SEMIGROUP STRUCTURE UNDERLYING EVOLUTIONS

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ABSTRACT. A member of a class of evolution systems is defined by averaging a one-parameter family of invertible transformations G with a semigroup T. The resulting evolution system, $U(t,s) = G(t)T(t-s)G(s)^{-1}$, preserves continuity and strong continuity, and in case G is a linear family, may have an identifiable generator and resolvent both of which are constructed from T. Occurrences of the class of evolutions are given to show possible applications.

KEY WORDS AND PHRASES. *Evolution system, semigroup of transformations, resolvent.*

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1. INTRODUCTION.

Given a subset C of a Banach space H, a strongly continuous evolution system of continuous transformations on C is a family of functions $U = \{ U(t,s) : t \geq s \geq 0 \}$ so that if $t \geq s \geq r \geq 0$, then

    i. $U(t,s)$ is a continuous function from C into C,

    ii. $U(s,s) = I$, the identity on C,

    iii. $U(t,s)U(s,r) = U(t,r)$, and

    iv. if $x \in C$, then $\{(p, U(p,s)x) : p \geq s \}$ is continuous.

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Such systems arise as a result of having unique solutions to the evolution

equation $y'(t) = A(t)(y(t))$, $y(s) = x$, where x is a point of C and if $t \geq 0$,

A(t) is a function from a subset of C into H.

   The study of evolution systems has in several papers been made through the

study of semigroups (see for instance Kato[5]). A strongly continuous semigroup

of continuous transformations on C is a family $T = \{ T(t): t \geq 0 \}$ of continuous

functions from C into C so that if t and s are in $[0, \infty)$, then

   i. $T(t+s) = T(t)(T(s))$,

   ii. $T(0) = I$, and

   iii. if x is an element of C, $\{(p,T(p)(x)) : p \geq 0 \}$
        is continuous.

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By the relationship $U(t,s) = T(t-s)$, each semigroup is a specialized evolution

system. One approach to the study of evolution systems has been to study semi-

groups and try to generalize to evolution systems properties the semigroups

display. A second approach (see Ball [1], or Neveu [8]) has been to embed U into

a semigroup T on $[0, \infty)xC$ by the action $T(t)(s,x) = (t+s,U(t+s,s)x)$ and study T.

Goldstein ([4]), however, has shown that even if U has linear, non-expansive maps

and analytic trajectories, T may not even have Lipschitz maps and hence may fail

to satisfy the hypothesis which is typically used in theorems about semigroups.

   This paper gives an alternative approach to studying evolution systems

through semigroups. Here the evolution systems is not studied by analogy but

rather by identifying a class of evolution systems for which semigroups are

 actual structural components. Results concern the structure of

 such evolution systems as they relate to the existing theory for

 evolution systems. Examples are given that illustrate under what

 conditions these evolution systems occur.

2. DEFINITIONS and THEOREMS.

   In addition to the notation established in the introduction,

the following notation will be used. L(H,H) will denote the

normed vector space of continuous linear transformations on H with,

for f in L(H,H), $|f| = glb \{m: if p \varepsilon H, \|f(p)\| \leq m \cdot \|p\| \}$.

R(H,H) will denote the subset of L(H,H) whose elements have

inverses that are in L(H,H). The identity function on the numbers

will be denoted by j. The domain (range) of a function f will be

denoted $D_f (R_f)$.

DEFINITION: Suppose that G is a family of functions so that

if t $\geq$ 0, G(t) : C $\to$ C. The statement that G is <u>strongly</u> <u>continuous</u>

at the point x of C means that g_x = {(t,G(t)(x)) : t $\geq$ 0} is

continuous. g_x will be called the <u>trajectory</u> of G from x.

DEFINITION: Suppose that U is a strongly continuous evolution

system of continuous   tranformations on C. The statement that

A is the <u>infinitesimal</u> <u>generator</u> for U means that A = {A(t):t $\geq$ 0}

is so that if t $\geq$ 0, A(t) = {(x,y) : x ε C and y =

$\lim_{h \to 0} h^{-1} (U(t+h,t)(x) - x)$}.

Theorem 1 establishes a way of combining a semigroup with a

one parameter family so that the result exhibits evolution system

structure.
Conditions are given under which the form of the infinitesimal

generator can be guaranteed.

