Internat. J. Math. & Math. Sci.
Vol. 5 No. 2 (1982) 289-299

GENERALIZATIONS OF p-VALENT FUNCTIONS VIA THE HADAMARD PRODUCT

ANIL K. SONI
Department of Mathematics and Statistics
Bowling Green State University
Bowling Green, Ohio 43403, U.S.A.

(Received July 1, 1981)

ABSTRACT. The classes of univalent prestarlike functions R_α, $\alpha \geq -1$, of Ruscheweyh [1] and a certain generalization of R_α were studied recently by Al-Amiri [2]. The author studies, among other things, the classes of p-valent functions $R(\alpha + p - 1)$ where p is a positive integer and α is any integer with $\alpha + p > 0$. The author shows in particular that $R(\alpha + p) \subset R(\alpha + p - 1)$ and also obtains the radius of $R(\alpha + p)$ for the class $R(\alpha + p - 1)$.

KEY WORDS AND PHRASES. *p-valent starlike functions, p-valent close-to-convex functions, Hadamard product.*

AMS (MOS) SUBJECT CLASSIFICATION (1980) CODES. *Primary 30C45.*

1. INTRODUCTION.

The classes of univalent prestarlike functions R_α, $\alpha \geq -1$, were studied by various authors [1,2]. The author extends these classes to the classes of p-valent starlike functions $R(\alpha + p - 1)$, where p is a positive integer and α is any integer greater that -p. The present studies give, along with other results, a method to determine the radius of $R(\alpha + p)$ for the class $R(\alpha + p - 1)$.

Let A_p denote the class of regular functions in the unit disc $D = \{z: |z| < 1\}$ having the power series

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad p \text{ a positive integer}, \quad z \in D. \tag{1.1}$$

We denote by $S^*(\beta)$, the subclass of A_1 whose members are starlike of order β, $0 \le \beta < 1$.

Ruscheweyh [1] introduced the following classes 'K_α' of univalent prestarlike functions:

$$K_\alpha = \{f(z) \mid f(z) \in A_1 \text{ and } \operatorname{Re} \frac{(z^\alpha f(z))^{(\alpha+1)}}{(z^{\alpha-1}f(z))^{(\alpha)}} > \frac{\alpha+1}{2}, \quad z \in D\},$$

$\alpha \in N_0 = \{0,1,2,\dots\}$; where $F^{(n)}$ denotes the n-th derivative of the function F. As observed by Ruscheweyh, $f \in K_\alpha$ if and only if $\operatorname{Re} \frac{D^{\alpha+1}f(z)}{D^\alpha f(z)} > \frac{1}{2}$, $z \in D$ where $D^\alpha f(z) = f(z) * \frac{z}{(1-z)^{\alpha+1}}$. Here '*' denotes the Hadamard product of two regular functions, that is to say if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, then $f(z)*g(z) = \sum_{n=0}^{\infty} a_n b_n z^n$.

Ruscheweyh proved that $K_{\alpha+1} \subset K_\alpha$ and $K_0 = S^*(\frac{1}{2})$. Hence for each $\alpha \in N_0$, K_α is a subclass of $S^*(\frac{1}{2})$. Recently, Al-Amiri [2] studied a certain generalization of K_α, in particular he obtained the radius of $K_{\alpha+1}$ in K_α, $\alpha \in N_0$. Further Singh and Singh [3] extended the classes K_α to the classes R_α, where

$$R_\alpha = \{f(z) \mid f(z) \in A_1 \text{ and } \operatorname{Re} \frac{D^{\alpha+1}f(z)}{D^\alpha f(z)} > \frac{\alpha}{\alpha+1}, \quad z \in D\}, \quad \alpha \in N_0.$$

They observed that R_α is a subclass of $S^*(0)$. In this note, we extend their ideas to the class of p-valent functions.

We call a function $f(z) \in A_p$ to be p-valent starlike if it satisfies $\operatorname{Re} \frac{zf'(z)}{f(z)} > 0$, $z \in D$. Further, we say that a function $f(z) \in A_p$ is p-valent close-to-convex if there exists a p-valent starlike function $g(z)$ for which $\operatorname{Re}\left(\frac{zf'(z)}{g(z)} \right) > 0$, $z \in D$.

