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TENSOR PRODUCTS OF COMMUTATIVE BANACH ALGEBRAS

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ABSTRACT. Let A_1, A_2 be commutative semisimple Banach algebras and $A_1 \otimes_\partial A_2$ be their projective tensor product. We prove that, if $A_1 \otimes_\partial A_2$ is a group algebra (measure algebra) of a locally compact abelian group, then so are A_1 and A_2. As a consequence, we prove that, if G is a locally compact abelian group and A is a commutative semi-simple Banach algebra, then the Banach algebra $L^1(G,A)$ of A-valued Bochner integrable functions on G is a group algebra if and only if A is a group algebra. Furthermore, if A has the Radon-Nikodym property, then the Banach algebra M(G,A) of A-valued regular Borel measures of bounded variation on G is a measure algebra only if A is a measure algebra.

KEY WORDS AND PHRASES. *Commutative semisimple Banach algebra, projective tensor product, group algebra, measure algebra, locally compact abelian group, Radon-Nikodym property.*

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1. INTRODUCTION.

Let A be a commutative Banach algebra. We shall say that A is a group algebra (measure algebra) if A is isometrically isomorphic to $L^1(G)$ (M(G)) for some locally compact abelian group G. Let G be a locally compact abelian group and A be a commutative semi-simple Banach algebra. The space $L^1(G,A)$ of A-valued Bochner integrable functions on G becomes a commutative Banach algebra (see [1], [2] and [3]). A natural question arises: when is $L^1(G,A)$ a group algebra? If $A = L^1(H)$ for some locally compact abelian group H, then it is well known (Theorem 3.2 of [3]) that $L^1(G,A)$ is isometrically isomorphic to $L^1(G \times H)$. Thus $L^1(G,A)$ is a group algebra if A is a group algebra. We shall prove that the converse is also true. There is another way of looking at this problem. It is known that $L^1(G,A)$ is isometrically isomorphic to $L^1(G) \otimes_\partial A$ (see 6.5 of [4]). Thus, if A_1 and A_2 are group algebras, then so is $A_1 \otimes_\partial A_2$. Conversely, we shall show that, if A_1 and A_2 are two commutative Banach algebras and $A_1 \otimes_\partial A_2$ is a group algebra, then so are A_1 and A_2. It seems proper to remark that we are concluding properties for A_1 and A_2, assuming corresponding properties for $A_1 \otimes_\partial A_2$. This is in contrast to the approach of Gelbaum [5] and [6]. Our result for $L^1(G,A)$ readily follows from this. The main tool in our investigation is a theorem of Rieffel [7] characterizing group algebras. In this paper, Rieffel also characterized measure algebras. Accordingly, we investigate whether the fact that $A_1 \otimes_\partial A_2$ is a measure algebra implies that A_1 and A_2 are measure algebras. We shall show that this is indeed the case. As a consequence, we shall show that, if A is a commutative Banach algebra having the Radon Nikodym property and M(G,A) is the Banach algebra of A-valued regular Borel measures of bounded variation on G, then M(G,A) is a measure algebra only if A is a measure algebra.

2. PRELIMINARIES.

Let E and F be Banach spaces. The projective tensor product of E and F (see [8]) is denoted by $E \otimes_\partial F$. Every element $t \in E \otimes_\partial F$ can be expressed as $t = \sum_{i=1}^{\infty} e_i \otimes f_i$, with each $e_i \in E$ and $f_i \in F$, such that $\sum_{i=1}^{\infty} \|e_i\| \, \|f_i\| < \infty$. The norm of t is given by

$$\|t\|_\partial = \inf \{\sum_i \|e_i\| \, \|f_i\| : t = \sum_i e_i \otimes f_i\},$$

where the infimum is taken over all possible expressions of t.

Let $f \in E^*$, $g \in F^*$, and $t \in E \otimes_\partial F$ with $t = \sum_{i=1}^{\infty} e_i \otimes f_i$. We define $f \circ t = \sum_1^{\infty} f(e_i) f_i$ and $g \circ t = \sum_1^{\infty} g(f_i) e_i$. The series defining $f \circ t$ and $g \circ t$ converge because $\sum_1^{\infty} \|e_i\| \, \|f_i\| < \infty$. It is obvious that $\|f \circ t\| \leq \|f\| \, \|t\|$ and $\|g \circ t\| \leq \|g\| \, \|t\|$. The norms here, as well as elsewhere, refer to the norms in the spaces containing the elements. $t \to g \circ t$ and $t \to f \circ t$ define bounded linear maps from $E \otimes_\partial F$ to E and $E \otimes_\partial F$ to F, respectively. These maps will be frequently used in the sequel.

