Internat. J. Math. & Math. Sci.
Vol. 5 No. 3 (1982) 417-439

ZEROS OF SMALLEST MODULUS OF FUNCTIONS RESEMBLING exp(z)

KENNETH B. STOLARSKY
Department of Mathematics, University of Illinois
Urbana, Illinois 61801 U.S.A.

(Received July 17, 1981)

ABSTRACT. To determine (in various senses) the zeros of the Laplace transform of a signed mass distribution is of great importance for many problems in classical analysis and number theory. For example, if the mass consists of finitely many atoms, the transform is an exponential polynomial. This survey studies what is known when the distribution is a probability density function of small variance, and examines in what sense the zeros must have large moduli. In particular, classical results on Bessel function zeros, of Szegö on zeros of partial sums of the exponential, of I. J. Schoenberg on k-times positive functions, and a result stemming from Graeffe's method, are all presented from a unified probabilistic point of view.

KEY WORDS AND PHRASES. *Bessel function, characteristic function, exponential function, exponential polynomial, Graeffe's method, Laplace transform, multiply positive function, normal density, probability density function, variance.*

A.M.S. CLASSIFICATION CODE. *30C15.*

§1. INTRODUCTION. It is fundamental that exp(az+b) is never zero for any complex number z. In fact, it is the only entire function of exponential type with this property. Now one could say that the zeros of the exponential lie "unborn" in the essential singularity at ∞. This leads us to ask how the zeros "stream out" of ∞ as the exponential is "slightly perturbed" in the space E of entire functions of exponential type.

Write

$$f(z) = \int_{-\infty}^{\infty} e^{tz} g(t)\,dt \tag{1.1}$$

where $g(t)$ is a probability density function of mean 1 and variance σ. When $\sigma = 0$ the $g(t)$ becomes the Dirac "delta function" at 1, and $f(z)$ is then $\exp z$. Thus, when all the mass is concentrated at 1, there are no zeros; when the mass spreads out from 1 the zeros move in from ∞ and may come closer and closer to $z = 0$, although they cannot attain it (nor any other point on the real axis).

In probability theory $f(iz)$ is known as the characteristic function of $g(t)$ [LU 1,2]. We note that if $g(t)$ consists of a finite number of atoms, then (1.1) becomes an exponential polynomial.

For an example, say $g(t) = g_a(t)$ has two atoms of mass $1/2$ located at $1 - a$ and $1 + a$ respectively. Then

$$f(z) = e^z \cosh az, \tag{1.2}$$

and as the parameter a increases the zeros become more and more dense along the imaginary axis, with the one of smallest modulus approaching 0.

For another example, note that as the parameter b increases from 0 to 1, the modulus of the zeros of

$$f(z) = b + (1-b)\cosh z \tag{1.3}$$

nearest to the origin tends to ∞; in fact, it can be shown to do so monotonically. Here we have two atoms of mass $(1-b)/2$ at ± 1 and one atom of mass b at 0, so the mean is 0 in this case.

The following informal assertion seems reasonable.

<u>The</u> $\delta\sigma$-<u>Hypothesis</u>. <u>The distance</u> δ <u>from the origin to the nearest zero tends to vary inversely with the variance</u> σ.

Our goal in this paper is to determine to what extent this hypothesis is true. To do this we assemble results scattered throughout the literature. However, the theorems of §3 and §4 may be new.

To achieve an appropriate level of generality, we shall study the hypothesis in the class of all functions of the form

$$f(z) = \int e^{wz} d\mu(w) \tag{1.4}$$

where $\mu(w)$ is a probability measure on the complex plane, not necessarily of compact support, with "mean 1" and "variance σ," i.e.,

$$\int w d\mu(w) = 1, \qquad \int |w-1|^2 d\mu(w) = \sigma^2. \tag{1.5}$$

For example, when $\mu(w)$ is purely atomic, symmetric about the real axis, and confined to $\mathrm{Re}(w) = 1$, then

$$f(z) = e^z \sum_j a_j \cos \lambda_j z \tag{1.6}$$

for appropriate real λ_j, $j = 1,2,3,\ldots$. If the mass is distributed on a circle of radius r about $z = 1$, with density $g(\theta)$ at $1 + re^{i\theta}$, then

$$\begin{aligned} f(z) &= \int e^{wz} d\mu(w) = \int_{-\pi}^{\pi} \exp[z(1+re^{i\theta})]g(\theta)d(\tfrac{r\theta}{2\pi r}) \\ &= \frac{e^z}{2\pi} \int_{-\pi}^{\pi} \exp[zre^{i\theta}]g(\theta)d\theta = e^z J(zr;g) \end{aligned} \tag{1.7}$$

where $J(z) = J(z;g)$ is an entire function such that $J(0) = 1$. If $J(z)$ is neither an exponential nor a constant, let z_o be its zero of smallest modulus. Then the $\delta\sigma$-hypothesis for $f(z)$ has in this case the very simple form

$$\delta\sigma = |z_o| > 0, \tag{1.8}$$

since $\sigma = r$. For (1.7) it is obvious that increasing r brings the nearest zero closer; for (1.6) it is mearly plausible that increasing one of the λ_j may have this effect.

REMARK 1. For a non-trivial characterization of $\exp z$, via its lack of zeros, in a function class much larger than E, see [HA, pp. 66-67].