Let $R(\alpha + p - 1)$ denote the class of functions $f(z) \in A_p$ satisfying

$$\operatorname{Re} \left[\frac{(z^\alpha f(z))^{(\alpha+p)}}{(z^{\alpha-1}f(z))^{(\alpha+p-1)}} \right] > \alpha + p - 1, \ z \in D, \tag{1.2}$$

where α is any integer greater that $-p$. In Section 2 we shall show that

$$R(\alpha + p) \subset R(\alpha + p - 1). \tag{1.3}$$

Since $R(0)$ is the class of functions which satisfy

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > p - 1 \ge 0 ,$$

it follows by our definition taken from [4] that such functions are p-valent starlike. Hence (1.3) implies that $R(\alpha + p - 1)$ contains p-valent starlike functions.

We denote by $H(\alpha + p - 1)$, the classes of functions $f(z) \in A_p$ that satisfy the condition

$$\operatorname{Re}\left[\frac{(z^{\alpha}f(z))^{(\alpha+p)} - \alpha(z^{\alpha-1}f(z))^{(\alpha+p-1)}}{(z^{\alpha-1}g(z))^{(\alpha+p-1)}}\right] > \frac{\alpha + p - 1}{\alpha + p}, \quad z \in D, \tag{1.4}$$

for some $g(z) \in R(\alpha + p - 1)$, α integer greater that $-p$.

In Section 4 we shall show that

$$H(\alpha + p) \subset H(\alpha + p - 1). \tag{1.5}$$

Again since $H(0)$ is the class of functions f that satisfy $\operatorname{Re} \frac{zf'(z)}{g(z)} > 0$, where g is starlike, (1.5) implies that $H(\alpha + p - 1)$ contains p-valent close-to-convex functions.

For $f \in A_p$, define

$$D^{\alpha+p-1}f(z) = f(z) * \frac{z^p}{(1 - z)^{\alpha+p}}, \tag{1.6}$$

where α is any integer greater than $-p$. Then

$$D^{\alpha+p-1}f(z) = \frac{z^p(z^{\alpha-1}f(z))^{(\alpha+p-1)}}{(\alpha + p - 1)!}. \tag{1.7}$$

It can be shown that (1.6) yields the following identity

$$z(D^{\alpha+p-1}f(z))' = (\alpha + p)D^{\alpha+p}f(z) - \alpha(D^{\alpha+p-1}f(z)). \tag{1.8}$$

From (1.2) and (1.7) it follows that a function f in A_p belongs to $R(\alpha + p - 1)$ if and only if

$$\operatorname{Re} \frac{D^{\alpha+p}f(z)}{D^{\alpha+p-1}f(z)} > \frac{\alpha + p - 1}{\alpha + p}. \tag{1.9}$$

Note that for $p = 1$, the classes $R(\alpha + p - 1)$ reduce to the classes R_{α} of Singh and Singh [3]. Hence our results are generalizations of Singh and Singh.

From (1.4) and (1.7), it follows that a function f in A_p belongs to $H(\alpha + p - 1)$ if and only if

$$\operatorname{Re}\left[\frac{z(D^{\alpha+p-1}f(z))'}{D^{\alpha+p-1}g(z)}\right] > \frac{\alpha + p - 1}{\alpha + p}, \tag{1.10}$$

for some $g \in R(\alpha + p - 1)$.

In Sections 3 and 4 we shall describe some special elements of $R(\alpha + p - 1)$ and $H(\alpha + p - 1)$, respectively. These elements have integral representations. In Section 5, we introduce the classes $R_{\frac{1}{2}}(\alpha + p - 1)$ via the Hadamard product. Also the radii of $R(\alpha + p)$ in $R(\alpha + p - 1)$ and of $R_{\frac{1}{2}}(\alpha + p)$ in $R_{\frac{1}{2}}(\alpha + p - 1)$ are determined. In Section 6, the classes $R_{\frac{1}{2}}(\alpha + p - 1,\beta)$ which are extensions of the classes $R_{\frac{1}{2}}(\alpha + p - 1)$, are given. Many authors have considered a variation of these classes, notably Ruscheweyh [1], Suffridge [5], Goel and Sohi [6]. However, this note basically uses the techniques given by Al-Amiri [2].

2. THE CLASSES $R(\alpha + p - 1)$.

We shall prove the following:

THEOREM 1. $R(\alpha + p) \subset R(\alpha + p - 1)$.

