ON ELATIONS IN SEMI-TRANSITIVE PLANES

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ABSTRACT. Let \( \pi \) be a semi-transitive translation plane of even order with reference to the subplane \( \pi_0 \). If \( \pi \) admits an affine elation fixing \( \pi_0 \) for each axis in \( \pi_0 \) and the order of \( \pi_0 \) is not 2 or 8, then \( \pi \) is a Hall plane.

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1. INTRODUCTION.

Kirkpatrick [9] and Rahilly [10] have characterized the Hall planes as those generalized Hall planes of order \( q^2 \) that admit \( q+1 \) central involutions.

In [7] the author has shown that the derived semifield planes of characteristic \# 3 and order \( q^2 \) are Hall planes precisely when they admit \( q+1 \) central involutions. This extends Kirkpatrick and Rahilly's work as generalized Hall planes are certain derived semifield planes.

If a translation plane \( \pi \) of order \( q^2 \) admits \( q+1 \) affine elations with distinct axes then the generated group \( \mathcal{G} \) contains \( \text{SL}(2,q) \), \( S_z(q) \) or contains a normal subgroup \( N \) of odd order and index 2 (Hering [5]). In the latter case, little is known about \( \mathcal{G} \) except that it is usually dihedral.
In this article, we study semi-transitive translation planes of order \( q^2 \) that admit \( q+1 \) affine elations.

In [8], the author introduces the concept of the generalized Hall planes of type 1. These are derivable translation planes that admit a particular collineation group which is transitive on the components outside the derivable net. In this situation the group is generated by Baer collineations.

More generally, Jha [6] has considered the "semi-transitive" translation planes.

(1.1) Let \( \pi \) be a translation plane with subplane \( \pi_0 \). If there is a collineation group \( \mathcal{G} \) such that

1) \( \mathcal{G} \) fixes \( \pi_0 \cap \ell_\infty \) pointwise,
2) leaves \( \pi_0 \) invariant, and
3) acts transitively on \( \ell_\infty - \pi_0 \cap \ell_\infty \),

then \( \pi \) is said to be a semi-transitive translation plane with reference to \( \pi_0 \) and with respect to \( \mathcal{G} \).

Our main result is that semi-transitive planes of order not 16 or 64 that admit elations with axis \( \mathcal{L} \) fixing \( \pi_0 \) for every component \( \mathcal{L} \) of \( \pi_0 \) are Hall planes. We also give a necessary and sufficient condition that a translation plane of order \( q^2 \neq 64 \) admitting \( q+1 \) elations with distinct axes is derivable.

2. TRANSLATION PLANES OF EVEN ORDER \( q^2 \) ADMITTING \( q+1 \) ELATIONS.

(2.1) THEOREM. Let \( \pi \) be a translation plane of even order \( q^2 \neq 64 \) that admits \( q+1 \) affine elations with distinct axes. Let \( \mathcal{N} \) denote the net of degree \( q+1 \) that is defined by the elation axes and assume the group \( D \) generated by these elations leaves \( \mathcal{N} \) invariant. Then \( \mathcal{N} \) is derivable if and only if \( D \) is either isomorphic to \( SL(2,q) \) or is dihedral of order \( 2(q+1) \) where the cyclic stem fixes at least two components.

PROOF. If \( D \) is isomorphic to \( SL(2,q) \) then \( \mathcal{N} \) is derivable and actually \( \pi \) is Desarguesian by Foulser-Johnson-Ostrom [3].

Let \( D = \langle \sigma, \chi \mid \sigma^2 = \chi^{q+1} = 1, \sigma \chi = \chi^{-1} \sigma \rangle \). If \( \langle \chi \rangle \) fixes the components \( X = \mathcal{O} \), \( Y = \mathcal{O} \) then we may choose coordinates so that \( \sigma \) is \( (x,y) \rightarrow (y,x) \) and \( \chi \) is \( (x,y) \rightarrow (xT, yT^{-1}) \) for some matrix \( T \) of order \( q+1 \).
By Ostrom [11], Theorem 3, there is a Desarguesian plane $\Sigma$ containing the two $\chi$-fixed components and $\gamma$. Clearly $\gamma$ is an André net in $\Sigma$ and thus derivable in $\pi$.