THEOREM 1. Suppose that T is a strongly continuous semi-

group of continuous transformations on C with infinitesimal gen-

erator B and G is a strongly continuous family of functions on C

so that if t $\geq$ 0, G(t) is Lipschitz and has a Lipschitz inverse

on C; and if a and b are numbers so that a $\leq$ b, {k: a $\leq$ p $\leq$ b and

k is the Lipschitz norm of G(p)} is a bounded set. For t $\geq$ s $\geq$ 0,

define U(t,s) = $G(t)T(t-s)G(s)^{-1}$.

Then U is a strongly continuous evolution system of continuous

transformations on C. Suppose, in addition, that C=H, x ε H,

G(s) ε R(H,H) for each s, and t is a number so that $G(t)^{-1}x$

is in the domain of B and the trajectory of G from $G(t)^{-1}x$ is

differentiable at t. Then, if A denotes the infinitesimal generator

for U, x is in the domain of $A(t)$ and $A(t)x = G(t)BG(t)^{-1}(x) + g_y'(t)$

where $y = G(t)^{-1}x$.

PROOF: To see that U is an evolution system of continuous

transformations on C, note that (i) $U(t,s) = G(t)T(t-s)G(s)^{-1}$ is

the composition of continuous functions each mapping C into

C; (ii) $U(s,s) = G(s)T(s-s)G(s)^{-1} = I$; and (iii) $U(t,s)U(s,r) =$

$G(t)T(t-s)G(s)^{-1}(G(s)T(s-r)G(r)^{-1}) = G(t)T(t-r)G(r)^{-1} = U(t,r)$.

For strong continuity, suppose that $s \geq 0$, $x \in C$, and

$\{t_n\}_{n=1}^{\infty}$ converges to t in $[s, \infty)$. $\| U(t_n,s)x - U(t,s)x \| =$

$\|G(t_n)T(t_n-s)G(s)^{-1}(x) - G(t)T(t-s)G(s)^{-1}(x)\| \leq \|G(t_n)T(t_n-s)G(s)^{-1}(x)$

$-G(t_n)T(t-s)G(s)^{-1}(x) \| + \| G(t_n)T(t-s)G(s)^{-1}(x) -$

$G(t)T(t-s)G(s)^{-1}(x) \| \leq |G(t_n)|_{Lip} \cdot \|T(t_n-s)G(s)^{-1}(x) - T(t-s)G(s)^{-1}(x) \|$

$+ \| G(t_n)T(t-s)G(s)^{-1}(x) - G(t)T(t-s)G(s)^{-1}(x) \|$. T is strongly

continuous at $G(s)^{-1}(x)$, $\{|G(t_n)|\}_{n=1}^{\infty}$ is a bounded number

sequence, and G is strongly continuous at $T(t-s)G(s)^{-1}(x)$. Thus

each term of the final inequality can be made arbitrarily small

and each trajectory of U is continuous.

With G satisfying the additional assumptions, suppose x is

an element of H so that $y = G(t)^{-1}(x)$ is in the domain of B and

consider $\lim_{h \to 0} h^{-1}(U(t+h,t)(x) - x) = \lim_{h \to 0} h^{-1}(G(t+h)T(h)G(t)^{-1}(x)$

$- x) = \lim_{h \to 0} h^{-1}(G(t+h)T(h)G(t)^{-1}(x) - G(t+h)G(t)^{-1}(x)) +$

$h^{-1}(G(t+h)G(t)^{-1}(x) - G(t)G(t)^{-1}(x)) = \lim_{h \to 0} G(t+h)(h^{-1}(T(h)-I)(G(t)^{-1}$

$(x))) + \lim_{h \to 0} h^{-1}(g_y(t+h) - g_y(t)) = G(t)BG(t)^{-1}(x) + g_y'(t)$.

Hence x is in the domain of $A(t)$ and $A(t)(x) = G(t)BG(t)^{-1}(x) +$

$g_y'(t)$.

Theorem 2 provides a partial converse to Theorem 1. The

thrust of the theorem is that a non-linear problem may, through

the structure from Theorem 1, be exchanged for a linear problem.

THEOREM 2: Suppose that $\{A(t)\}_{t \geq 0}$ is so that there is a unique solution to $v'(t) = A(t)(v(t))$, $v(s) = x$ for each point x in the domain of A(s). Suppose also that there is a function G with domain $[0, \infty)$ and range in R(H,H) and a strongly continuous semigroup T of continuous transformations on H with infinitesimal generator B so that

    i. if $t > 0$ and $x \epsilon H$, then $T(t)(x) \epsilon D_B$;

    ii. if $t \epsilon [0, \infty)$, then $G(t)(D_B) = D_{A(t)}$;

    iii. if $x \epsilon D_B$ and $y_x : [0, \infty) \to H$ is defined by
        $y_x(t) = G(t)(x)$, then $y_x{}'(t) = A(t)G(t)(x) - G(t)B(x)$; and

    iv. if $t \geq s \geq 0$, $\{|G(r)| : s \leq r \leq t\}$ is a bounded set.