PROOF. Let $f \in R(\alpha + p)$. Define $w(z)$ by

$$\frac{D^{\alpha+p}f(z)}{D^{\alpha+p-1}f(z)} = \frac{\alpha + p - 1}{\alpha + p} + \frac{1}{\alpha + p}\,\frac{1 - w(z)}{1 + w(z)}\ . \tag{2.1}$$

Here $w(z)$ is a regular function in D with $w(0) = 0$, $w(z) \neq -1$ for $z \in D$. It suffices to show that $|w(z)| < 1$, $z \in D$, since then (2.1) would imply that $f \in R(\alpha + p - 1)$.

Taking logarithmic derivative of both sides of (2.1) and using the identity (1.8) the following is obtained.

$$\frac{D^{\alpha+p+1}f(z)}{D^{\alpha+p}f(z)} = \frac{1}{(\alpha + p + 1)}\left[1 + \frac{(\alpha+p) + (\alpha+p-2)w(z)}{1 + w(z)} - \frac{2zw'(z)}{(1+w(z))(\alpha+p+(\alpha+p-2)w(z))}\right]. \tag{2.2}$$

The above equation must yield $|w(z)| < 1$ for all $z \in D$, for otherwise by using a lemma of Jack [7] one can obtain $z_0 \in D$ such that $z_0w'(z_0) = Kw(z_0)$, $|w(z_0)| = 1$ and $K \geq 1$. Consequently (2.2) would yield

$$\frac{D^{\alpha+p+1}f(z_0)}{D^{\alpha+p}f(z_0)} = \frac{1}{(\alpha+p+1)} + \frac{(\alpha+p) + (\alpha+p-2)w(z_0)}{(\alpha+p+1)(1+w(z_0))} - \frac{2Kw(z_0)}{(\alpha+p+1)(1+w(z_0))}\,\frac{(\alpha+p+(\alpha+p-2)\overline{w(z_0)})}{|\alpha+p+(\alpha+p-2)w(z_0)|^2}\ .$$

Since

$$\operatorname{Re} \frac{1}{1 + w(z_0)} = \frac{1}{2}, \quad \operatorname{Re} \frac{w(z_0)}{1 + w(z_0)} = \frac{1}{2},$$

the above equation implies

$$\operatorname{Re} \frac{D^{\alpha+p+1} f(z_0)}{D^{\alpha+p} f(z_0)} \leq \frac{\alpha + p}{\alpha + p + 1}.$$

This is a contradiction to the assumption that $f \in R(\alpha + p)$. Hence $f \in R(\alpha + p - 1)$. This completes the proof of Theorem 1.

3. SPECIAL ELEMENTS OF $R(\alpha + p - 1)$.

In this section we form special elements of the classes $R(\alpha + p - 1)$ via the Hadamard product of elements of $R(\alpha + p - 1)$ and $h_\gamma(z)$, where

$$h_\gamma(z) = \sum_{j=p}^{\infty} \frac{\gamma + p}{\gamma + j} z^j, \quad \operatorname{Re} \gamma > -p.$$

THEOREM 2. Let $f \in A_p$ satisfy the condition

$$\operatorname{Re} \frac{D^{\alpha+p} f(z)}{D^{\alpha+p-1} f(z)} > \frac{2(\gamma+p-1)(\alpha+p-1) - 1}{2(\alpha+p)(\gamma+p-1)}, \quad z \in D, \tag{3.1}$$

p a positive integer, α any integer greater than $-p$ and $\gamma \geq -p + 2$. Then

$$F(z) = f(z)*h_\gamma(z) = \frac{\gamma + p}{z^\gamma} \cdot \int_0^z t^{\gamma-1} f(t)\,dt \tag{3.2}$$

belongs to $R(\alpha + p - 1)$.

PROOF. Let $f \in A_p$ satisfy the condition (3.1). From (3.2) we obtain

$$z(D^{\alpha+p}F(z))' + \gamma(D^{\gamma+p}F(z)) = (p+\gamma)D^{\alpha+p}f(z), \tag{3.3}$$

and

$$z(D^{\alpha+p-1}F(z))' + \gamma(D^{\alpha+p-1}F(Z)) = (p+\gamma)D^{\alpha+p-1}f(z). \tag{3.4}$$

Define $w(z)$ by

$$\frac{D^{\alpha+p}F(z)}{D^{\alpha+p-1}F(z)} = \frac{\alpha + p - 1}{\alpha + p} + \frac{1}{\alpha + p} \cdot \frac{1 - w(z)}{1 + w(z)}. \tag{3.5}$$

Here $w(z)$ is a regular function in D with $w(0) = 0$, $w(z) \neq -1$ for $z \in D$. It suffices to show that $|w(z)| < 1$, $z \in D$.