Conversely, suppose $\gamma$ is derivable. Since each elation fixes $\gamma$, $D$ must fix each Baer subplane of $\gamma$ incident with $O$. By Foulser [2], Theorem 3, $D \leq \text{GL}(2, q)$ in its action on $\pi$ so that $D \leq \text{SL}(2, q)$ (each elation is then in $\text{SL}(2, q)$). By Gleason [4], $D$ is transitive on the elation axes so $q+1 \mid |D|$. Thus, $D$ is clearly $\text{SL}(2, q)$ or is dihedral of order $2(q+1)$. Moreover, if $\gamma$ is derivable then $\chi$ fixes at least two infinite points of $\pi - \gamma$. Let $\gamma$ replace $\gamma$ so $\chi$ fixes $\gamma$ componentwise in the derived plane $\bar{\pi}$. Let $\langle \chi \rangle \triangleleft \langle \bar{x} \rangle$ such that $|\bar{x}|$ is a prime 2-primitive divisor of $q^2 - 1$ (one exists since $q^2 \neq 64$). Then $\bar{\chi}$ fixes at least two infinite points of $\pi - \bar{\gamma}$ so there is a unique Desarguesian plane $\Sigma$ containing the $\bar{\chi}$-fixed components of $\bar{\pi}$ (see Ostrom [11], Cor. to Theorem 1—uniqueness comes from the fact that the degree of $\Sigma \cap \bar{\pi}$ is greater than $q+1$). Since $\bar{\chi}$ permutes the components of $\Sigma \cap \bar{\pi}$ (i.e., $\langle \bar{\chi} \rangle$ is characteristic in $\langle \chi \rangle$), $\bar{\chi}$ is a collineation group of $\Sigma$. The collineation $\chi$ has the form $(x, y) \rightarrow (x^a y^{-\phi(a)}, y^a)$ where $\phi$ is an automorphism of $\text{GF}(q^2)$ and $a \in \text{GF}(q^2)$. (Note $\chi$ fixes $\gamma$ componentwise.) Since $q+1$ is odd, $\langle \chi^2 \rangle = \langle \chi \rangle$. Choosing coordinates so that the components of $\bar{\gamma}$ are $X = O$, $Y = O$, $y = xa$, $a \in \text{GF}(q^2)$ then $\chi$ fixes $y = xa$ for all $a \in \text{GF}(q^2)$ if and only if $a^\phi = a$. Since $\langle \chi^2 \rangle = \langle \chi \rangle$, we may assume $\phi = 1$. Thus, $\chi$ fixes $\ell_\infty$ of $\Sigma$ pointwise. Since $\Sigma$ and $\pi$ share at least two components (those fixed by $\bar{\chi}$), $\chi$ must fix at least two components of $\pi$.

3. SEMI-TRANSITIVE TRANSLATION PLANES OF EVEN ORDER.

Let $\pi$ be a translation plane of even order $q^2$ that admits $q+1$ elations as in section 2. Then, $\pi$ is a derivable plane provided the generated group $D$ is dihedral and the cyclic stem fixes at least 2 points or $\text{SL}(2, q)$. In any case let $\gamma$ denote the net defined by the elation axes. Let $\mathcal{G}$ be a collection group that commutes with $D$. Then clearly, $\mathcal{G}$ must fix $\gamma \cap \ell_\infty$ pointwise.

(3.1) THEOREM. Let $\pi$ be a translation plane of even order $q^2 \neq 64$ that admits $q+1$ elations with distinct axes. Assume the group $D$ generated by these
q+1 elations leaves the net \( \mathcal{H} \) of the elation axes invariant. Let \( \mathcal{S} \) be a collineation group which commutes with \( D \) and is transitive on \( l_\infty - \mathcal{H} \cap l_\infty \). Then \( \pi \) is a Hall plane.

**PROOF.** Since \( q^2 \neq 64 \), there is a prime 2-primitive divisor \( m \) of \( q^2 - 1 \). By Gleason [4], \( q+1 \mid |D| \). Clearly, \( m \mid q+1 \). Let \( \chi \) be an element of \( D \) of order \( m \). \( \chi \) acts on the \( q(q-1) \) points of \( l_\infty - \mathcal{H} \cap l_\infty \) so must fix at least two points of \( l_\infty - \mathcal{H} \cap l_\infty \). Since \( \mathcal{S} \) commutes with \( \chi \) and \( \mathcal{S} \) is transitive on \( l_\infty - \mathcal{H} \cap l_\infty \), \( \chi \) must fix \( l_\infty - \mathcal{H} \cap l_\infty \) pointwise.

