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OPERATOR REPRESENTATION OF WEAKLY CAUCHY SEQUENCES IN PROJECTIVE TENSOR PRODUCTS OF BANACH SPACES

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ABSTRACT. It is shown that the above sequences always determine linear transformations and if the sequences are bounded under the least cross norm, that the transformations are continuous. Such operators are characterized to within algebraic isomorphism with the weak-star sequential closure of the tensor product space in its second dual, and consequently certain classes of weakly sequentially complete projective tensor products are exhibited.

KEY WORDS AND PHRASES. Tensor product, Weak topology, Operators, Sequential limits.

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1. INTRODUCTION.

Let E and F be normed linear spaces and G and H subspaces of their duals E* and F*, respectively. Let λ be the least cross norm [8] (operator norm, norm giving the inductive topolgy), and consider $E \hat{\otimes}_\lambda F$, the λ-completion of the tensor

product $E \otimes_\lambda F$. Then $G \otimes H$ is algebraically isomorphic to a subspace of $(E \hat{\otimes}_\lambda F)^*$ by $(\sum_i g_i \otimes h_i)(\sum_j x_j \otimes y_j) = \sum_{i,j} g_i(x_j)h_i(y_j)$ where $g_i \in G, h_i \in H$, $x_j \in E$, $y_j \in F$. As such, $G \otimes H$ carries the dual norm λ^0 of λ which is itself a cross norm different, in general, from the greatest cross norm γ[8] (nuclear norm, norm giving the projective topology).

It is easy to show that in order for $E \hat{\otimes}_\lambda F$ to be $\sigma(E \hat{\otimes}_\lambda F, G \otimes_\lambda{}^0 H)$-sequentially complete (or sequentially complete in its weak topology) it is necessary that E and F be (weakly) sequentially complete in their respective weak topologies $\sigma(E,G)$ and $\sigma(F,H)$. To motivate our work, let $E = \ell_1$, the space of absolutely summable sequences, $G = c$, the space of convergent sequences embedded in the bounded sequences, and F a weakly sequentially complete Banach space. Then $\ell_1 \hat{\otimes}_\lambda F$ is sequentially complete in its $\sigma(\ell_1 \hat{\otimes}_\lambda F, c \otimes_\lambda{}^0 F^*)$ topology, and the canonical map $\ell_1 \otimes_\lambda F \to L(c,F)$ is surjective, where $L(c,F)$ is the space of bounded linear transformations from c into F. This example and others of λ-tensor products which are sequentially complete under a weak topology and coincide with an associated operator space are found in [1]. Similar examples exist where the sequential completeness is under the weak topology. For instance, let E and F be reflexive Banach spaces with bases such that every operator from E^* into F is compact. Then $E \hat{\otimes}_\lambda F$ is reflexive [4, p. 188], whence $E \hat{\otimes}_\lambda F$ is weakly sequentially complete. Moreover, since F has a basis it also has the approximation property (a.p.) [7, p. 115] so that the canonical injection $E \hat{\otimes}_\lambda F \to L(E^*,F) = C(E^*,F)$ is surjective, where $C(E^*,F)$ is the space of compact operators from E^* into F.

Thus, we will show that the above examples are special cases of more general properties enjoyed by weakly Cauchy sequences in projective tensor product spaces and that the equivalence class of such a sequence (definition follows) always defines a linear transformation from G into H^* (Theorem 1)

which is continuous if the equivalence class is, in a sense, bounded (Theorem 3). Further, if G and H contain the extreme points of the unit cells S_{E^*} and S_{F^*}, we show that these equivalence classes are, algebraically, precisely the functionals in the weak star sequential closure of $E \otimes_\lambda F$ in its second dual and if H is norm-closed that these functionals are operators from G into H^* (Theorem 5). Consequently, for reflexive spaces E and weakly sequentially complete spaces F such that every operator from E^* into F is compact, we obtain that $E \otimes_\lambda F$ is weakly sequentially complete if E or F has the a.p. (Corollary 6).