Then if U is the evolution system generated by A and $t \geq s$,

$$U(t,s) = G(t)T(t-s)G(s)^{-1}.$$

PROOF: Suppose that $t \geq s$ and that $x \epsilon D_{A(s)}$. By assumption, there is a unique function v so that $v'(t) = A(t)(v(t))$ and $v(s) = x$. Also, $U(t,s)(x)$ is defined to be $v(t)$. Consider $f_x : [0, \infty) \to H$ defined by $f_x(t) = G(t)T(t-s)G(s)^{-1}(x)$. $f_x(s) = x$.

$\| h^{-1}(f_x(t+h) - f_x(t)) - A(t)(f_x(t)) \| = \| h^{-1}(G(t+h)T(t+h-s)G(s)^{-1}(x)$
$- G(t)T(t-s)G(s)^{-1}(x)) - A(t)G(t)T(t-s)G(s)^{-1}(x)) \|$.

Notice that $A(t)(f_x(t))$ makes sense since by assumption x is in $D_{A(s)}$ and if $t > s$, $R_{T(t-s)}$ is a subset of D_B and $G(t)(D_B) = D_{A(t)}$. From assumption iii. on the strong differentiability of G, and denoting $T(t-s)G(s)^{-1}x$ by y , the above norm can be written $\| h^{-1}(G(t+h)T(t+h-s)G(s)^{-1}(x) - G(t)T(t-s)G(s)^{-1}(x)) -$
$(g_y{}'(t) - G(t)BT(t-s)G(s)^{-1}(x) \| \leq \| G(t+h)(h^{-1}(T(h)(T(t-s)G(s)^{-1}(x)) -$
$T(t-s)G(s)^{-1}(x)) - G(t+h)(B(T(t-s)G(s)^{-1}(x)) \| +$
$\| h^{-1}(G(t+h)(T(t-s)G(s)^{-1}(x)) - G(t+h)(B(T(t-s)G(s)^{-1}(x))) - g_y{}'(t) \|$
$+ \| G(t+h)(BT(t-s)G(s)^{-1}(x)) - G(t)BT(t-s)G(s)^{-1}(x) \|$.

The first term converges to 0 by the differentiability of T at

$T(t-s)G(s)^{-1}(x)$ and the boundedness of $\{|G(r)| : r \in [t, t+m]\}$ the second converges to 0 by the strong differentiability of G at $T(t-s)G(s)^{-1}(x)$; and the third converges to 0 by the strong contin- uity of G at $BT(t-s)G(s)^{-1}(x)$. Thus $f_x'(t) = A(t)(f_x(t))$, $f_x(s)=x$ and by the uniqueness of solutions $U(t,s)(x)=v(t)=f_x(t) = G(t)T(t-s)G(s)^{-1}(x)$.

In earlier work on nonlinear evolution systems (see for instance Crandall [3]) the assumption of accretiveness on the argu- ments of A provides existence of resolvents for A from which product formulas for the evolution system can be constructed. Theorem 3 shows that if the linear differentiability of the hypo- thesis of Theorem 2 is strengthened, then the resolvent structure for the underlying semigroup is carried forward to the evolution system.

THEOREM 3: Suppose that G: $[0, \infty) \to R(H,H)$ is differentiable in $L(H,H)$ and that T is a strongly continuous semigroup of contin- uous transformations on H with generator B so that if $t>0$, $(I-tB)^{-1}$ is a Lipschitz transformation on H, and that there is a number M so that the Lipschitz norms for $\{(I - tB)^{-1}: 0 < t \le M\}$ is a bounded set. Define U(t,s) to be $G(t)T(t-s)G(s)^{-1}$ and let A denote the infinitesimal generator for U. Then if $t > 0$, there is $k > 0$ so that if $0 < m < k$, $(I - mA(t))^{-1}$ is a Lipschitz transformation with domain H and Lipschitz norm no greater than
$(|G(t)| |(I - mB)^{-1}| |G(t)^{-1}|) / (1 - m|G(t)| |(I - mB)^{-1}| |G(t)^{-1}| |G'(t)|)$.