Let (S,Σ,λ) be a measure space and X be a Banach space. $L^1(S,X)$ denotes the Banach space of X-valued functions integrable with respect to λ. We shall often use the fact that $L^1(S) \otimes_\partial X$ is isometrically isomorphic to $L^1(S,X)$.

Gelbaum [5] and Tomiyama [9] have shown that, if A and B are commutative Banach algebras, then $A \otimes_\partial B$ forms a commutative Banach algebra whose maximal ideal space is homeomorphic to the cartesian product of the maximal ideal spaces of A and B. The maximal ideal space of a commutative Banach algebra A will be denoted by $\Delta(A)$. An element of $\Delta(A)$ will be regarded as a multiplicative linear functional (m.l.f.) of A. All the Banach algebras in our discussion will be taken to be commutative and semisimple. It is proved in [6] that $A \otimes_\partial B$ has an identity if and only if both A and B have identities. It is also known [6] that $A \otimes_\partial B$ is Tauberian if A and B are Tauberian. The following lemma, though simple, does not seem to have appeared in print.

LEMMA 2.1. If $A \otimes_\partial B$ is Tauberian, then so are A and B.

PROOF. Let us show that B is Tauberian. It can be shown in the same way that A is Tauberian. Let $b \in B$ and $\epsilon > 0$. Take $\phi \in \Delta(A)$ and $a \in A$ such that $\phi(a) = 1$. Let $t = a \otimes b$. Choose $s \in A \otimes_\partial B$ such that $\hat{s}$ has compact support K and $||s-t|| < \epsilon$. Let $K_1 = \{\psi \in \Delta(B): (\phi,\psi) \in K\}$. Then K_1 is compact. Let $x = \phi \circ s$. Then $\hat{x}$ is supported in K_1 and

$$||b-x|| = ||\phi \circ t - \phi \circ s|| \leq ||t-s|| < \epsilon .$$

This proves that B is Tauberian.

Let (S,Σ) be a measurable space and X be a Banach space. Let μ be an X-valued set function on Σ. The total variation $V(\mu)$ of μ is defined for any $E \subset S$ as follows.

$$V(\mu)(E) = \text{Sup } \{\sum_1^n ||\mu(E_i)||: E_i\text{'s disjoint, } E_i \subset E \text{ for } i \leq i \leq n\},$$

the supremum being taken for all possible choices of E_i's.

An X-valued measure on (S,Σ) is a countably additive set function from Σ into X. μ is said to be of bounded variation if $V(\mu)$ is finite. The space $\overline{M}(S,\Sigma,X)$ of X-valued measures of bounded variation on S forms a Banach space under the norm $||\mu||_V = V(\mu)(S)$.

Let λ be a positive measure on (S,Σ) and $L^1(S,X)$ be the Banach space of X-valued functions on S, integrable with respect to λ. If $F \in L^1(S,X)$, then we can define the mapping μ_F: $\Sigma \to X$ by $\mu_F(E) = \int_E F\, d\lambda$. Then μ_F is an X-valued measure of bounded variation on S. Let $\mu \in \overline{M}(S,\Sigma,X)$. We say that μ has the derivative F with respect to λ if μ equals μ_F for $F \in L^1(S,X)$. We say that X has the Radon-Nikodym property (X has RNP) if every X-valued measure μ of bounded variation on an arbitrary measurable space (S,Σ) has a derivative with respect to $V(\mu)$. If X is separable and the dual of a Banach space or is reflexive, then X has RNP (see [10] and [11]). An example of a separable Banach space which does not have RNP is $L^1[0,1]$ (see [12]).