By the corollary to Theorem 1, Ostrom [11], there is a Desarguesian plane \( \Sigma \) such that the components fixed by \( \chi \) in \( \pi \) are exactly the common components of \( \Sigma \) and \( \pi \). Let \( \pi = \mathcal{H} \cup \mathcal{M} \) where \( \mathcal{M} \) is the net complementary to \( \mathcal{H} \) in \( \pi \). Then \( \Sigma = \mathcal{H} \cup \mathcal{M} \) for some net \( \mathcal{H} \) of degree \( q+1 \). So \( \Sigma \) and \( \pi \) are two extensions of a net \( \mathcal{M} \) of critical deficiency (see Ostrom [12]). Then \( \pi \) must be Hall since \( \Sigma \) and \( \pi \) must be related by derivation (i.e., \( \pi \) cannot be itself Desarguesian) by Ostrom [12].

The conditions of (3.1) are close to giving the definition of a "semi-transitive" translation plane (see (1.1)). In (3.1), it is possible that \( \mathcal{S} \) may not satisfy condition 2. Also, it is not clear that a semi-transitive translation plane is derivable. However, Jha [6] shows if \( \pi \) has order not 16 and there is a nontrivial kern homology in \( \pi \) then \( \pi \) is derivable and \( \pi_0 \) is a Baer subplane.

We may overcome this restriction on the kern in our situation:

**(3.2) THEOREM.** Let \( \pi \) be a semi-transitive translation plane of even order with respect to a collineation group \( \mathcal{S} \) and with reference to a subplane \( \pi_0 \). Let \( \pi \) admit an affine elation for each axis in \( \pi_0 \).

1) If the order of \( \pi_0 \) is not 8 then \( \pi \) is derivable.

2) If the order of \( \pi_0 \) is not 2 or 8 then \( \pi \) is a Hall plane.

**PROOF.** Following Jha's [6] ideas, let \( \pi_1 \) be a minimal subplane of \( \pi \) properly containing \( \pi_0 \). Clearly, the stabilizer \( \mathcal{S}_{\pi_1} \) of \( \pi_1 \) is a semi-transitive collineation group of \( \pi_1 \) with reference to \( \pi_0 \). Moreover, a sylow 2-subgroup of \( \mathcal{S}_{\pi_1} \) must leave \( \pi_0 \) pointwise fixed since \( \mathcal{S} \) fixes \( \pi_0 \) and fixes \( \pi_0 \cap l_\infty \) pointwise. (Note \( |\mathcal{S}_{\pi_1}| \) is divisible by \( (2^r+1)-(2^s+1) \) for some \( r,s \).) Clearly, \( \pi_0 \) is a Baer subplane of \( \pi_1 \).
Every elation which leaves $\pi_0$ invariant must also leave any superplane invariant. So the group $D$ generated by the elations leaves $\pi_1$ invariant and, clearly, $D$ commutes with $D$ since $D$ fixes $\pi_0 \cap \mathcal{L}_\infty$ pointwise ($D$ must commute with each central collineation fixing $\pi_0$).

By (3.1), if the order of $\pi_0$ is not 8 then $\pi_1$ is a Hall plane and $\pi_1$ is derivable. We may now directly use Jha [6] to show that if the order of $\pi_0$ is not 2 then $\pi_1 = \pi$ (that is, Jha uses the hypothesis that there is a kern homology to show that $\pi_1$ is derivable).

Actually, our proof of (3.2) proves the following more general theorem for arbitrary order.

(3.3) THEOREM. Let $\pi$ be a semi-transitive translation plane with reference to $\pi_0$ and with respect to $D$ and order $p^r$. Let $\chi$ be a collineation generated by central collineations leaving $\pi_0$ invariant such that $|\chi|$ is a prime $p$-primitive divisor of $(\text{order } \pi_0)^2 - 1$ (where the order of $\pi_0$ is not 2). Then $\pi$ is a Hall plane.

Note that a semi-transitive plane of odd order $p^{2r}$ must admit Baer $p$-elements (see Jha [6]). By Foulser [1], we could then not have both Baer $p$-elements and elations so we could restate our Theorem (3.2) without reference to order.

(3.2)2 is also valid if the order $\pi_0$ is 8. The arguments supporting this will appear in a related article.

REFERENCES