PROOF:

Initially note that the differentiability assumed for Theorem 3 is in the norm topology of $L(H,H)$, that is, if $t \ge 0$, $G'(t)$ is in $L(H,H)$. Hence, from Theorem 1, A(t) can be written as $T(t)BG(t)^{-1} + G'(t)G(t)^{-1}$. Pick x from H. For $I - mA(t)$

to be invertible with domain H there must be exactly one point y so that $(I - mA(t))(y) = x$. For $m > 0$, define a function

F_m: $H \to H$ by $F_m(p) = G(t)(I - mB)^{-1}(mG(t)^{-1}G'(t)G(t)^{-1}(p) + G(t)^{-1}(x))$. Given $c > 0$, let M denote an upper bound for $\{(I - nB)^{-1} : n \leq c\}$.

$\| F_m(p) - F_m(q) \| \leq m|G(t)| \ |(I - mB)^{-1}| \ |G(t)^{-1}|^2|G'(t)| \ \| p - q \|$.

Thus $m < \min\{(|G(t)| M \ |G(t)^{-1}|^2|G'(t))^{-1}|, c\}$ implies that F_m is a strict contraction and has a unique fixed point y.

Since $y = G(t)(I - mB)^{-1}(mG(t)^{-1}G'(t)G(t)^{-1}(y) + G(t)^{-1}(x))$, it follows that $(I - mB)G(t)^{-1}(y) = mG(t)^{-1}G'(t)G(t)^{-1}(y) + G(t)^{-1}(x)$,

$G(t)^{-1}(y) - m(BG(t)^{-1}(y) + G(t)^{-1}G'(t)G(t)^{-1}(y)) = G(t)^{-1}(x)$, and

$y - m((G(t)BG(t)^{-1} + G'(t)G(t)^{-1})(y)) = x$, that is, $(I - mA(t))(y) = x$. Reversing the calculations shows that any such point must be a fixed point of F_m. Thus $y = (I - mA(t))^{-1}(x)$.

For $y_1 = (I - mA(t))^{-1}(x_1)$ and $y_2 = (I - mA(t))^{-1}(x_2)$,

$\|(I - mA(t))^{-1}(x_1) - (I - mA(t))^{-1}(x_2)\| = \| G(t)(I - mB)^{-1}$

$(mG(t)^{-1}G'(t)G(t)^{-1}(y_1) + G(t)^{-1}(x_1)) - G(t)(I - mB)^{-1}$

$(mG(t)^{-1}G'(t)G(t)^{-1}(y_2) + G(t)^{-1}(x_2)) \| \leq |G(t)| \ |(I - mB)^{-1}|$

$(m|G(t)^{-1}|^2|G'(t)| \ \|y_1 - y_2\| + |G(t)^{-1}| \ \| x_1 - x_2 \|)$.

Thus $\| y_1 - y_2\| \leq ((|G(t)| \ |(I - mB)^{-1}\|G(t)^{-1}|)/(1 - m|G(t)|$

$|(I - mB)^{-1}\|G(t)^{-1}|^2 \ |G'(t)|)) \| x_1 - x_2 \|$.

3. OCCURRENCES.

   There appear to be several possibilities for the application of the evolution structure identified in the preceding theorems.

Example 1. In [7], Neuberger studied semigroups of Lipschitz transformations so that if T is one of them, then there is $c \geq 0$ so that if x and y are elements of H, and $t \geq 0$, then $\| T(t)(x) - T(t)(y)\| \leq e^{ct} \| x - y \|$. By taking $G(t) = e^{-ct}$ I, Neuberger's special evolution system which produced product formulas for the semigroup can be seen to be an instance of the theory.

Example 2. Suppose that T is a semigroup of continuous transformations on H with infinitesimal generator B and G is a group of continuous linear trans-

formations on H with generator C.   Then there is an evolution system U on H

with generator A so that $B + C \subset A(0)$.

PROOF: Define U by $U(t,s) = G(t)^{-1}T(t-s)G(s)^{-1}$. If $x \in D_{B+C}$ the hypothesis

to Theorem 1 is satisfied and $A(0)(x) = G(0)BG(0)^{-1}(x) + g_{G(0)-1(x)}{}'(0)$. Hence

$A(0)(x) = B(x) + \lim_{h \to 0} h^{-1}(G(h)(x) - x) = B(x) + C(x)$.

