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ON THE PERIODIC SOLUTIONS OF LINEAR HOMOGENEOUS SYSTEMS OF DIFFERENTIAL EQUATIONS

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ABSTRACT. Given a fundamental matrix $\phi(x)$ of an n-th order system of linear homogeneous differential equations $Y' = A(x)Y$, a necessary and sufficient condition for the existence of a k-dimensional ($k \leq n$) periodic sub-space (of period T) of the solution space of the above system is obtained in terms of the rank of the scalar matrix $\phi(T) - \phi(0)$.

KEY WORDS AND PHRASES. *Linear homogeneous system of differential equations, Fundamental matrix, Periodic solutions, periodic sub-spaces of (period T), Rank of the scalar matrix, Linearly independent vectors.*

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1. INTRODUCTION.

Consider the n-th order system of linear homogeneous differential equations

$$Y' = A(x)Y, \tag{1.1}$$

where $Y = \text{col}\ (y_1(x), y_2(x), \ldots, y_n(x))$, $Y' = \text{col}\ (Y_1'(x), \ldots, y_n'(x))$, $A(x) = ((a_{ij}(x)))$ is a square matrix of order n, each element $a_{ij}(x)$ of $A(x)$ is a real-valued function continuous on the real line R. Let S_n be the solution space of the system of equations (1.1) on the real line R and $T > 0$ be a real number. Let

$$\phi(x) = \begin{pmatrix} y_{11}(x) & y_{21}(x) & \cdots & y_{n1}(x) \\ y_{12}(x) & y_{22}(x) & \cdots & y_{n2}(x) \\ \vdots & \vdots & & \vdots \\ y_{1n}(x) & y_{2n}(x) & & y_{nn}(x) \end{pmatrix} \tag{1.2}$$

be a fundamental matrix of the system (1.1). The column vectors of $\phi(x)$ are linearly independent solutions of (1.1).

The purpose of this note is to deduce a necessary and sufficient condition for the existence of periodic sub-spaces (of period T) of the solution space S_n of the system (1.1) and to show that the existence and dimensions of these periodic sub-spaces depend not on any prior assumption about the periodicity (of period T) of the elements $a_{ij}(x)$ of the coefficient matrix $A(x)$ of the system (1.1) (that is, all the elements $a_{ij}(x)$ of $A(x)$ need not be periodic of period T), but precisely on the rank of the scalar matrix

$$\phi(T) - \phi(0). \tag{1.3}$$

2. MAIN RESULTS.

The condition (1.3) is stated more explicitly in the following theorem:

THEOREM. Let k be a non-negative integer, $0 \le k \le n$. There exists a k-dimensional sub-space S_k of the solution space S_n of the linear homogeneous system (1.1) such that each member of S_k is periodic of period T and no member of $S_n - S_k$ is periodic of period T if and only if the rank of the scalar matrix $\phi(T) - \phi(0)$ is n-k.

The above theorem can also be phrased is terms of the eigen values of the scalar matrix $\phi^{-1}(0)\phi(T)$ as follows.

COROLLARY. Let k be a non-negative integer, $0 \le k \le n$. There exists a k-dimensional sub-space S_k of the solution space S_n of the linear homogeneous system (1.1) such that each member of S_k is periodic of period T and no member of $S_n - S_k$ is periodic of period T if and only if $\lambda = 1$ is an eigen value of the scalar matrix $\phi^{-1}(0)\phi(T)$ of multiplicity k.

PROOF OF THE THEOREM. Let k be a non-negative integer, $0 \le k \le n$ and rank of $\phi(T) - \phi(0)$ is n-k. Then the dimension of the kernel of $\phi(T) - \phi(0)$ is k. Hence there exists k linearly independent vectors

$$v_i = \text{col }(c_{i1}, c_{i2}, \ldots, c_{in}), \quad i = 1,2,\ldots,k,$$

belonging to R^n such that

$$(\phi(T) - \phi(0))v_i = 0, \quad i = 1,2,\ldots,k \tag{2.1}$$

Let $f_i(x) = \phi(x)v_i$, $i = 1,2,\ldots,k$. The linear independence of the vectors $v_1, v_2,\ldots, v_k$ implies the linear independence of the k solution vectors $f_1(x), f_2(x), \ldots, f_k(x)$ of the system (1.1). Also,

$$f_i(T) - f_i(0) = (\phi(T) - \phi(0))v_i = 0, \quad i = 1,2,\ldots,k, \tag{2.2}$$

implies by the uniqueness of the solutions of initial value problem that $f_i(x+T) - f_i(x) = 0$ for all x, $i = 1,2,\ldots,k$. Hence each solution vector $f_i(x)$, $i = 1,2,\ldots,k$, is periodic of period T. Let S_k be the k-dimensional periodic (of period T) sub-space of S_n generated by $f_1(x),\ldots,f_k(x)$. We need to show that no member of $S_n - S_k$ is periodic of period T. Let $g_1(x) \in S_n - S_k$. Then $g_1(x)$ is non-trivial and $f_1(x), f_2(x),\ldots,f_k(x)$, $g_1(x)$ are k+1 linearly independent members of S_n. Let

$$g_1(x) = \phi(x)v_{k+1}, \quad \text{where } v_{k+1} = \mathrm{col}(c_{k+1\,1}c_{k+1\,2},\ldots,c_{k+1\,n}).$$