Let G be a locally compact abelian group and let A be a commutative Banach algebra. M(G,A) denotes the Banach space of A-valued regular Borel measures of bounded variation on G. Suppose the range of every $\mu \in M(G,A)$ is separable. This is true if A has RNP or if G is second countable. Under these conditions, we can define the convolution of measures μ and ν belonging to M(G,A). This makes M(G,A) a commutative Banach algebra (see [13]). The algebra $L^1(G,A)$ is an ideal in M(G,A) (see [14]). There is a natural isometric isomorphism from $M(G) \otimes_\partial A$ into M(G,A) (Theorem 4.2 of [15]). This is a Banach algebra isomorphism and, if A has RNP, then it is onto (Theorem 4.4 of [15]).

Let A be a commutative and semisimple Banach algebra and $m \in \Delta(A)$. Let $P_m = \{a \in A\colon\ m(a) = ||m||\ ||a||\}$. Then P_m is a cone in A and therefore introduces an order in A. Let $R_m = \{a-b\colon\ a,b \in P_m\}$. m is said to be L'-inducing if the following conditions are satisfied:

(1) $||m|| = 1$.

(2) P_m is a lattice.

(3) If $a,b \in R_m$ and $a \wedge b = 0$, then $||a+b|| = ||a-b||$.

(4) If $a \in A$, then there exists unique elements, a_1, $a_2 \in R_m$, such that $a = a_1 + i\, a_2$.

(5) Let $|a| = V\{\mathrm{Re}\,(e^{i\theta} a): \theta \in [0,2\pi]\}$. Then $||a|| = ||\ |a|\ ||$.

[V and $\wedge$ respectively denote supremum and infimum. Re (a) = a_1 where $a = a_1 + i\, a_2$, $a_i \in R_m$]. We note that if (1) - (3) hold, then R_m forms a real abstract L-space in the sense of Kakutani [16], and hence R_m is a boundedly complete lattice (see page 35 of [7]). Therefore, $|a|$ is well defined.

In [7], a L'-inducing m.l.f. is defined to be a m.l.f. which satisfies the following condition in addition to (1) - (5).

(6) For $a,b \in A$, $|a.b| \leq |a| \cdot |b|$.

However, White [17] has shown that a m.l.f. satisfying (1) - (5) automatically satisfies (6), and hence our definition is equivalent to that of [7]. We now state Rieffel's characterization of a group algebra.

THEOREM R_1. Let A be a commutative semisimple Banach algebra. A is a group algebra if and only if

(a) every m.l.f. of A is L'-inducing, and

(b) A is Tauberian.

Let A be a commutative semisimple Banach algebra and let D be the collection of L'-inducing m.l.f.'s of A. Consider the ω^*-topology on D. A continuous function p on D is said to be a D-Eberlein function if there exists a constant $k > 0$ such that for any choice of points $m_1,\dots,m_n$ of D and scalars $\alpha_1,\dots,\alpha_n$; we have

$$\left|\sum_1^n \alpha_i\, p(m_i)\right| \leq k \left|\left|\sum_1^n \alpha_i\, m_i\right|\right|_{A^*}.$$

The following theorem of Rieffel characterizes a measure algebra.

THEOREM R_2. Let A be a commutative Banach algebra and let D be the set of L'-inducing m.l.f.'s of A. Then A is a measure algebra if and only if

(i) D is a separating family of linear functionals of A,

(ii) D is locally compact in the ω^*-topology, and

(iii) every D-Eberlein function is the restriction to D of the Gelfand transform of some element of A.

The 'if' part is nothing but Theorem B of [7]. The 'only if' part follows from the following and the familiar properties of Fourier-Stieltjes transforms.

PROPOSITION 2.1. The L'-inducing m.l.f.'s of M(G) are precisely those that are given by Γ, the dual of G.

PROOF. Let S be the structure semigroup of M(G) (see 4.3 of [18]). M(G) can be identified (3.2 of [18]) with a weak*-dense subalgebra of M(S). Under this identification, the m.l.f.'s of M(G) are given by $\hat{S}$, the collection of semicharacters of S. Let $f \in \hat{S}$. Then, using the arguments of Proposition 2.5 of [7] (see also Proposition 2.8 of [7]), we can prove that f represents an L'-inducing m.l.f. if and only if $|f(s)| = 1$ for all $s \in S$. By 4.3.3 of [18], $\{f \in \hat{S}: |f| = 1\}$ is the canonical image of Γ in S. This proves our proposition.

3. MAIN RESULT.

Our main result is the following theorem. All other results are derived as a consequence of this.