In addition to notation already introduced, J will be the usual embedding of a normed linear space F in its second dual. If H is a subspace of F^*, $\phi = F \to H^*$ is defined by $\phi(y)h = h(y)$, $y \in F$, $h \in H$. K(F) will be the weak-star ($\sigma(F^{**}, F^*)$) sequential closure of JF in F^{**}, and $K_H(F)$ the $\sigma(H^*,H)$-sequential closure of ϕF in H^* (this last space arising in a natural way in our work). Thus, $K_{F^*}(F) = K(F)$ and $K_H(F) = F$ if F is $\sigma(F,H)$-sequentially complete. The sense of the last equality (algebraic isomorphism, homeomorphism, isometric isomorphism) depends on results in [10] and its bibliography which can be used to generate corollaries to Theorems 1,3 and 5.

RESULTS:

It is simple to show that G and H are total subspaces ($G_0 = \{0\}$) of the duals of the normed linear spaces E and F, respectively, if and only if $\sigma(E \otimes F, G \otimes H)$ is Hausdorff on $E \otimes F$. Also, if at least G is total over E, then $E \otimes F$ is algebraically isomorphic to a subspace of the linear transformations from G into F by

$$(\Sigma\, x_i \otimes y_i)g = \Sigma\, g(x_i)y_i$$

where $x_i \in E$, $y_i \in F$, $g \in G$.

We define $\omega(E,F,G,H)$ to be the set of equivalence classes of $\sigma(E \otimes F, G \otimes H)$ - Cauchy sequences in $E \otimes F$ where equivalence of sequences

means agreement in the limit at points of $G \otimes H$. This becomes a vector space when given the natural addition and scalar multiplication and as such contains a copy of $E \otimes F$:

$$E \otimes F \to \omega(E,F,G,H)$$

defined by

$$t \to z \text{ where } (t,t,t,\cdots) \in z$$

THEOREM 1: Let E and F be normed linear spaces, G a total subspace of E^* and H a norm-closed total subspace of F^*. Then $\omega(E,F,G,H)$ is algebraically isomorphic to a subspace of the linear transformations from G into $K_H(F)$.

PROOF: Let $\{z_i\} \in z \in \omega(E,F,G,H)$, where

$$z_i = \sum_{k=1}^{s_i} x_{k,i} \otimes y_{k,i}$$

for $x_{k,i} \in E$, $y_{k,i} \in F$, $i = 1,2,\cdots$. Then if $\varepsilon > 0$, $g \in G$ and $h \in H$ there exists $N > 0$ such that

$$\left| h\left(\sum_{k=1}^{s_n} g(x_{k,n})y_{k,n}\right) - h\left(\sum_{k=1}^{s_m} g(x_{k,m})y_{k,m}\right)\right| < \varepsilon$$

for $m,n > N$. Fixing $g \in G$ and varying ε and h, we see that for each $g \in G$ the sequence $\{\sum_{k=1}^{s_n} g(x_{k,i})y_{k,i}\}$ in F is $\sigma(F,H)$ - Cauchy.

Thus

$$h^* = \sigma(H^*,H) - \lim_i \phi\left[\sum_{k=1}^{s_i} g(x_{k,i})y_{k,i}\right]$$

is uniquely defined since H is total, is independent of the choice $\{z_i\} \in z$, and lies in $K_H(F)$. Define $(U(z))g = h^*$. It is straight forward that $U(z)$ is a linear (not necessarily continuous) map from G into $K_H(F)$ and that U is linear.

Put $U(z) = 0$, $z \in \omega(E,F,G,H)$. Then for all $g \in G$, and consequently for every $h \in H$,

$$\lim_i \sum_{k=1}^{s_i} g(x_{k,i})h(y_{k,i}) = 0$$

showing that $\{\sum_{k=1}^{s_i} x_{k,i} \otimes y_{k,i}\}$ converges to the null sequence in the $\sigma(E \otimes F, G \otimes H)$ topology. Thus, $z = 0$ and U is injective.