REMARK 2. It may be of interest to investigate the angular distribution of the zeros of $J(z)$. If $g(\theta)$ is considerably larger for $|\theta| < \pi/2$ than for

$|\theta| \geq \pi/2$ (say $-\pi \leq \theta \leq \pi$), will J have more zeros in the left half plane than in the right? The results of Pólya [PO 4] on Mittag-Leffler's function suggest that a nearly uniform $g(\theta)$ will have its zeros distributed almost uniformly with respect to angle.

§2. COUNTEREXAMPLES TO THE $\delta\sigma$-HYPOTHESIS. When $\mu(w)$ is chosen so that $f(z)$ has the form (1.6), one can sometimes create a larger zero free region about the origin by dispersing some of the mass arbitrarily far away from $w = 1$. We exhibit such an example.

Let D be the closed disc about 0 of radius π, and set

$$f(z;\varepsilon,\lambda) = k[\cos z + \varepsilon \cos \lambda z], \tag{2.1}$$

where k is chosen so

$$\int d\mu(w) = 1. \tag{2.2}$$

Fix a positive integer n. For $|\lambda| \leq 5n$, the functions of (2.1) converge equiuniformly on D to $\cos z$ as $\varepsilon \to 0$. Hence for small ε they have only two zeros in D, one near $\pi/2$ and the other near $-\pi/2$. Since f is even, we need only consider the zero z_o near $\pi/2$. For $\lambda = 1$ this z_o is $\pi/2$. For $\lambda = 4n$ write $z_o = \pi/2 + \alpha$ so

$$\varepsilon \cos 4n\alpha = \sin \alpha \tag{2.3}$$

and (since α is small)

$$\varepsilon = \alpha + \frac{24n^2-1}{3}\alpha^3 + \cdots, \quad \text{or} \quad \alpha = \varepsilon - \frac{24n^2-1}{3}\varepsilon^3 + \cdots . \tag{2.4}$$

Thus for $\varepsilon = \varepsilon(n)$ sufficiently small, the zeros of $f(z;\varepsilon,4n)$ nearest the origin are more distant than the nearest zero of $f(z;\varepsilon,1)$.

For further examples of a similar nature (involving cosine sums on the real line) see [NU]. One might doubt, however, the existence of far more extreme counterexamples. For $t \geq 0$ let $\mu_t(w) = \mu(w;t)$ be a family of mass distributions of compact support, continuous in t, such that for $t_0 < t_1$ the distribution

$\mu(w;t_1)$ is obtained from $\mu(w;t_0)$ by moving some of the mass of $\mu(w;t_0)$ further away from 1 in the radial direction. We shall even make the assumption (A): all the μ_t are centrally summetric about 1, and μ_0 has no mass in some neighborhood U of 1. Let R_t be the radius of the largest disc D_t about 0 such that

$$f(z) = f_t(z) = \int e^{zw} d\mu_t(w) \tag{2.5}$$

has no zeros interior to D_t.

PROBLEM. Is the quantity

$$R_\infty = \limsup_{t\to\infty} R_t \tag{2.6}$$

finite?

If $\mu_0(w)$ is the mas distribution corresponding to $e^z \cos z$, it seems safe to conjecture that R_∞ is rather less than $3\pi/2$, and that any mass distribution corresponding to a nearly maximal R_t will have a large proportion of its mass very near the points $1 \pm i$.

REMARK 1. For mass distributions along the real axis, the answer can be negative, no matter how large U is! See §7.

REMARK 2. In the example (2.1), the highest frequency was attached to the smallest mass. The distribution of the roots of

$$f(z) = \int_a^b e^{tz} g(t)\,dt \tag{2.7}$$

where $g(t)$ is continuous is sometimes more peculiar or more difficult to analyze when $g(a) = g(b) = 0$; see [CART, TI]. When $g(t)$ is monotone, or

$$f(z) = \sum_{k=N}^{M} a_k \cos kz, \quad a_N < a_{N+1} < \cdots < a_M, \tag{2.8}$$

the situation is far simpler; see [PO 2].

The simplest quantitative formulation of the $\delta\sigma$-hypothesis is that

$$\delta\sigma \gg 1. \tag{2.9}$$

More precisely, this means that given a family G of probability measures the product $\delta\sigma$ exceeds a positive constant depending only on G. We now show that this is not true unless G is somehow restricted. Let

$$f(z) = e^z[\varepsilon e^{-az} + \varepsilon e^{az} + 1 - 2\varepsilon], \quad a,\varepsilon > 0. \tag{2.10}$$

Here

$$\sigma = a\sqrt{2\varepsilon}. \tag{2.11}$$

By solving a quadratic equation, we find there are roots as close as

$$a^{-1}|\ell n[-\varepsilon + 0(\varepsilon^2)]|, \quad \text{and} \quad a^{-1}|\ell n[-(4\varepsilon)^{-1} + 0(1)]|. \tag{2.12}$$

Since

$$\varepsilon^{1/2}\ell n\ \varepsilon \to 0 \quad \text{as} \quad \varepsilon \to 0, \tag{2.13}$$

the formulation (2.9) cannot be correct. In fact, if $P(z)$ is any function that assumes positive values, we cannot have a universal rule of the form

$$\delta >> P(\sigma) \tag{2.14}$$

since for

$$a = c_0/\sqrt{2\varepsilon} \tag{2.15}$$

it would imply that

$$\delta \sim \frac{\sqrt{2\varepsilon}}{c_0}\ \ell n\ \varepsilon^{-1} >> P(c_0), \tag{2.16}$$

a contradiction for an appropriate c_0 by (2.13).