Thus, even though such sums may fail to guarantee the generation of semigroups

(see Chernoff[2]), they must always support the generation of globally defined

evolutions.

Generation theories for evolution systems typically require that $\bigcap_{s \le t} D_{A(s)}$

be dense in C and that $(I - mA(t))^{-1}$ be Lipschitz with domain C. These

restrictions need not apply for the theory discussed in Theorem 1 to be viable.

Example 3. Let H be the Banach space of bounded uniformly continuous functions

from $[0, \infty)$ into R with for $f \in H$, $\| f \| = \mathrm{lub} \{ |f(x)| \; x \ge 0 \}$. Define A(s) by

$$(A(s)(f))(x) = \begin{cases} f'(x) \text{ if } x \le s \\ 0 \text{ if } x > s \end{cases}$$

By taking T to be the semigroup generated by $\{(f,g) : f \in H, g \in H, \text{ and } g = f'\}$

and G defined by

$$(G(t)(f))(x) = \begin{cases} f(x) \text{ if } x \ge t \\ 2f(x) - f(t) \text{ if } x < t \end{cases},$$

U, defined by $U(t,s) = G(t)T(t-s)G(s)^{-1}$ has generator A. Notice that the two

conditions on the domain of A(s) are provided separately by the differentiability

of T and G respectively. Here U(t,s) is defined globally although

$\bigcap_{0 \le a \le b} D_{A(a)}$ is nowhere dense in any neighborhood and $I - mA(t)$ does not

have range H.

Even when existing theories cover a given evolution system, the structures

studied here may provide alternative computation possibilities.

Example 4. Consider the ordinary differential equation

$y'(t) = A/((Bt + C)y(t)) + -Dy(t)/(Bt + C), \; y(s) = x, \; D/B = \frac{1}{2}$.

A typical solution might involve solving for y^2 in the associated linear equation.

However, by letting G solve the linear term, a semigroup generator can be found

by purely algebraic means from which $U(t,s) = G(t)T(t-s)G(s)^{-1}$ can be built to

solve the original equation. Here the forms in which the two methods give the

solution are computationally different.

Example 5. Suppose that A is a linear group generator and that C is a continuous

function from H into the compact subset K of H. Consider, for x ε H, the

problem

$$y'(t) = (A + C)(y(t)), \quad y(0) = x$$

which can be solved by the method of convergence of approximate solutions as

employed by Martin[6].

   Using the approach suggested in the theorems, let G denote the group gener-

ated by A and look for a function f from [0, d] into H so that

$y = \{(t, G(t)(f(t))) : 0 \le t \le d\}$ solves the problem. For such a function to be

a solution it must be true that $f'(t) = G(t)^{-1}CG(t)(f(t))$. Letting

$X = C([0, \infty), H)$, this in turn translates into a fixed point problem for the

function F defined by, for k ε X,

$$(F(k))(t) = x + \int_0^t G(j)^{-1} CG(j) k$$

and $G(j)^{-1} CG(j)$ is a compact map on [0,d] xH. If f is a fixed point for F, then

$$\lim_{h \to 0} h^{-1}(G(t+h)(f(t+h)) - G(t)(f(t))) = \lim_{h \to 0} h^{-1}(G(h)G(t)(f(t)) - G(t)(f(t))) +$$

$G(t+h)(h^{-1}(f(t+h) - f(t)) = AG(t)(f(t)) + G(t)(f'(t)) = AG(t)(f(t)) +$

$G(t)G(t)^{-1}CG(t)(f(t))$. Hence $y'(t) = (A + C)(y(t))$ and $y(0) = x$. Notice that

the burden for solution is shifted from evaluating the convergence of approximate

solutions to finding a fixed point associated with a compact map.

4. QUESTIONS.

   In Example 4, when G is produced by solving the linear problem

$u'(t) = -Du(t)/(Bt+c)$, finding the appropriate associated semigroup generator

depends on finding a function f so that $A/((Bt + C)x) = C^{D/B} f(C^{-D/B}(Bt + C)x)$.

In this case $f(p) = A/p$ works. Any systematic application for the evolution

system in this paper must include an analysis of the conditions under which a

family $\{B(t)\}_{t \le 0}$ can be algebraically resolved to the form $\{G(t)fG(t)^{-1}\}_{t \le 0}$.

   Theorem 1 identifies a closed form for the evolution generator in case G is

a linear family. Can other forms of the generator for $G(t)T(t-s)G(s)^{-1}$ be

identified when G is not linear?

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