Taking the logarithmic derivative of (3.5) and using (1.8) for $F(z)$ one can get

$$z(D^{\alpha+p}F(z))' = D^{\alpha+p}F(z) \cdot \left[(\alpha+p) \frac{D^{\alpha+p}F(z)}{D^{\alpha+p-1}F(z)} - \alpha - \frac{2zw'(z)}{(1+w(z))(\alpha+p+(\alpha+p-2)w(z))} \right]. \tag{3.6}$$

Now (3.3) and (3.6) yield

$$(p+\gamma)D^{\alpha+p}f(z) = D^{\alpha+p}F(z) \cdot \left[\gamma-\alpha + \frac{(\alpha+p)+(\alpha+p-2)w(z)}{1+w(z)} - \frac{2zw'(z)}{(1+w(z))(\alpha+p+(\alpha+p-2)w(z))} \right]. \tag{3.7}$$

Use (3.4) and (1.8) to eliminate the derivative and then apply (3.5) to get

$$(p+\gamma)D^{\alpha+p-1}f(z) = D^{\alpha+p-1}F(z) \cdot \left[\gamma - \alpha + \frac{(\alpha+p) + (\alpha+p-2)w(z)}{1+w(z)} \right]. \tag{3.8}$$

Therefore (3.7), (3.8) and (3.5) give

$$\frac{D^{\alpha+p}f(z)}{D^{\alpha+p-1}f(z)} = \frac{\alpha+p-1}{\alpha+p} + \frac{1}{\alpha+p} \frac{1-w(z)}{(1+w(z))} - \frac{2zw'(z)}{(\alpha+p)(1+w(z))} \frac{(\gamma+p) + (\gamma+p-2)\overline{w(z)}}{|\gamma+p+(\gamma+p-2)w(z)|^2}. \tag{3.9}$$

Equation (3.9) should yield $|w(z)| < 1$ for all $z \in D$, for otherwise by Jack's lemma there exists $z_0 \in D$ with $z_0w'(z_0) = Kw(z_0)$, $K \geq 1$, and $|w(z_0)| = 1$. Applying this to (3.9) it follows that

$$\operatorname{Re}\left[\frac{D^{\alpha+p}f(z_0)}{D^{\alpha+p-1}f(z_0)}\right] \leq \frac{\alpha+p-1}{\alpha+p} - \frac{2}{(\alpha+p)} \frac{\gamma+p-1}{4(\gamma+p-1)^2}$$

$$= \frac{2(\gamma+p-1)(\alpha+p-1) - 1}{2(\alpha+p)(\gamma+p-1)}.$$

This contradicts the assumption on f given by (3.1). Hence $F \in R(\alpha + p - 1)$. This completes the proof of Theorem 2.

REMARK 1. For $\gamma = 1$ and $p = 1$, Theorem 2 reduces to a result given in [3].

The following special cases of Theorem 2 represent some improvement on theorems due to Libera [8] in the sense that much weaker assumptions produce the same results.

By taking $\alpha = 0$, $p = 1$ in Theorem 2 we get

COROLLARY 1. Let $f \in A_1$ be such that $\operatorname{Re} \frac{zf'(z)}{f(z)} > \frac{-1}{2\gamma}$, $\gamma \geq 1$, $z \in D$. Then F is starlike in D, where

$$F(z) = \frac{\gamma + 1}{z^\gamma} \cdot \int_0^z t^{\gamma-1} f(t)\,dt. \tag{3.10}$$

For $\alpha = 1$, $p = 1$, Theorem 2 reduces to

COROLLARY 2. Let $f \in A_1$ be such that $\mathrm{Re}[1 + \frac{zf''(z)}{f'(z)}] > -\frac{1}{2\gamma}$, $\gamma \geq 1$, $z \in D$. Then $F(z)$ as given in (3.10) above is convex in D.

Using the technique employed in the proof of Theorem 1 and Corollary 2 we obtain the following result.