§3. THE $\delta\sigma$-HYPOTHESIS FOR SYMMETRIC PEAKED DISTRIBUTIONS. The $\delta\sigma$-hypothesis is true for functions $f(z)$ of the form (1.1) when $g(t)$ is a probability density function symmetric about 1 and "strongly" peaked near 1. The result (3.3) asserts that δ is large if σ is very small.

Note that here and elsewhere we use "peaked" as an informal adjective rather than as a concept with a precise logical definition.

THEOREM. If

$$g(1-t) = g(1+t) \tag{3.1}$$

and

$$g(1+t) \le Ke^{-t^2}, \quad t \ge 0, \tag{3.2}$$

then

$$\delta \ge \min[F(\sigma), \sqrt{F(\sigma)}] \tag{3.3}$$

where

$$F(\sigma) = \frac{|\ell n\ 2\sigma|}{2k+1} . \tag{3.4}$$

PROOF. Let z_0 be a zero of minimal modulus of

$$e^{-z}f(z) = \int_{-\infty}^{\infty} e^{zt}g(1+t)dt; \tag{3.5}$$

thus $\delta = |z_0|$. Now

$$\begin{aligned} &= |f(0) - e^{-z_0}f(z_0)| = |\int_{-\infty}^{\infty} (e^{z_0t}-1)g(1+t)dt| \qquad (3.6)\\ &= |\int_{-\infty}^{0} + \int_{0}^{\infty}| = 2|\int_0^{\infty} \left(\frac{e^{z_0t}+e^{-z_0t}}{2} - 1\right) g(1+t)dt| \\ &= 2|\int_0^{\infty} t^2 g(1+t)\left(\frac{\cosh z_0t-1}{t^2}\right)dt|, \end{aligned}$$

and by expanding the hyperbolic cosine into an infinite series we easily see that this is

$$\begin{aligned} &\le 2\int_0^T t^2g(1+t)\ \frac{\cosh \delta t-1}{t^2}\ dt + 2\int_T^{\infty} g(1+t)[\cosh \delta t-1]dt \qquad (3.7)\\ &\le 2\sigma^2\ \frac{\cosh \delta T-1}{T} + 2K\int_T^{\infty} e^{\delta t-t^2} dt. \end{aligned}$$

Now choose

$$T = 1 + \delta + 4K = (1 + (4K+1)\delta^{-1})\delta = M\delta. \tag{3.8}$$

Then the last integral on the right of (3.7) is bounded by

$$\{\exp[-M(M-1)\delta^2]\}/(M-1)\delta. \tag{3.9}$$

Hence

$$1 \leq 2\sigma^2 \exp[\delta^2 + (4K+1)\delta] + 1/2 \tag{3.10}$$

and

$$\begin{aligned} 1/4\sigma^2 &\leq \exp[\delta^2 + (4K+1)\delta] \\ &\leq \begin{cases} \exp(4K+2)\delta^2, & \delta > 1, \\ \exp(4K+2)\delta, & \delta \leq 1. \end{cases} \end{aligned} \tag{3.11}$$

The result follows.

Note that if g were replaced by $|g|$ in (3.2), the proof would make no essential use of the nonnegativity of g.

4. GRAEFFE'S METHOD. The first naive formulation of the $\delta\sigma$-hypothesis, (2.9), does seem to hold when the $g(t)$ in (1.1) is very close to a smooth unimodal distribution that is "somewhat" peaked at 1. For example, when $g(t)$ is a rectangular pulse of width 2ε centered at 1 (this can be approximated arbitrarily well by a C^∞ function) we have

$$\sigma = \varepsilon/\sqrt{3}, \quad \text{and} \quad z_0 = \pm i\pi/\varepsilon \tag{4.1}$$

where z_0 denotes zeros of minimal modulus. Hence

$$\delta\sigma = \pi/\sqrt{3}. \tag{4.2}$$

We now show that something like (2.9) holds when $g(t)$ is suitably concentrated near 1 and δ is small. In this case the old root-squaring method (Graeffe's method) that has been popular in the past for determining minimum modulus zeros of polynomials is quite usable. In fact, this method has been used previously on certain transcendental entire functions; see [PO1, 5] and [DIR]. We shall require that the power moments of $g(t)$ do not grow too rapidly; our condition (4.3) seems quite natural since it is satisfied for

$$g(t) = \exp(-|t-1|) \quad \text{and} \quad g(t) = \exp[-(t-1)^2],$$

and commonly occurs in the study of characteristic functions [LU1, pp. 19ff.,27ff].

THEOREM 4.1. If the $g(t)$ of (1.1) satisfies

$$\int |t-1|^j g(t)\,dt \le AK^j j!, \quad j = 0,1,2,\ldots, \tag{4.3}$$

for positive constants A and K, and z_0 is any zero of $f(z)$ less than $1/K$ in modulus, then

$$|1-z_0^2\sigma^2| \le |Kz_0|^4 \frac{A^2(5+3K^2|z_0|^2)}{(1-K^2|z_0|^2)^2}. \tag{4.4}$$

