in the form (2.4), the exponents $-\lambda_j$ satisfy $\lambda_j \le 2\gamma_m$ ($j=1, \ldots q$: $m > 0$). Hence, using the Lemma, we have

$$\int_0^{2\pi} |\phi_m(z)g(z)|^2 \, d\theta \le A\, n^{2\gamma_m - 1}, \quad (n\to\infty). \tag{2.7}$$

Also

$$\frac{1}{2\pi} \int_0^{2\pi} |h(z)|^2 d\theta = \sum_{n=0}^{\infty} |d_n|^2 r^{2n} \quad (d_o=1),\ \text{Re}h(z) > 0$$

But $|d_n| \le 2$, $n \ge 1$, and so

$$\frac{1}{2\pi} \int_0^{2\pi} |h(z)|^2 d\theta \le 1+4 \sum_{n=1}^{\infty} r^2 n = \frac{1+3r^2}{1-r^2} \le An,\ n \ge 1 \tag{2.8}$$

From (2.7) and (2.8), we have

$$(n+m)|a_{mn}| \le An^{\gamma_m} \quad (n\to\infty)$$

i.e. $a_{mn} = 0(1)\, n^{\gamma_m - 1}$ $\quad (n\to\infty)$.

This proves the theorem in this case.

(b) Let g in (2.2) be not of the form (2.3). Then using arguments like those in [2], it follows that, for $z = re^{i\theta}$

$$\int_0^{2\pi} |\phi_m(z)g(z)h(z)| d\theta = 0(1)(1-r)^{-\gamma_m} .$$

Hence from (2.6), we have

$$a_{mn} = 0(1)n^{\gamma_m - 1} \quad (n\to\infty),$$

where a_{mn} is defined by (5).

The function f_o: $f_o(z) = z(1-z^q)^{-2/q} = \sum_{\nu=0}^{\infty} \binom{2/q+\nu-1}{\nu} z^{\nu q+1}$, shows that the bounds (1) are best possible. We also note that except in the case where $m=(q-1)$ and g in (2.2) is not of the form (2.3), one can choose $0 \le \gamma_m > \frac{2}{m+1}$ from theorem (1) and Pommerenke's method [2], we can now easily prove the following

THEOREM 2. Let $f \in K$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$.

Then for $q \ge 1$, $n \ge 1$,

$$H_q(n) = 0(1)n^{2-q} \quad (n\to\infty)$$

This estimate is best possible. In particular, if g in (2.2) is not of the form (2.3), there exists a $\delta = \delta(q,g) > 0$ such that $H_q(n) = 0(1)n^{2-q-\delta}$ $(n\to\infty)$.

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