THEOREM 3.1. Let A_1, A_2 be commutative semisimple Banach algebras and $A = A_1 \otimes_\gamma A_2$. Let $\eta \in \Delta(A)$ be given by (ϕ, ψ) for $\phi \in \Delta(A_1)$ and $\psi \in \Delta(A_2)$. Then η is L'-inducing if and only if ϕ and ψ are L'-inducing.

PROOF. Suppose η is L'-inducing. We shall show that ϕ satisfies (1) - (5) for ϕ to be L'-inducing. Since $1 = \|\eta\| \leq \|\phi\| \, \|\psi\| \leq 1$, if follows that $\|\phi\| = \|\psi\| = 1$. Let $P = \{t \in A: \eta(t) = \|t\|\}$ and $P_\phi = \{r \in A_1: \phi(r) = \|r\|\}$. Choose a fixed $s \in A_2$ such that $\psi(s) = \|s\| = 1$. Let $t \in P_\eta$ and $r = \psi \circ t$. Then $\phi(r) = \phi(\psi \circ t) = \eta(t) = \|t\| \geq \|r\|$. Therefore, $r \in P_\phi$. On the other hand, if $r \in P_\phi$, then $\eta(r \otimes s) = \phi(r)\psi(s) = \phi(r) = \|r\| = \|r\| \, \|s\| = \|r \otimes s\|$ and so $r \otimes s \in P_\eta$. Thus we have shown that, if $t_1, t_2 \in A$ and $t_1 \geq t_2$, then $\psi \circ t_1 \geq \psi \circ t_2$ and, if $r_1, r_2 \in A_1$ and $r_1 \geq r_2$, then $r_1 \otimes s \geq r_2 \otimes s$.

Now, let $r_1, r_2 \in P_\phi$. Then it is easy to see that $r_1 \vee r_2 = \psi \circ ((r_1 \otimes s) \vee (r_2 \otimes s))$ and $r_1 \wedge r_2 = \psi \circ ((r_1 \otimes s) \wedge (r_2 \otimes s))$. For example, if $r = \psi \circ ((r_1 \otimes s) \vee (r_2 \otimes s))$, then, since $(r_1 \otimes s) \vee (r_2 \otimes s) \geq r_1 \otimes s$, if follows that $r \geq r_1$. Similarly, $r \geq r_2$. On the other hand, if $r' \geq r_1$ and $r' \geq r_2$, then $r' \otimes s \geq r_1 \otimes s$ and $r' \otimes s \geq r_2 \otimes s$. Therefore, $r' \geq r$. Note that $r_1 \vee r_2$ and

$r_1 \wedge r_2$ depend only on r_1 and r_2 and not on s. Therefore, P_ϕ is a lattice. We can also see that $(r_1 \vee r_2) \otimes s = (r_1 \otimes s) \vee (r_2 \otimes s)$ and $(r_1 \wedge r_2) \otimes s = (r_1 \otimes s)$ $(r_2 \otimes s)$. For example, it is obvious that $(r_1 \vee r_2) \otimes s \geq (r_1 \otimes s) \vee (r_2 \otimes s)$ and furthermore,

$$\|(r_1 \vee r_2) \otimes s - (r_1 \otimes s) \vee (r_2 \otimes s)\|$$
$$= \eta\, [(r_1 \vee r_2) \otimes s - (r_1 \otimes s) \vee (r_2 \otimes s)]$$
$$= \phi\, [r_1 \vee r_2 - \psi \circ ((r_1 \otimes s) \vee (r_2 \otimes s))] = 0.$$

Next, if $t \in R_\eta$ and $r \in R_\phi$, then $\psi \circ t \in R_\phi$ and $r \otimes s \in R_\eta$. Moreover, all the above relations are true for $r_1 \vee r_2$ and $r_1 \wedge r_2$ for $r_1, r_2 \in R_\phi$. Now, let $r_1, r_2 \in R_\phi$ and $r_1 \wedge r_2 = 0$. Then $r_1 \otimes s$, $r_2 \otimes s \in R_\eta$ and $(r_1 \otimes s) \wedge (r_2 \otimes s) = 0$. Therefore, $\|r_1 \otimes s + r_2 \otimes s\| = \|r_1 \otimes s - r_2 \otimes s\|$, and hence $\|r_1 + r_2\| = \|r_1 - r_2\|$. Hence ϕ satisfies (3).