The following points out that if $G = E^*$ and $H = F^*$, every functional in $K(F)$ can be reached by a $U\omega$ map.

PROPOSITION 2: Let E and F be normed linear spaces. Then U has the property that given $y^{**} \in K(F)$ there exists $z \in \omega(E,F,E^*,F^*)$ and $x^* \in E^*$, $\|x^*\| = 1$, such that $(U(z))x^* = y^{**}$. Further, there exists a sequence $\{z_i\} \in z$ such that for all cross norms τ on $E \otimes F$, $\tau(z_i) = \|y^{**}\|$, $i = 1,2,\cdots$.

PROOF: Let $\{y_n\} \subseteq F$ converge $\sigma(F^{**},F^*)$ to $y^{**} \in K(F)$, fix $x \in E$, $\|x\| = 1$, and let $z_i = x \otimes y_i$. Then $\{z_i\}$ is $\sigma(E \otimes F, E^* \otimes F^*)$ - Cauchy in $E \otimes F$. There exists $x^* \in S_{E^*}$, $\|x^*\| = 1$, such that $x^*(x) = 1$, whence $(U(z))x^* = y^{**}$. Since [5, Lemma 2] holds for normed linear spaces, we may assume $\|y^{**}\| = \|y_n\|$ so that $\tau(z_n) = \|x\| \, \|y_n\| = \|y^{**}\|$ for any cross norm τ.

Below, $\omega_\lambda(E,F,G,H)$ will denote the subspace of those $z \in \omega(E,F,G,H)$ such that $\sup_i \lambda(z_i) < +\infty$ for some $\{z_i\} \in z$.

THEOREM 3: Let G and H be total over the normed linear spaces E and F and H be norm closed in F^*. Then $\omega_\lambda(E,F,G,H)$ is algebraically isomorphic to a subspace of $L(G,K_H(F))$ by the mapping U, and

$$\|U(z)\| \leq \inf_{\{z_i\} \in z} \sup_i \lambda(z_i).$$

PROOF: Let $\{z_i\} \varepsilon z \varepsilon \omega_\lambda(E,F,G,H)$, with

$$z_i = \sum_{k=1}^{s_i} x_{k,i} \otimes y_{k,i},\ i = 1,2, \cdots. \text{ Then}$$

$$\sup_{g\varepsilon S_G} \|(U(z))g\| = \sup_{g\varepsilon S_G} \sup_{h\varepsilon S_H} \lim_i \left|\sum_{k=1}^{s_i} g(x_{k,i})h(y_{k,i})\right|$$

$$\leq \sup_{g\varepsilon S_G} \sup_{h\varepsilon S_H} \overline{\lim_i}\, \lambda(z_i)$$

$$\leq \sup_i \lambda(z_i)$$

The central result, Theorem 5, provides an algebraic characterization of the weak-star sequential closure of $E \otimes_\lambda F$ in $(E \otimes_\lambda F)^{**}$ (i.e. in the dual of the space of integral bilinear forms on E x F [3]). The proof keys on [6] and [9] , and we cite [6] explicitly:

LEMMA A: (Rainwater): Let $\{x_n\}$ be a norm bounded sequence in a normed linear space X and M the set of extreme points of S_{X^*}. If $\{x_n\}$ is M-Cauchy, then $\{x_n\}$ is weakly Cauchy.

LEMMA 4: Let X be a normed linear space and W a total subspace of X^* which contains the extreme points of S_{X^*}, and $\{x_n\}$, $\{y_n\}$ two norm bounded $\sigma(X,W)$-Cauchy sequences in X such that $\lim f(x_n) = \lim f(y_n)$ for every $f \varepsilon W$. Then $\{Jx_n\}$ and $\{Jy_n\}$ are $\sigma(X^{**},X^*)$ - convergent to the same functional in X^{**}.

PROOF: Both sequences converge in X^{**} to the same limit as the weakly Cauchy sequence $\{w_k\}$, where $w_{2k-1} = x_k$, $w_{2k} = y_k$, $k = 1,2, \cdots$.