PROOF. Define

$$G(z) = \int e^{izt} g(t)\,dt. \tag{4.5}$$

Since

$$\int g(t)\,dt = \int tg(t)\,dt = 1, \quad \int (t-1)^2 g(t)\,dt = \sigma^2 \tag{4.6}$$

we find for the <u>even</u> function below on the left that

$$\begin{aligned} G(z)G(-z) &= \iint e^{iz(t-s)} g(t)g(s)\,dtds \qquad (4.7)\\ &= \iint \Big[1 - \frac{z^2}{2!}[(t-1) - (s-1)]^2 + \sum_{m=2} \frac{(-1)^m z^{2m}(t-s)^{2m}}{(2m)!}\Big] g(t)g(s)\,dtds \\ &= 1 - \frac{z^2}{2!}[\sigma^2 - 2\cdot 0 + \sigma^2] + R(z) \end{aligned}$$

where $R(z)$ is an infinite sum of double integrals. Since

$$\iint [(t-1)-(s-1)]^{2m} g(t)g(s)\,dt\,ds \tag{4.8}$$

$$\leq \sum_j \binom{2m}{j} \int |t-1|^{2m-j} g(t)\,dt \int |s-1|^j g(s)\,ds$$

$$\leq A^2K^{2m} \sum_{j=0}^{2m} \binom{2m}{j}(2m-j)!j! \leq A^2K^{2m}(2m+1)!,$$

we have

$$|R(z)| \leq A^2 \sum_{m=2}^{\infty} |Kz|^{2m}(2m+1). \tag{4.9}$$

Since $G(z_0) = 0$, the result follows by summing the series on the right of (4.9).

For example, if $A = K = 1$ and $f(z)$ has a zero in $|z| < 1/4$, inequality (4.4) shows that the variance of $g(t)$ must exceed 15.

If $g(t)$ has compact support, we can prove much more.

THEOREM 4.2. If the support of $g(t)$ lies in the interval $[a,b]$, then

$$\delta > 1/(b-a). \tag{4.10}$$

PROOF. Modify the proof of Theorem 4.1 by estimating $R(z)$ as follows:

$$|R(z)| \leq \sum_{m=2}^{\infty} \frac{|z(b-a)|^{2m}}{(2m)!} \iint g(t)g(s)\,dt\,ds. \tag{4.11}$$

say that contrary to (4.10) there is a zero z_0 with

$$|z_0(b-a)| \leq 1. \tag{4.12}$$

Since the double integral on the right of (4.11) equals one,

$$|R(z_0)| \leq \delta^4(b-a)^4\left(\frac{1}{4!}+\frac{1}{6!}+\cdots\right) \leq \frac{1}{23}\,\delta^4(b-a)^4 \leq \frac{1}{23}\,. \tag{4.13}$$

Now the variance σ^2 is maximal when the mass consists of two equal atoms located at a and b respectively, in which case

$$\sigma = (b-a)/2. \tag{4.14}$$

From (4.7) and (4.13) with $z = z_0$ we obtain

$$1 - \delta^2\sigma^2 \le 1/23 \tag{4.15}$$

and so

$$\sqrt{22/23} \le \delta(b-a)/2. \tag{4.16}$$

But this contradicts (4.12).

Note that the proofs of the above theorems use (4.6), but not the nonnegativity of g.

Since the function

$$f(z) = (e^{az}+e^{bz})/2 \tag{4.17}$$

corresponding to two equal atoms at a and b has zeros at $\pm i\pi/(b-a)$, Theorem 4.2 is within a factor of π of being best possible. For this function

$$\delta\sigma = \pi/2. \tag{4.18}$$

CONJECTURE. If the mass is in $[a,b]$, then

$$\delta \ge \pi/(b-a). \tag{4.19}$$

We add that if the mass is not supported by any inteval smaller than $[a,b]$, then $(b-a)/\pi$ is the "linear density" of the zeros of $f(z)$; see [TI].

§5. PARTIAL SUMS OF THE EXPONENTIAL SERIES. By Hurwitz's Theorem, the partial sums

$$s_N(z) = \sum_{k=0}^{N} z^k/k! \tag{5.1}$$

have no zeros in a disc of radius r_N about 0, where $r_N \to \infty$ as $N \to \infty$. This statement has been made very precise by Szegö [SZ] who has shown that if the zeros of $s_N(z)$ are divided by N, these normalized zeros, for large N, lie very near the subset H of the curve C defined by

$$|ze^{1-z}| = 1 \tag{5.2}$$

for which $\mathrm{Re}(z) \le 1$. In fact, H is a loop that encircles the origin. The curve

(5.2) roughly resembles the Greek letter "α". It intersects itself at right angles at $z = 1$, and to the right of $z = 1$ it consists of two curves, mirror images in the z-axis of each other, with the one in the first quadrant being convex, and having an ordinate that grows exponentially with its abscissa. We speak of the parts of (5.2) to the left and right of $z = 1$ as being the "head H" and the "tail T", respectively of C.

Curiously enough, Szegö also proved that the normalized zeros of the "tail" of the exponential series

$$T_N(z) = \sum_{k=N+1}^{\infty} \frac{z^k}{k!} \tag{5.3}$$

lie very near T! In particular, all the zeros are at least N in absolute value, and this is asymptotically best possible. We now show how this bears on the present investigations.