COROLLARY 3. Let $f \in A_1$ be such that $\mathrm{Re}\,\frac{f'(z)}{g'(z)} > 0$, $z \in D$ and g be such that $\mathrm{Re}[1 + \frac{zg''(z)}{g'(z)}] > -\frac{1}{2\gamma}$, $\gamma \geq 1$, $z \in D$. Then $F(z)$ as given by (3.10), is close-to-convex, i.e., $\mathrm{Re}\,\frac{F'(z)}{G'(z)} > 0$, $z \in D$ and where $G(z)$ is the convex function given by

$$G(z) = \frac{\gamma + 1}{z^\gamma} \cdot \int_0^z t^{\gamma-1} g(t)\,dt.$$

We state without proof the following theorem since its method of proof is similar to that of Theorem 2.

THEOREM 3. Let p be a positive integer and α be an integer greater than $-p$ and let $\mathrm{Re}\,\gamma \geq -p + 1$. Then $F(z) = f(z)*h_\gamma(z)$, as given by (3.2), belongs to $R(\alpha + p - 1)$ for all $f \in R(\alpha + p - 1)$.

In case $\gamma = \alpha$, Theorem 3 can be improved as follows:

THEOREM 4. Let p be a positive integer, and α be any integer greater than $-p$. Then for $f(z) \in R(\alpha + p - 1)$,

$$F(z) = f(z)*h_\alpha(z) = \frac{p + \alpha}{z^\alpha} \cdot \int_0^z t^{\alpha-1} f(t)\,dt \in R(\alpha + p). \tag{3.11}$$

PROOF. Let $f(z) \in R(\alpha + p - 1)$. Differentiating (3.11) and then applying the operators $D^{\alpha+p}$, $D^{\alpha+p-1}$ we get, respectively, by using (1.8)

$$(\alpha + p) \cdot D^{\alpha+p} f(z) = (\alpha + p + 1) D^{\alpha+p+1} F(z) - D^{\alpha+p} F(z)$$

and

$$(\alpha + p) D^{\alpha+p-1} f(z) = (\alpha + p) D^{\alpha+p} F(z).$$

Therefore

$$\mathrm{Re}\left[\frac{\alpha + p + 1}{\alpha + p} \frac{D^{\alpha+p+1} F(z)}{D^{\alpha+p} F(z)} - \frac{1}{\alpha + p}\right] = \mathrm{Re}\,\frac{D^{\alpha+p} f(z)}{D^{\alpha+p-1} f(z)} > \frac{\alpha + p - 1}{\alpha + p}.$$

This implies that

$$\mathrm{Re}\,\frac{D^{\alpha+p+1}F(z)}{D^{\alpha+p}F(z)} > \frac{\alpha + p}{\alpha + p + 1}, \quad z \in D.$$

Hence $F(z) \in R(\alpha + p)$, and this completes the proof of Theorem 4.

REMARK 2. For $p = 1$, Theorem 4 reduces to a result of Singh and Singh [3].

4. THE CLASSES $H(\alpha + p - 1)$.

We state without proof Theorems 5 and 6 since their proofs use the same technique employed in Theorem 1. See Section 1 for the definition of the classes $H(\alpha + p - 1)$.

THEOREM 5. $H(\alpha + p) \subset H(\alpha + p - 1)$.

THEOREM 6. If p is any positive integer, α is any integer greater than $-p$, and $\mathrm{Re}\,\gamma \geq -p + 1$, then

$$F(z) = f(z)*h_\gamma(z) = \frac{p+\gamma}{z^\gamma} \cdot \int_0^z t^{\gamma-1} f(t)\,dt \in H(\alpha + p - 1)$$

whenever $f(z) \in H(\alpha + p - 1)$.

5. RADII OF THE CLASSES $R(\alpha + p)$ AND $R_{\frac{1}{2}}(\alpha + p)$.

Because discussing the problem concerning the radii of the classes $R(\alpha + p)$ and $R_{\frac{1}{2}}(\alpha + p)$ we define the classes $R_{\frac{1}{2}}(\alpha + p - 1)$. $R_{\frac{1}{2}}(\alpha + p - 1)$ contains functions $f(z) \in A_p$ that satisfy the condition

$$\mathrm{Re}\left[\frac{(z^\alpha f(z))^{(\alpha+p)}}{(z^{\alpha-1} f(z))^{(\alpha+p-1)}}\right] > \frac{\alpha + p}{2}, \quad z \in D, \tag{5.1}$$

where α is any integer greater than $-p$. These classes have been studied by Goel and Sohi [6].

