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REMARK ON FUNCTIONS WITH ALL DERIVATIVES UNIVALENT

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ABSTRACT. An attractive conjecture is discounted for the class of normalized univalent functions whose derivatives are also univalent.

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1. INTRODUCTION.

Let S be the family of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic and univalent in the unit disk D: $|z| < 1$. Denote by E the set of functions f(z) in S for which the n-th derivative $f^{(n)}(z)$ is univalent in D for each n = 1, 2, 3, $\cdots$. Next we set

$$\alpha := \sup\{|a_2| : f(z) \in E\}. \tag{2}$$

It is known (cf. [2], [3]) that

$$\pi/2 \leq \alpha < 1.7208, \tag{3}$$

the left hand inequality being a consequence of the fact that $(e^{\pi z} - 1)/\pi \in E$. It was the belief of several authors that the function $(e^{\pi z} - 1)/\pi$ was extremal for (2). The purpose of the present note is to show that $\pi/2$ is not sharp in the inequality (3). This is accomplished by exhibiting other members of the set E which improve the lower estimate on α in (3).

In particular, we consider perturbations of $e^{\pi z}$ of the form

$$F(z) = F(z;\, a,b) := e^{\pi z} + a(z + bz^2). \tag{4}$$

for real parameters $a \geq 0$ and $b \geq \pi/2$. For certain values of a and b, the analytic function $(F(z;\, a,b) - 1)/(\pi + a)$ can be shown to be in E, which yields the estimate

$$\frac{\pi}{2} + \frac{a(b - \pi/2)}{\pi + a} \leq \alpha. \tag{5}$$

This leads us to the following

PROPOSITION. Let $a = \pi e^{-\pi}/35$ and $b = 18$. Then

$$\frac{F(z;\, a,b) - 1}{\pi + a} \in E. \tag{6}$$

Consequently we obtain the following improvement in (3),

$$\pi/2 + .02 < 1.5910 < \alpha. \tag{7}$$

Before discussing the proof of the above proposition, we motivate the particular choice of parameters a and b by examining what restrictions are imposed upon a and b by the univalence of $F(z;\, a,b)$ defined in (4).

As $F'(z;\, a,b)$ cannot vanish for z in D, and, in particular, for z in (-1,1), it follows that

$$0 \leq a \leq \pi e^{-\pi}/(2b - 1). \tag{8}$$

Furthermore, since $F(\bar{z}; a,b) = \overline{F(z; a,b)}$, the imaginary part of $F(z; a,b)$ must remain positive for all z in D with $\operatorname{Im} z > 0$. After some simple computations, this last remark implies, for b fixed and $x \in (-1, -1/2b)$, that

$$a \leq \frac{-e^{\pi x} \sin \pi\sqrt{1 - x^2}}{(1 + 2\,bx)\sqrt{1 - x^2}} \tag{9}$$

Hence to optimize the lower bound on α provided by (5) we select $a = \pi e^{-\pi}/(2b - 1)$ and choose b as large as possible so that the inequality of (9) remains valid. This maximal value appears to be near 18.9851. However, as little improvement is gained in (7) by this extreme choice for b, we make the more convenient special choice of $b = 18$ and $a = \pi e^{-\pi}/35$ in the Proposition.

PROOF OF PROPOSITION We now return to the univalence question for $F(z; \pi e^{-\pi}/35, 18)$ and its derivatives. From the definition of $F(z; a,b)$ in (4), for $[F(z; \pi e^{-\pi}/35, 18) - 1]/[\pi + \pi e^{-\pi}/35]$ to be in E, we need only show that $F(z; \pi e^{-\pi}/35, 18)$ and its first derivative are univalent in D. Moreover, it suffices to show that each of these functions is univalent on the unit circle $|z| = 1$ (cf. [1]). As the proof is rather technical, we only sketch the procedure for the univalence of $F(z; \pi e^{-\pi}/35, 18)$.

For θ real we define

$$u(\theta) := \operatorname{Re}\{F(e^{i\theta}; \pi e^{-\pi}/35, 18)\};$$
$$v(\theta) := \operatorname{Im}\{F(e^{i\theta}; \pi e^{-\pi}/35, 18)\}. \tag{10}$$

It can be verified (cf. Figure 1) that $u(\theta)$ is strictly decreasing on $(0,\theta_2)$ and strictly increasing on (θ_2,π) for some $\theta_2 \in (0,\pi)$. Hence there exists a unique $\theta_1 \in (0,\theta_2)$ for which $u(\theta_1) = u(\pi)$. Next it can be shown that $v(\theta) > v(\theta_2)$ for $\theta \in (\theta_1,\theta_2)$ and that $v(\theta_2) > v(\theta) > 0$ for $\theta \in (\theta_2,\pi)$. This behavior of the real and imaginary parts of $F(e^{i\theta}; a,b)$ guarantees the univalence of F(z; a,b) on the unit circle and hence in D, for the particular choices of a and b stated in the Proposition. Similarly, we may establish the univalence of F'(z; a,b). The Figure 1 is slightly exaggerated to demonstrate the behavior of v(θ) near the real line.

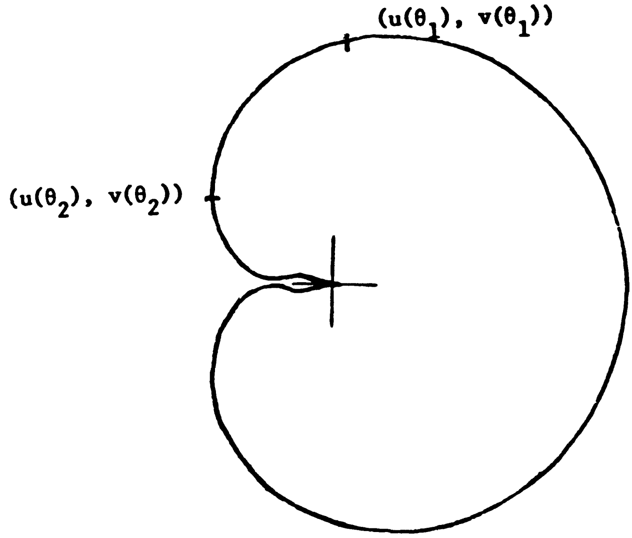

Figure 1. $F(e^{i\theta}; \pi e^{-\pi}/35, 18)$

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