Consider the probability density given by

$$g(t) = \begin{cases} (N+1)t^N & 0 \le t \le 1 \\ 0 & \text{otherwise.} \end{cases} \tag{5.4}$$

Then $g(t)$ can be approximated arbitrarily well by sharply peaked C^∞ unimodal densities. A simple calculation yields

$$\int_0^1 tg(t)\,dt = \frac{N+1}{N+2}, \quad \sigma^2 = \frac{N+1}{(N+2)^2(N+3)}, \tag{5.5}$$

so for N large it is very nearly the sort of $g(t)$ we considered in (1.1). Since

$$T_N(z) = \frac{e^z}{N!}\int_0^z e^{-s}s^N\,ds \tag{5.6}$$

and (by change of variable) we have

$$f(z) = \int_0^1 e^{zt}g(t)\,dt = \frac{N+1}{(-z)^{N+1}}\int_0^{-z} e^{-s}s^N\,ds, \tag{5.7}$$

it follows from (5.6) and (5.7) that

$$f(z) = \frac{(N+1)!\,e^z}{(-z)^{N+1}}\,T_N(-z). \tag{5.8}$$

Thus

$$\sigma\delta >> 1, \quad \text{and} \quad \sigma\delta \to 1 \quad \text{as} \quad N \to \infty. \tag{5.9}$$

This provides another example where (2.9) is valid, and with a smaller constant than in (4.2) or (4.18).

REMARK. Szegö also determined the asymptotic angular distribution of the zeros. His results (particularly in a neighborhood of $z = 1$) have been both greatly refined and also extended to entries of the Padé table for $\exp(z)$. See [BU, NEW, SA1-3].

§6. SMALL ZEROS OF BESSEL FUNCTIONS. The rule (2.9) seems to hold quite generally in the absence of the sort of "side lobes" that played a role in constructing the counterexamples of §2. This leads us to suspect it may be satisfied, or even "over-satisfied," when $g(t)$ is convex aside from some small neighborhood of its maximum. For an example of this we turn to the book of B. C. Carlson, Special Functions of Applied Mathematics [CAR], that emphasizes the utility of the fact that most of the commonly occurring special functions are expectations of elementary functions (sometimes exponentials) with respect to common probability measures. For example, the Bessel functions $J_r(x)$ have the form

$$J_r(x) = \frac{(x/2)^r}{\Gamma(r+1)} S(r+1/2, r+1/2; ix, -ix) \tag{6.1}$$

where [CAR, pp. 93-96]

$$\begin{aligned} S(b_1, b_2; z_1, z_2) &= \int \exp[wz_1 + (1-w)z_2] d\ (w) \\ &= \int_0^1 \exp[uz_1 + (1-u)z_2] \frac{u^{b_1-1}(1-u)^{b_2-1}}{B(b_1, b_2)} du. \end{aligned} \tag{6.2}$$

Here $B(x,y)$ is the usual beta-function. For a random variable X with the distribution $\mu(w)$ one has

$$E(X) = \frac{b_1}{b_1+b_2}, \quad E(X^2) = \frac{b_1(b_1+1)}{(b_1+b_2)(b_1+b_2+1)} \tag{6.3}$$

so

$$\sigma^2 = \frac{b_1 b_2}{(b_1+b_2)^2(b_1+b_2+1)} \quad . \tag{6.4}$$

Hence

$$S(r+1/2, r+1/2; z/2, -z/2) = \frac{\Gamma(r+1)}{(z/4i)^r} J_r(z/2i) \tag{6.5}$$

is a function $f(z)$ of the form (1.1) with

$$\sigma = [8(r+1)]^{-1/2}. \tag{6.6}$$

Thus the rule (2.9) yields

$$\delta >> r^{1/2} \tag{6.7}$$

for $J_r(z)$. But Watson's treatise on Bessel functions [WA] tells us that the smallest zero of $J_r(z)$ is $r + O(r^{1/3})$, so in this case

$$\delta >> \frac{1}{\sigma^2} \ . \tag{6.8}$$

§7. SOME IDEAS FROM PROBABILITY THEORY. The example of §6 leads us to ask if there are some further hypotheses that can be made on the shape of a unitary spike so that the nearest zero would have to be even further away for σ small, with perhaps a rule such as

$$\delta\sigma^m >> 1 \tag{7.1}$$

where m is some positive integer.

If we think of $g(t)$ as the distribution function of a random variable X, it is natural to also consider the distribution function for the average

$$\overline{X} = \frac{X_1 + \cdots + X_n}{n} \tag{7.2}$$

where $X_1, \ldots, X_n$ are random variables with the same distribution as X itself. Let $f(z;Y)$ denote the moment generating function for a given random variable Y, let $\sigma(Y)$ denote its variance, etc. Then

$$f(z;\overline{X}) = [f(\tfrac{z}{n};X)]^n, \quad \sigma(\overline{X}) = \sigma(X)/\sqrt{n}, \tag{7.3}$$

and

$$\delta(\overline{X}) = n\delta(X). \tag{7.4}$$

Thus if we begin with a class of functions $g(t)$ for which the rule (2.9) holds, and replace each $g(t)$ by the density for the corresponding $\overline{X}$, the rule holds with "much more room to spare." What (7.3) and (7.4) suggest, of course, is that for "nicely shaped" pulses, the appropriate rule to investigate is

$$\delta\sigma^2 >> 1, \tag{7.5}$$

a rule we already stumbled upon in §6.

We next observe that for $g(t)$ with infinite support, it may be that σ is large, and $f(z)$ is <u>zero free</u>. For example, let $g(t)$ be the normal density

$$g(t) = \frac{1}{\sigma\sqrt{2\pi}} \exp[-\frac{(t-1)^2}{2\sigma^2}]. \tag{7.6}$$

Since

$$f(z) = \exp(-z+z^2\sigma^2/2), \tag{7.7}$$

there are no zeros! We can now construct densities of compact support that are nearly as good. Simply multiply $g(t)$ by a number $1 + \varepsilon$ where $\varepsilon > 0$ and redefine it to be zero outside of some large interval $[-L,L]$, so the resulting function again has total mass one. Various densities other than the $g(t)$ of (7.6) are also available; in fact, when an infinitely divisible distribution has an entire characteristic function $f(z)$, then $f(z)$ is zero free [LU1, p. 187].