Suppose now that $r \in A_1$. Then $r \otimes s \in A$ and $r \otimes s = t_1 + it_2$, with $t_1 t_2 \in R_\eta$. Then $r = \psi \circ (r \otimes s) = \psi \circ t_1 + i\, \psi \circ t_2$. Also, if $r = r_1 + ir_2 = r_3 + ir_4$ for $r_i \in R_\phi$, then $r \otimes s = r_1 \otimes s + i\, r_2 \otimes s = r_3 \otimes s + i\, r_4 \otimes s$. Therefore, $r_1 \circ s = r_3 \circ s$ and $r_2 \otimes s = r_4 \otimes s$. Hence, $r_1 = r_3$ and $r_2 = r_4$. We have also shown that $(\mathrm{Re}\ r) \otimes s = \mathrm{Re}\ (r \otimes s)$. Thus ϕ satisfies (4). We now show that ϕ satisfies (5). Let $r \in A_1$. First, we show that $|r| = \psi \circ |r \otimes s|$ and $|r \otimes s| = |r| \otimes s$. We have

$$\psi \circ |r \otimes s| - \mathrm{Re}\ (e^{i\theta} r) = \psi \circ |r \otimes s| - \psi \circ (\mathrm{Re}\ (e^{i\theta}\ r \otimes s))$$
$$= \psi \circ [|r \otimes s| - \mathrm{Re}\ (e^{i\theta}\ r \otimes s)],$$

for every $\theta \in [0, 2\pi]$. Therefore, $\psi \circ |r \otimes s| \geq |r|$. On the other hand, $|r| \geq \mathrm{Re}\ (e^{i\theta}\ r)$. Hence $|r| \otimes s \geq \mathrm{Re}\ (e^{i\theta}\ r) \otimes s = \mathrm{Re}(e^{i\theta}\ (r \otimes s))$. Therefore, $|r| \otimes s \geq |r \otimes s|$, so that $|r| \geq \psi \circ |r \otimes s|$. Thus we have $|r| = \psi \circ |r \otimes s|$. Also, since $|r| \otimes s \geq |r \otimes s|$, we get

$$\|\, |r| \otimes s - |r \otimes s|\, \| = \eta[|r| \otimes s - |r \otimes s|]$$
$$= \phi[|r| - \psi \circ |r \otimes s|] = 0.$$

Therefore, $|r| \otimes s = |r \otimes s|$. Now $\|r\| = \|r\|\, \|s\| = \|r \otimes s\| = \|\, |r \otimes s|\, \| = \|\, |r| \otimes s\| = \|\, |r|\, \|\, \|s\| = \|\, |r|\, \|$. This proves that ϕ satisfies (5). Hence ϕ is L'-inducing. We can show similarly that ψ is L'-inducing.

Conversely, suppose ϕ and ψ are L'-inducing. We shall show that η is L'-inducing. It is obvious that $\|\eta\| = 1$. Since ϕ is L'-inducing m.l.f. of A_1, by Proposition 2.3 of [7], there exists a locally compact Hausdorff space X and a positive regular Borel measure μ on X such that A_1 is isometrically linear isomorphic and, under the order induced by ϕ, order isomorphic to $L^1(X,\mu)$. The dual of A_1 is then represented by $L^\infty(X,\mu)$ and, under this representation, ϕ is represented by the constant function $\|\phi\| = 1$ on X. Now, $A_1 \otimes_\partial A_2 \simeq L^1(X,\mu) \otimes_\partial A_2 \simeq L^1(X,\mu,A_2)$. Hereafter, we shall not distinguish between elements of A and $L^1(X,\mu,A_2)$ and, for $F \in L^1(X,\mu,A_2)$, statements like "$F \in A$" will be used without explanation. For $F \in A$, we observe that $F \in P_\eta$ if and only if $\phi \circ F \in P_\psi$. This is so, because $\|F\| \geq \|\phi \circ F\| \geq |\psi(\phi \circ F)|$. We also have

$$\|F\| = \int \|F(x)\| \, d\mu(x) \geq \left|\psi\left[\int F(x) \, d\mu(x)\right]\right|$$
$$= \left|\int \psi(F(x)) \, d\mu(x)\right|.$$

This shows that $F \in P_\eta$ if and only if $F(x) \in P_\psi$ a.e. (μ). Let $F_1, F_2 \in P_\eta$. Using the continuity and other properties of the lattice operations, it is easy to prove that the function $F_1 \vee F_2$ defined a.e. (μ) by $(F_1 \vee F_2)(x) = F_1(x) \vee F_2(x)$, belongs to $L^1(X,\mu,A_2)$ and consequently defines an element of P_η. This proves that P_η is a lattice. Other details involved in showing that η is L'-inducing are also now easy to verify and hence we omit them. This completes the proof of our Theorem.