THEOREM 5: Let E and F be normed linear spaces with G and H total subspaces of E^* and F^*, respectively. If G and H contain the extreme points of S_{E^*} and S_{F^*}, then $\omega_\lambda(E,F,G,H)$ is algebraically isomorphic to $K(E \otimes_\lambda F)$. If also H is norm closed in F^*, $K(E \otimes_\lambda F)$ is algebraically isomorphic to a

subspace of $L(G,K_H(F))$ by a mapping T which is continuous with $\|T\| \leq 1$. If G and H determine the norm in E and F (i.e. have Dixmier characteristic one), then $\|T\| = 1$.

PROOF: The extreme points of the unit ball of $(E \otimes_\lambda F)^*$ are precisely those functionals of the form $x^* \otimes y^*$, where x^* and y^* are extreme points of S_{E^*} and S_{F^*}, respectively [9]. Let $z \in \omega_\lambda(E,F,G,H)$ and choose $\{z_i\} \in z$ so that $\sup_i \lambda(z_i) < +\infty$. By Lemma A, $\{z_i\}$ is weakly Cauchy in $E \otimes_\lambda F$. Define $V: \omega_\lambda(E,F,G,H) \to K(E \otimes_\lambda F)$ by $V(z) = \lim J(z_i)$, where the limit is in the weak-star topology of $(E \otimes_\lambda F)^{**}$. By Lemma 4, V is well-defined, and clearly it is linear and injective. Moreover, V is surjective. For if $z^{**} \in K(E \otimes_\lambda F)$ then z^{**} is the weak-star limit of a weakly Cauchy, hence norm bounded, sequence $\{z_i\}$ in $E \otimes_\lambda F$. Thus, for some z, $\{z_i\} \in z \in \omega_\lambda(E,F,G,H)$, and $V(z) = z^{**}$.

To establish the second claim, consider that V^{-1} is an algebraic isomorphism of $K(E \otimes_\lambda F)$ onto $\omega_\lambda(E,F,G,H)$, and by Theorem 3 U is injective from $\omega_\lambda(E,F,G,H)$ into $L(G,K_H(F))$. We take $T = UV^{-1}$. Thus, T is the required isomorphism, and we claim that T when restricted to $J(E \otimes_\lambda F)$ is J^{-1} (considering $E \otimes_\lambda F$ <u>algebraically</u> embedded in $L(G,K_H(F))$). Let $t \in J(E \otimes_\lambda F)$. Now $V^{-1}(t) = z$, where $\{J^{-1}(t), J^{-1}(t), \cdots\} \in z$. Consider $U(z) \in L(G,K_H(F))$. Recalling that the action of $U(z)$ on $g \in G$ is independent of the choice of sequence $\{z_i\} \in z$, we choose $\{J^{-1}(t), J^{-1}(t), \cdots\} \in z$. Then if $t = J(\Sigma x_k \otimes y_k)$, $(U(Z))g = \sigma(H^*,H) - \lim \phi[\Sigma g(x_k)y_k] = (\Sigma x_k \otimes y_k)g$ for each $g \in G$, whence $T(J(\Sigma x_k \otimes y_k)) = \Sigma x_k \otimes y_k$ for every $\Sigma x_k \otimes y_k \in E \otimes_\lambda F$.

Let $\{z_i\} \in E \otimes_\lambda F$ converge weak-star to z^{**} in $K(E \otimes_\lambda F)$. We may take $\lambda(z_i) = \|z^{**}\|$ [5] and from Theorem 3 obtain $\|Tz^{**}\| = \|UV^{-1}z^{**}\| \leq \|z^{**}\|$, whence $\|T\| \leq 1$. If G and H determine the norms in E and F and $x \otimes y \in E \otimes F$,

$$\begin{aligned}
\| T(J(x \otimes y)) \| &= \sup_{g \varepsilon S_G} \| (x \otimes y)g \| \\
&= \sup_{g \varepsilon S_G} \| g(x) \ \phi \ (y) \| \\
&= \sup_{g \varepsilon S_G} \ \sup_{h \varepsilon S_H} | g(x) \ h(y) | \\
&= \sup_{g \varepsilon S_{E^*}} \ \sup_{h \varepsilon S_{F^*}} | g(x) \ h(y) | \\
&= \lambda(x \otimes y) \\
&= \|J(x \otimes y)\| ,
\end{aligned}$$

whence $\|T\| = 1$.