The graph of the normal density is <u>bell-shaped</u>; this means that its nth derivative changes signs n times. If this holds merely for $n \leq M$ we can say it is M-bell shaped. It is conceivable that some such subtle measure of the shape of the spike and not merely its "peakedness" may determine the extent to which (2.9) or (7.5) may be replaced by a sharper rule. The concept of bell-shaped will also occur below in §8.

We now return to the problem of §2. Let $g_\sigma(t)$ be a "normal density distribution" with $\mu = 1$ and σ extremely large, but modified as above to be of compact support. Fix $\varepsilon > 0$ and let $r(t;\eta)$ be a rectangular pulse distribution with $\mu = 1$ and support $[1-\eta, 1+\eta]$ where

$$\int_{1-\eta}^{1+\eta} g_\sigma(t)\,dt = \alpha < \varepsilon. \tag{7.8}$$

First shift $\alpha/2$ of the mass of $r(t;\eta)$ to each of the points $1 - \eta$ and $1 + \eta$, and then shift the remaining mass of $r(t;\eta)$ outwards so it becomes the mass distribution of $g_\sigma(t)$. By choice of ε we can insure that before the second mass shift the nearest zero is at a distance of approximately π/η while after that shift it is far away as desired. Thus the problem of §2 has a negative answer in this case.

PROBLEM. Can the mass of the rectangular pulse be moved out to infinity continuously with respect to time so that the distance of the nearest zero from the origin is monotonically increasing?

§8. MULTIPLY POSITIVE FUNCTIONS. In [SCHO3] Schoenberg defines a real measurable function $g(x)$ such that

$$0 < \int_{-\infty}^{\infty} g(x)\,dx < \infty \tag{8.1}$$

to be k-times positive if for

$$x_1 < x_2 < \cdots < x_n; \quad y_1 < y_2 < \cdots < y_n \tag{8.2}$$

we have

$$\det[g(x_i - y_j)]_{i,j=1}^{n} \geq 0 \tag{8.3}$$

for $1 \leq n \leq k$. If $g(x)$ is k-times positive for every k, it is said to be totally positive. For example,

$$\exp(-x^2), \quad \exp(-x-e^{-x}), \quad (\cosh x)^{-1} \tag{8.4}$$

are known to be totally positive. Schoenberg [SCHO2] has proved the

THEOREM. If $g(x)$ is totally positive, but not the exponential of a linear function, then

$$\int_{-\infty}^{\infty} e^{zt} g(t)\,dt = \frac{1}{E(-z)} \tag{8.5}$$

where

$$E(z) = Ce^{-\gamma z^2+az} \prod_{m=1}^{\infty} (1+a_m z)e^{-a_m z}. \tag{8.6}$$

Here the parameters C,γ,a,a_m are real, with $\gamma \geq 0$ and

$$\sum a_m^2 < \infty. \tag{8.7}$$

Thus the Laplace transform of such a function is zero free. Moreover, it is known that the totally positive functions are bell-shaped [SCHO1, HIR1-2]! But of greatest interest in connection with the $\delta\sigma$-hypothesis is Schoenberg's theorem [SCHO3] on k-times positive functions of compact support.

THEOREM. Let $g(x)$ be k-times positive and identically zero outside of $[0,\pi]$. Then

$$f(z) = \int_0^{\pi} e^{zt} g(t)\,dt \tag{8.8}$$

has no zeros in the strip

$$|\mathrm{Im}(z)| < k, \tag{8.9}$$

and this is best possible.

Schoenberg's example that shows the result is best possible is particularly illuminating. He defines the function

$$g(t) = g_\alpha(t) = \begin{cases} (\sin t)^\alpha & 0 \leq x \leq \pi, \\ \\ 0 & \text{otherwise}, \end{cases} \tag{8.10}$$

and observes that

$$e^{-z\pi/2} \int_0^{\pi} e^{zt} g(t)\,dt = \frac{\pi\Gamma(\alpha+1)}{2^\alpha \Gamma(\frac{1}{2}(\alpha+2+iz))\Gamma(\frac{1}{2}(\alpha+2-iz))} \tag{8.11}$$

for $\alpha > -1$ (note that $g(t)$ has mean $\pi/2$ rather than 1). For $\alpha \geq k-2$ he shows that $g(t)$ is k-times positive. Now by (8.11) the zeros of its transform are

$$\pm(\alpha+2+m)i, \quad m = 0,1,\ldots, \tag{8.12}$$

and (set $m = 0$) it follows that the strict inequality in (8.9) cannot be relaxed.

Scale $g(t)$ so that it becomes a probability density. Then as $\alpha \to \infty$ its variance tends to 0, and all the mass becomes concentrated at its mean $\pi/2$. In terms of variance we have, just as for the Bessel functions of §6, that

$$\delta \gg \sigma^{-2}. \tag{8.13}$$

REMARK 1. I know of no paper or monograph that discusses infinitely divisible distributions and totally positive functions together, although they seem to be similar in some ways. Note that $\exp(-t^2)$ belongs to both of these function classes.

REMARK 2. The property of being k times positive is very delicate and can be destroyed by a small perturbation of $g(t)$. However, the parameters δ and σ are at most slightly changed by such perturbations, so a law such as (8.13) is "robust."