Having proved our main theorem, we now proceed to give its consequences.

THEOREM 3.2. Let A_1 and A_2 be commutative semisimple Banach algebras. Then $A_1 \otimes_\partial A_2$ is a group algebra if and only if A_1 and A_2 are group algebras.

PROOF. As mentioned in the introduction, it is well known that, if A_1 and A_2 are group algebras, then so is $A_1 \otimes_\partial A_2$. The converse follows from Lemma 2.1, Theorem R_1 and Theorem 3.1.

The following is an immediate consequence of Theorem 3.2.

THEOREM 3.3. Let G be a locally compact abelian group and let A be a commutative semisimple Banach algebra. Then $L^1(G,A)$ is a group algebra iff A is a group algebra.

PROOF. The result follows from Theorem 3.2 and the fact that the Banach algebras $L^1(G,A)$ and $L^1(G) \otimes_\partial A$ are isometrically isomorphic.

THEOREM 3.4. Let A_1 and A_2 be commutative semisimple Banach algebras and $A = A_1 \otimes_\partial A_2$. If A is a measure algebra, then A_1 and A_2 are measure algebras.

PROOF. Let D, D_1, D_2 be the set of L'-inducing m.l.f.'s of A, A_1, and A_2 respectively. Theorem 3.1 implies that $D = D_1 \times D_2$. Since D satisfies condition (i) of Theorem R_2, it easily follows that D_1 and D_2 also satisfy this condition. Since D is locally compact in the ω^*-topology, D_1 and D_2 are also locally compact in the ω^*-topology. It remains to show that A_1 and A_2 satisfy condition (iii) of Theorem R_2. We shall do this for A_2, the case of A_1 being similar. Since A is a measure algebra, it has an identity. It follows that A_1 and A_2 have identities. Let e be the identity of A_1. Let p be a D_2-Eberlein function. Define the function P on D by $P(\phi,\psi) = p(\psi)$. Obviously, P is continuous. Moreover,

$$\left|\sum_1^n \alpha_i \, P(\phi_i,\psi_i)\right| = \left|\sum_1^n \alpha_i \, p(\psi_i)\right| \le k\left\|\sum_1^n \alpha_i \, \psi_i\right\|_{A_2^*} .$$

However, for any $a \in A_2$,

$$\begin{aligned} \langle a, \sum_1^n \alpha_i \, \psi_i\rangle &= \langle e \otimes a, \sum_1^n \alpha_i(\phi_i,\psi_i)\rangle \\ &\le \left\|\sum_1^n \alpha_i(\phi_i,\psi_i)\right\|_{A^*}\|a\|. \end{aligned}$$

Therefore, $\|\sum_1^n \alpha_i \, \psi_i\|_{A_2^*} \le \|\sum_1^n \alpha_i(\phi_i,\psi_i)\|_{A^*}$. This shows that P is a D-Eberlein function and therefore there exists $t \in A$ such that $\hat{t}(\eta) = P(\eta)$ for every $t \in \Delta(A)$. Choose $\phi \in \Delta(A_1)$ and let $b = \phi \circ t$. Then $\hat{b}(\psi) = p(\psi)$ for every $\psi \in \Delta(A_2)$. This shows that A_2 satisfies condition (iii) of Theorem R_2 and the proof of our theorem is complete.

THEOREM 3.5. Let G be a locally compact abelian group and A be a commutative semisimple Banach algebra having RNP. Then M(G,A) is a measure algebra only if A is a measure algebra.

PROOF. The theorem follows from Theorem 3.4 and the fact that the algebras $M(G) \otimes_\partial A$ and M(G,A) are isometrically isomorphic under the hypothesis of our theorem.

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