From (1.7) and (5.1), it follows that a function f in A_p belongs to $R_{\frac{1}{2}}(\alpha + p - 1)$ if and only if

$$\mathrm{Re}\,\frac{D^{\alpha+p}f(z)}{D^{\alpha+p-1}f(z)} > \frac{1}{2}. \tag{5.2}$$

Our main interest is to determine the radius of the largest disc $D(r) = \{z: |z| < r\}$, $0 < r \leq 1$ so that if $f \in R(\alpha + p - 1)$ then $\mathrm{Re}\,\frac{D^{\beta+p}f(z)}{D^{\beta+p-1}f(z)} > \frac{\beta + p - 1}{\beta + p}$, $\beta > \alpha$, z $D(r)$. A partial answer to this problem can be deduced by a simple application of a lemma due to (Ruscheweyh and Singh) [9]:

LEMMA 1. If $p(z)$ is an analytic function in the unit disc D with $p(0) = 1$, $\mathrm{Re}\, p(z) > 0$ and also

$$|z| < \frac{|\mu + 1|}{[A + (A^2 - |\mu^2 - 1|^2)^{\frac{1}{2}}]^{\frac{1}{2}}}, \tag{5.3}$$

$$A = 2(S + 1)^2 + |\mu|^2 - 1.$$

Then we have

$$\mathrm{Re}\left[p(z) + S\frac{zp'(z)}{p(z) + \mu}\right] > 0.$$

The bound given by (5.3) is best possible.

THEOREM 7. Let p be any positive integer, α any integer greater than $-p$. If $f(z) \in R(\alpha + p - 1)$ then

$$\mathrm{Re}\,\frac{D^{\alpha+p+1}f(z)}{D^{\alpha+p}f(z)} > \frac{\alpha + p}{\alpha + p + 1} \quad \text{for} \quad |z| < r_{\alpha,p},$$

where

$$r_{\alpha,p} = \frac{\alpha + p}{2 + \sqrt{3 + (\alpha+p-1)^2}}. \tag{5.4}$$

This result is sharp.

PROOF. Let $f(z) \in R(\alpha + p - 1)$. We define the regular function $q(z)$ on D by

$$\frac{D^{\alpha+p}f(z)}{D^{\alpha+p-1}f(z)} = \frac{1}{(\alpha + p)}\,(q(z) + \alpha + p - 1), \quad z \in D. \tag{5.5}$$

Therefore $q(0) = 1$ and $\mathrm{Re}\, q(z) > 0$ in D.

Taking logarithmic derivative of (5.5) and using (1.8) we get

$$\frac{D^{\alpha+p+1}f(z)}{D^{\alpha+p}f(z)} - \frac{\alpha + p}{\alpha + p + 1} = \frac{1}{(\alpha+p+1)}\left[q(z) + \frac{zq'(z)}{q(z) + \alpha + p - 1}\right]. \tag{5.6}$$

Using Lemma (1) with $S = 1$, $\mu = \alpha + p - 1$, (5.6) and (5.3) show that

$$\mathrm{Re}\left[\frac{D^{\alpha+p+1}f(z)}{D^{\alpha+p}f(z)}\right] > \frac{\alpha + p}{\alpha + p + 1} \quad \text{for}$$

$$|z| < \frac{\alpha + p}{[A + (A^2 - ((\alpha+p-1)^2 - 1)^2)^{\frac{1}{2}}]^{\frac{1}{2}}}, \tag{5.7}$$

where

$$A = (\alpha + p)^2 - 2(\alpha + p) + 8.$$

Minor computations yield the following:

$$A + (A^2 - ((\alpha+p-1)^2 - 1)^2)^{\frac{1}{2}} = (2 + \sqrt{3 + (\alpha+p-1)^2}\,)^2. \qquad (5.8)$$

Thus (5.7) yields the radius $r_{\alpha,p}$ as given by (5.4).

The method of Al-Amiri [2] is used to determine the extremal functions. The extremal functions thus obtained for this theorem are rotations of f(z) where f(z) is given by

$$\frac{D^{\alpha+p}f(z)}{D^{\alpha+p-1}f(z)} = \frac{1}{(\alpha + p)}\left[\frac{1 + z}{1 - z} + \alpha + p - 1\right], \quad z \in D.$$

This completes the proof of Theorem 7.