§9. SEVERAL COMPLEX VARIABLES. The nonvanishing of the exponential has an analogue in several complex variables, namely that

$$\sum_{j=1}^{n} \exp z_j = 0 \tag{9.1}$$

cannot occur if the z_j exhibit only "small deviations from the mean." To make this precise, define the distance $d(A,P)$ between any points

$$A = (a_1,\ldots,a_n), \quad P = (z_1,\ldots,z_n) \tag{9.2}$$

in complex n-space by

$$d^2(A,P) = \Sigma|a_j-z_j|^2 . \tag{9.3}$$

The diagonal of complex n-space is the set of complex numbers having all components identical.

THEOREM. If (9.1) holds and $n \geq 2$, then the point $P = (z_1, \ldots, z_n)$ has distance at least

$$d_n = (1+n^{-1})\ell n\, n \tag{9.4}$$

from the diagonal. On the other hand, there is a P with distance at most

$$D_n = [1+O(\ell n\, n)^{-1}]\ell n\, n \tag{9.5}$$

from the diagonal for which (9.1) holds.

The above result again comfirms the "$\delta\sigma$-hypothesis," though in a rather different setting. For the proof, and further comments on the relation of this result to various statistical estimates, see [ST].

§10. OSCILLATION THEORY. It is curious that the $\delta\sigma$-hypothesis does not seem to be directly addressed by any previous literature. However, if we think of the parameter t in

$$h(z) = h_t(z) = e^{tz} + e^{-tz} \tag{10.1}$$

as being expressed by square roots of q in the differential equation

$$h''(z) - qh(z) = 0 \tag{10.2}$$

or by fourth roots of p in

$$h^{(4)}(z) - ph(z) = 0, \tag{10.3}$$

then the oscillation theory of ordinary differential equations, starting with the comparison theorems of Sturm (see [HI2, pp. 373-384; pp. 576-644]), is possibly relevant. The literature in this direction is large; we mention merely [HI1-2, LE, NE1-2].

To illustrate its bearing on the $\delta\sigma$-hypothesis, consider

$$y'' - qy = 0 \tag{10.4}$$

where $q = q(z)$ is analytic in a neighborhood of $z = 0$ and real for real z. Let

$f(z)$ be the solution with $f(0) = 1$, $f'(0) = 0$. If $d\mu(w)$ is a measure on the real axis satisfying (1.4), then a simple calculation (note that $f''(0) = q(0)$) shows that (1.5) is valid with

$$\sigma^2 = q(0) + 1. \tag{10.5}$$

Now a theorem of Hille [HI2] asserts that for $q = q(z)$ analytic in $|z| < R$ and satisfying $|q(z)| \leq M$ there, we have

$$\delta \geq \min(R, \pi/2\sqrt{M}). \tag{10.6}$$

Thus, in some loose sense, Hille's oscillation theorem gives the same lower bound as the $\delta\sigma$ hypothesis for slowly varying q. Both are quantitative versions of the classical principle that the larger the potential the more rapidly the solution oscillates.

§11. REMARKS. The location of minimum modulus zeros has been extensively investigated for polynomials [MA], but the literature for analytic functions does not seem terribly large, despite the importance of the related problem of location of minimal eigenvalues. The literature on zero-free half planes, at least for exponential polynomials, is perhaps larger owing to its importance for stability theory. See [PON] and also [BE] for many further references.

For the asymptotic distribution of zeros of exponential polynomials and closely related functions see [BE, CART, DI1-2, LA, ME, MO, PO2-4, POO2, SCHW, TI, TU]. Also relevant here, as well as to §5, is [GA]. For the local distribution of zeros of exponential polynomials in the complex plane see [POO1, TIJ, VO1].

ACKNOWLEDGEMENT. I thank the referee for a number of valuable suggestions, and also for removing many misprints and obscurities.

REFERENCES

[BE] BELLMAN, RICHARD and COOKE, KENNETH L., Differential-Difference Equations, Academic Press, New York, 1963.

[BER] BERENSTEIN, CARLOS A. and TAYLOR, B. A., Mean Periodic Functions, Int. J. of Math. and Math. Sci. 3(1980), 199-235.

[BU] BUCKHOLTZ, J. D., A characterization of the exponential series, Amer. Math. Monthly 73(1966), 121-123.

[CAR] CARLSON, B. C., Special Functions of Applied Mathematics, Academic Press, New York, 1977.

[CART] CARTWRIGHT, M. L., The zeros of certain integral functions, Quart. J. Math. 1(1930), 38-59; II, ibid 2(1931), 113-129.

[DI1] DICKSON, D. G., The asymptotic distribution of zeros of exponential sums, Publ. Math. Debrecen 11(1964), 295-300.

[DI2] ____________, Zeros of exponential sums, Proc. Amer. Math. Soc. 73(1965), 84-89.

[DIR] DIRSCHMID, H., Bemerkungen zu einer Arbeit von G. Pólya zur Bestimmung der Nullstellen ganzer Funktionen, Numer. Math. 13(1969), 344-348.

[GA] GANELIUS, T., Sequences of analytic functions and their zeros, Arkiv. Mat. 3(1953), 1-50.

[HA] HAYMAN, W. K., Meromorphic Functions, Clarendon Press, Oxford, 1964.

[HI1] HILLE, E., Nonoscillation theorems, Trans. Amer. Math. Soc. 64(1948), 234-252.