If possible, let $g_1(x)$ be periodic of period T. That is

$$g_1(x + T) - g_1(x) = 0 \quad \text{for all } x.$$

Then

$$g_1(T) - g_1(0) = 0. \tag{2.3}$$

Since any set of linearly independent members of S_n form a part of a basis of S_n, let $f_1(x),f_2(x),\ldots,f_k(x),g_1(x),g_2(x),\ldots,g_{n-k}(x)$ be a basis of S_n and

$$g_i(x) = \phi(x)v_{k+i}, \quad i = 2,3,\ldots,n-k,$$

where $v_{k+i} = \mathrm{col}(c_{k+i\,1},c_{k+i\,2},\ldots,c_{k+i\,n})$, $i = 2,3,\ldots,n-k$. The linear independence of the basis vectors $f_1(x), f_2(x),\ldots,f_k(x),g_1(x),\ldots,g_{n-k}(x)$ implies that the matrix

$$C = \begin{pmatrix} c_{11} & c_{21} & c_{n1} \\ c_{12} & c_{22} & c_{n2} \\ & & \\ c_{1n} & c_{2n} & c_{nn} \end{pmatrix}$$

is non-singular and hence

$$\text{rank of } (\phi(T) - \phi(0)) = \text{rank of } (\phi(T) - \phi(0))C, \quad \text{[see, 2]}, \tag{2.4}$$

But, by actual multiplication and using (2.2) and (2.3), we see that the first k+1 column vectors of $(\phi(T) - \phi(0))C$ are zero-vectors and hence the rank of $(\phi(T) - \phi(0))C$ is at most n-k-1. Therefore, from (2.4)

$$n-k = \text{rank of } (\phi(T) - \phi(0)) = \text{rank of } (\phi(T) - \phi(0))C \leq n-k-1$$

implying a contradiction. Hence $g_1(x)$ cannot be periodic of period T. That is no member of $S_n - S_k$ is periodic of period T.

Conversely, suppose that S_k be a k-dimensional sub-space of S_n such that every member of S_k is periodic of period T and no member of $S_n - S_k$ is periodic of period T. We need to show that the rank of $\phi(T) - \phi(0)$ is n-k.

Let $f_1(x), f_2(x), \ldots, f_k(x)$ be a basis of S_k and $f_1(x), f_2(x), \ldots, f_k(x), g_1(x), \ldots g_{n-k}(x)$ be a basis of S_n. Clearly $g_i(x) \in S_n - S_k$, $i = 1,2,\ldots,n-k$. Hence each $g_i(x)$, $i = 1,2,\ldots,n-k$, is not periodic of period T. Again the n-k vectors

$$g_1(T) - g_1(0),\ g_2(T) - g_2(0), \ldots,\ g_{n-k}(T) - g_{n-k}(0)$$

are linearly independent. For

$$\ell_1(g_1(T) - g_1(0)) + \ell_2(g_2(T) - g_2(0)) + \ldots + \ell_{n-k}(g_{n-k}(T) - g_{n-k}(0)) = 0$$

implies by the uniqueness of the solutions of initial value problem that

$$g(x) = \ell_1 g_1(x) + \ldots. + \ell_{n-k} g_{n-k}(x)$$

is a periodic solution of the system (1.1) of period T. Hence by our hypothesis $g(x) \in S_k$ and therefore

$$g(x) = \ell_1 g_1(x) + \ldots. + \ell_{n-k} g_{n-k} = b_1 f_1(x) + \ldots. + b_k f_k(x),$$

for all x, where $b_1, b_2, \ldots, b_k$ are real constants.

Since $f_1(x), \ldots, f_k(x), g_1(x), \ldots, g_{n-k}(x)$ form a basis of S_n, it follows that

$$\ell_i = 0, \quad i = 1,2,\ldots,n-k$$
$$b_j = 0, \quad j - 1,2,\ldots,k.$$

Hence the n-k vectors

$$g_1(T) - g_1(0), g_2(T) - g_2(0), \ldots,\ g_{n-k}(T) - g_{n-k}(0)$$

are linearly independent.

Let H(x) be the fundamental matrix of the linear system (1.1) whose column vectors are

$$f_1(x),\ f_2(x),\ldots,\ f_k(x),\ g_1(x),\ldots,\ g_{n-k}(x)$$

and C be a non-singular scalar matrix such that

$$H(x) = \phi(x)C.$$

Then

$$H(T) - H(0) = (\phi(T) - \phi(0))C \tag{2.5}$$

Since C is non-singular,

$$\text{rank of } (\phi(T) - \phi(0)) = \text{rank of } (\phi(T) - \phi(0))C = \text{rank of } (H(T) - H(0)).$$

But, the first k columns of $H(T) - H(0)$ are zero vectors by the periodicity of $f_1(x),\ f_2(x),\ldots,f_k(x)$ and the last n-k column vectors

$$g_1(T) - g_1(0),\ g_2(T) - g_2(0),\ldots,g_{n-k}(T) - g_{n-k}(0)$$

of $H(T) - H(0)$ are linearly independent as proved before. Hence the rank of $H(T) - H(0)$ is n-k. That is, the rank of $\phi(T) - \phi(0)$ is n-k. This completes the proof of the theorem.

To prove the corollary, we see, from (2.1) that

$$\phi^{-1}(0)\phi(T)v_i = v_i\ ,\quad i = 1,2,\ldots,k.$$

That is,

$$(\phi^{-1}(0)\phi(T) - I)v_i = 0,\quad i = 1,2,\ldots,k.$$

where I is the identity matrix. This means that $\lambda = 1$ must be an eigen value of the scalar matrix $\phi^{-1}(0)\phi(T)$ of multiplicity k. Hence arguing similarly as in the proof of the theorem one can prove the corollary easily.

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