REMARK 3. For $\alpha = 0$, $p = 1$, Theorem 7 gives the well-known radius of convexity for the class of starlike functions: $r_{0,1} = 2 - \sqrt{3}$.

Now an easy modification of the method used by Al-Amiri [2, Theorem 4] gives the following result.

THEOREM 8. Let p be any positive integer, α any integer greater than -p. If $f(z) \in R_{\frac{1}{2}}(\alpha + p - 1)$, then f(z) satisfies (5.2) with α replaced by $\alpha + 1$ for $|z| < r_{\alpha,p}$ where

$$r_{\alpha,p} = \left[\frac{(\alpha + p - 1) + 2(\alpha + p + 2)^{\frac{1}{2}}}{(\alpha + p + 3) + 2(\alpha + p + 2)^{\frac{1}{2}}}\right]^{\frac{1}{2}}$$

This result is sharp.

REMARK 4. For $p = 1$, Theorem 8 becomes a special case of a result due to Al-Amiri [2, Theorem 4].

6. THE CLASSES $R_{\frac{1}{2}}(\alpha + p - 1,\beta)$.

By $R_{\frac{1}{2}}(\alpha + p - 1,\beta)$, we denote the classes of functions $f(z) \in A_p$ that satisfy

$$\mathrm{Re}\left[(1 - \beta)\frac{D^{\alpha+p}f(z)}{D^{\alpha+p-1}f(z)} + \beta\frac{D^{\alpha+p+1}f(z)}{D^{\alpha+p}f(z)}\right] > \frac{1}{2}, \quad z \in D, \qquad (6.1)$$

for some $\beta \geq 0$, p any positive integer and α any integer greater than -p. Again using the technique employed in [2], the following theorem is obtained.

THEOREM 9. Let p be any positive integer, α any integer greater than -p. If $f(z) \in R_{\frac{1}{2}}(\alpha + p - 1)$, then f(z) satisfies (6.1) for $|z| < r_{\alpha,p,\beta}$ where

$$r_{\alpha,p,\beta} = \left[\frac{(\alpha + p + 1 - 2\beta) + 2(\beta(\alpha + p + 1 + \beta))^{\frac{1}{2}}}{(\alpha + p + 1 + 2\beta) + 2(\beta(\alpha + p + 1 + \beta))^{\frac{1}{2}}}\right]^{\frac{1}{2}} .$$

This result is sharp.

REMARK 5. For $\beta = 1$, Theorem 9 reduces to Theorem 8. Also for $p = 1$, Theorem 9 represents a special case of a theorem due to Al-Amiri [2, Theorem 8].

ACKNOWLEDGEMENTS. This paper forms a part of the author's doctoral thesis written at Bowling Green State University of Ohio at Bowling Green. The author would like to thank Professor Hassoon S. Al-Amiri for his guidance and direction.

REFERENCES

1. RUSCHEWEYH, S. New Criteria for Univalent Functions, Proc. Amer. Math. Soc. 49 (1975), 109-115.

2. AL-AMIRI, H.S. Certain Generalizations of Prestarlike Functions, J. Aust. Math. Soc. (Series A) 28 (1979), 325-334.

3. SINGH, R. and SINGH, S. Integrals of Certain Univalent Functions, Proc. Amer. Math. Soc. 77 (1979), 336-340.

4. UMEZAWA, T. Multivalently Close-to-Convex Functions, Proc. Amer. Math. Soc. 8 (1957), 869-874.

5. SUFFRIDGE, T.J. Starlike Functions as Limits of Polynomials, Advances in Complex Function Theory, (Lecture Notes in Mathematics 505, Springer-Verlag, Berlin) (1976), 164-202.

6. GOEL, R.M. and SOHI, N.S. A New Criterion for p-Valent Functions, Proc. Amer. Math. Soc. 78 (1980), 353-357.

7. JACK, I.S. Functions Starlike and Convex of Order α, J. London Math. Soc. (2) 3 (1971), 469-474.

8. LIBERA, R.J. Some Classes of Regular Univalent Functions, Proc. Amer. Math. Soc. 16 (1965), 755-758.

9. RUSCHEWEYH, S. and SINGH, V. On Certain Extremal Problems for Functions with Positive Real Part, Proc. Amer. Math. Soc. 61 (1976), 329-334.