[HI2] ____________, Lectures on Ordinary Differential Equations, Addison-Wesley, Reading, 1969.

[HIR1] HIRSCHMAN, I. I. and WIDDER, D. V., The inversion of a general class of convolution transforms, Trans. Amer. Math. Soc. 66(1949), 135-201.

[HIR2] ____________, Proof of a conjecture of I. J. Schoenberg, Proc. Amer. Math. Soc. 1(1950), 63-65.

[LA] LANGER, R. E., On the zeros of exponential sums and integrals, Bull. Amer. Math. Soc. 37(1931), 213-239.

[LE] LEIGHTON, W. and NEHARI, Z., On the oscillation of solutions of self-adjoint linear differential equations of the 4th order, Trans. Amer. Math. Soc. 89(1958), 325-377.

[LU1] LUKACS, E., Characteristic Functions, Hafner, New York, 1960.

[LU2] ____________, Characteristic Functions, 2nd. end., Griffin, London, 1970.

[MA] MARDEN, M. Geometry of Polynomials, 2nd. ed., Providence, Amer. Math. Soc., 1966.

[ME] MEYER, G. P., Über die Nulstellen von Exponentialsummen längs einer logarithmischen Wechsellinie, Archiv der Math. 32(1979), 479-486.

[MO] MORENO, C. J., The zeros of exponential polynomials (I), Compositio Math. 26(1973), 69-78.

[NE1] NEHARI, Z., Non-oscillation criteria for n-th order linear differential equations, Duke Math. J. 32(1965), 607-616.

[NE2] ______________, Disconjugacy criteria for linear differential equations, J. Differential Equations 4(1968), 604-611.

[NEW] NEWMAN, D. J. and RIVLIN, T. J., The zeros of partial sums of the exponential function, J. Approximation Theory 5(1972), 405-412.

[NU] NULTON, J. D. and STOLARSKY, K. B., The first sign change of a cosine polynomial, Proc. Amer. Math. Soc., to appear.

[PO1] PÓLYA, G., Sur la méthode de Graeffe, C. R. 156(1913), 1145-1147.

[PO2] ______________, Über die Nullstellen gewisser ganzer Funktionen, Math. Zeit. 2(1918), 352-383.

[PO3] ______________, Geometrisches über die Verteilung der Nullstellen gewisser ganzer transzendenter Funktionen, S. B. Bayer Akad. Wiss. (1920), 285-290.

[PO4] ______________, Bemerkung über die Mittag-Lefflerschen Funktionen $E_\alpha(z)$, Tôhoku Math. J. 19(1921), 241-248.

[PO5] ______________, Graeffe's method for eigenvalues, Numer. Math. 11(1968), 315-319.

[PON] PONTRYAGIN, L. S., On the zeros of some elementary transcendental functions, Izv. Akad. Nauk SSSR, Ser. Mat. 6(1942), 115-134 (Russian); Amer. Math. Soc. Translations 1(Ser. 2), Providence (1955), 95-110.

[POO1] VAN DER POORTEN, A. J., On the number of zeros of functions, L'Enseigenement Math. 23(1977), 19-38.

[POO2] ______________, A note on the zeros of exponential polynomials, Comp. Math. 31(1975), 109-113.

[SA1] SAFF, E. B. and VARGA, R. S., Zero-free parabolic regions for sequences of polynomials, SIAM J. Math. Anal. 7(1976), 344-357.

[SA2] ______________, The behavior of the Padé table for the exponential, in Approximation Theory, Vol. II (G. G. Lorentz, C. K. Chui, L. L. Shumaker, eds.), New York, Academic Press, 1976, pp. 195-213.

[SA3] ______________, On the zeros and poles of Padé approxiamnts to e^z, III, Numer. Math. 30(1978), 241-266.

[SCHO1] SCHOENBERG, I. J., On smoothing operations and their generating functions, Bull. Amer. Math. Soc. 59(1953), 199-230.

[SCHO2] ______________, On Pólya frequency functions I: The totally positive functions and their Laplace transforms, Journal d'Analyse Math. 1(1951), 331-374.

[SCHO3] ______________, On the zeros of the generating functions of multiply positive sequences and functions, Ann. Math. 62(1955), 447-471.

[SCHW] SCHWENGELER, E., Goemetrisches über die Verteilung der Nullstellen spezieller ganzer Funktionen, Thesis, Zürich, 1925.

[ST] STOLARSKY, K. B., Zero-free regions for exponential sums, Proc. Amer. Math. Soc., to appear.

[SZ] SZEGÖ, G., Über eine Eigenschaft der Exponentialreihe, Berlin Math. Ges. Sitzungsber 23(1924), 50-64.

[TI] TITCHMARSH, E. C., The zeros of certain integral functions, Proc. London Math. Soc. 25(1926), 283-302.

[TIJ] TIJDEMAN, R., On the number of zeros of general exponential polynomials, Proc. Nederl. Akad. Wetensch. Ser. A 74(1971), 1-7.

[TU] TURÁN, P., Eine Neue Methode in der Analysis und Deren Anwendungen, Akadémiai Kiadó, Budapest, 1953.

[VO1] VOORHOEVE, M., On the oscillation of exponential polynomials, Math. Zeit. 151(1976), 277-294.

[VO2] ______________, Angular variation and the zeros of certain functions, preprint.

[WA] WATSON, G. N., A treatise on the Theory of Bessel Functions, 2nd. ed., Cambridge Univ. Press, London, 1944.