COROLLARY 6: Let E and F be Banach spaces such that every operator from E^* into F is compact with E or F having the a.p. Then if E is reflexive and F is weakly sequentially complete, $E \hat{\otimes}_\lambda F$ is weakly sequentially complete.

PROOF: The a.p. insures that $E \hat{\otimes}_\lambda F$ coincides with the space of compact operators from E^* into F [7, p. 113], whence $L(E^*,F) = E \hat{\otimes}_\lambda F$. In Theorem 5 use $G = E^*$ and $H = F^*$. Since the weak-star sequential closure of a normed linear space is equal to that of its norm completion, $K(E \hat{\otimes}_\lambda F)$ is algebraically isomorphic under T to a subspace of $L(E^*,F)$. Since T, when restricted to $J(E \otimes_\lambda F)$, is the canonical map j of $E \otimes_\lambda F$ into $L(E^*,F)$, it follows that T, when restricted to the subspace $E \hat{\otimes}_\lambda F$ of $K(E \otimes_\lambda F)$, is j extended to $E \hat{\otimes}_\lambda F$. Thus,

$$j(E \hat{\otimes}_\lambda F) \subseteq TK(E \hat{\otimes}_\lambda F) \subseteq L(E^*,F) = E \hat{\otimes}_\lambda F$$

where each inclusion is isometric. It follows that T is an isometric isomorphism of $K(E \hat{\otimes}_\lambda F)$ onto $E \hat{\otimes}_\lambda F$, completing the proof.

Theorem 5 and Corollary 6 give information in a variety of special cases. For instance, every operator from a reflexive space E into ℓ_1 is compact [2, p. 515], and ℓ_1, by having a Schauder basis, has the a.p. Thus,

$E \hat{\otimes}_\lambda \ell_1$ is weakly sequentially complete by Corollary 6. In particular, the spaces $\ell_q \hat{\otimes}_\lambda \ell_1$ are weakly sequentially complete ($q > 1$). For $\frac{q}{q-1} > r > 1$, the spaces $\ell_q \hat{\otimes}_\lambda \ell_r$ are reflexive [4, p. 189], and thus weakly sequentially complete. However, if $p = \frac{q}{q-1} \leq r$, the spaces $\ell_q \hat{\otimes}_\lambda \ell_r$ are not reflexive and there exists a non-compact operator from ℓ_p into ℓ_r [4, p. 189]. Thus $L(\ell_p, \ell_r) \neq \ell_q \hat{\otimes}_\lambda \ell_r$. But by [3, p. 122] we have $(\ell_q \hat{\otimes}_\lambda \ell_r)^{**} = (\ell_p \hat{\otimes}_\gamma \ell_{r/r-1})^* = L(\ell_p, \ell_r)$ so that $K(\ell_q \hat{\otimes}_\lambda \ell_r) = L(\ell_p, \ell_r)$. Therefore, $\ell_q \hat{\otimes}_\lambda \ell_r$ is not weakly sequentially complete, $\frac{q}{q-1} \leq r$.

Following Theorem 5 one naturally seeks conditions under which T is a homeomorphism. Among the more interesting questions is that of $K(E \otimes_\lambda F)$ being homeomorphic to the whole of $L(G, K_H(F))$. Aside from Corollary 6, in dealing with some of the possibilities surrounding G and H we obtain results which are accessible through the piecing together of several theorems in [3]. For instance, let E be a Banach space with separable dual E and F a separable reflexive Banach space. Put $G = E^*$ and $H = F^*$. If E^* or F^* has the a.p. one can show via Theorem 5 or [3] that $K(E \otimes_\lambda F)$ is linearly homeomorphic to $L(E^*, F)$.

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