

## STABILITY IMPLICATIONS ON THE ASYMPTOTIC BEHAVIOR OF NONLINEAR SYSTEMS

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**ABSTRACT.** In this paper we generalize Boudns' Theorems (1) to the systems  $\frac{dY(t)}{dt} = A(t) Y(t)$  and  $\frac{dX(t)}{dt} = A(t) X(t) + F(t, X(t))$ . Moreover, we also show that there always exists a solution  $X(t)$  of  $\frac{dX}{dt} = A(t)X + B(t)$  for which  $\limsup_{t \rightarrow \infty} \|X(t)\| > 0 (= \infty)$  if there exists a solution  $Y(t)$  for which  $\limsup_{t \rightarrow \infty} \|Y(t)\| > 0 (= \infty)$ .

**KEY WORDS AND PHRASES.** *stable, norm, linear systems, null solution, Schauder-Tychonoff Theorem, uniformly converges, equicontinuous.*

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### 1. INTRODUCTION.

In this paper we shall study the stability behavior of the following systems

$$\frac{dY(t)}{dt} = A(t)Y(t), \quad 0 \leq t < \infty \quad (1.1)$$

and

$$\frac{dX(t)}{dt} = A(t)X(t) + F(t, X(t)), \quad 0 \leq t < \infty \quad (1.2)$$

where  $A(t)$  is a continuous matrix on  $\mathbb{R}^n$  for all  $0 \leq t < \infty$ ,  $F(t, X(t))$  is a real valued continuous  $n$ -vector defined on  $[0, \infty) \times \mathbb{R}^n$  and  $X(t)$  and  $Y(t)$  are  $n$ -vectors.

Consider special equations of (1.1) and (1.2)

$$y'' + a(t)y = 0, \quad 0 \leq t < \infty \quad (1.3)$$

and

$$x'' + a(t)x = g(t, x, x'), \quad 0 \leq t < \infty \quad (1.4)$$

where  $a(t) \in C[0, \infty)$  and  $g(t, x, x')$  is continuous on  $[0, \infty) \times R \times R$ . From some theorems of stability theory, Bownds [1] showed that (1.3) has a solution  $y(t)$  with property

$$\limsup_{t \rightarrow \infty} (|y(t)| + |y'(t)|) > 0 \quad (1.5)$$

He also established that (1.4) has the property (1.5) provided that the zero solution of (1.3) is stable and there exists a function  $\gamma(t) \in L[0, \infty)$  such that

$$|g(t, x, x')| \leq \gamma(t) (|x| + |x'|)$$

for  $(t, x, x') \in [0, \infty) \times R \times R$ .

Thus in the following section we shall extend the above results to systems (1.1) and (1.2). In section 3 we shall consider a nonhomogeneous system

$$\frac{dX(t)}{dt} = A(t)X(t) + B(t), \quad 0 \leq t < \infty \quad (1.6)$$

where  $B(t)$  is a continuous vector for  $0 \leq t < \infty$ . We shall prove that there always exists a solution  $X(t)$  of (1.6) for which  $\limsup_{t \rightarrow \infty} \|X(t)\| > 0 (= \infty)$ , if there exists a solution  $Y(t)$  of (1.1) for which  $\limsup_{t \rightarrow \infty} \|Y(t)\| > 0 (= \infty)$ . Here  $\|\cdot\|$  is an appropriate vector (or matrix) norm.

## 2. ASYMPTOTIC BEHAVIOR FOR (1.1) AND (1.2).

Before stating main theorems, let us recall a theorem from Coppel [2, p. 60].

**THEOREM 2.1.** (Hartman [2, p. 60]). Suppose that, for every solution  $Y(t)$  of (1.1), the limit

$$\lim_{t \rightarrow \infty} \|Y(t)\| \quad (2.1)$$

exists and is finite. If there exists a nontrivial solution  $Y(t)$  of (1.1) for which the limit (2.1) is zero, then

$$\int_{t_0}^t t_r A(s) ds \rightarrow -\infty \quad \text{as } t \rightarrow \infty.$$

From the above theorem we will obtain the following corollary which is a generalization of Theorem 1 in [1].

**COROLLARY 2.1.** Suppose that

$$\int_{t_0}^{\infty} t_r A(s) ds < \infty.$$

Then there exists a nontrivial solution  $Y(t)$  of (1.1) for which

$$\limsup_{t \rightarrow \infty} \|Y(t)\| > 0.$$

PROOF. Suppose, to the contrary, that all solutions  $Y(t)$  of (1.1) satisfy

$$\limsup_{t \rightarrow \infty} \|Y(t)\| = 0.$$

From Theorem 2.1 we obtain  $\int_{t_0}^t \tau_r A(s) ds \rightarrow -\infty$  as  $t \rightarrow \infty$ . This leads to a contra-

dition. The corollary then follows.

Throughout this paper we shall denote  $\Phi(t)$ , the fundamental matrix of (1.1) with initial condition  $\Phi(0) = I$  (identity matrix).

Now we shall prove the following theorem via the Schauder-Tychonoff Theorem [2, p. 9].

THEOREM 2.2. Suppose that the null solution of (1.1) is stable and that there exists a solution  $Y(t)$  of (1.1) for which

$$\limsup_{t \rightarrow \infty} \|Y(t)\| > 0. \quad (2.2)$$

Suppose also that there exists  $\gamma(t) \in L_1[t_0, \infty)$  such that for some positive constant  $\ell$

$$\|F(t, x)\| \leq \gamma(t) \|x\|^\ell. \quad (2.3)$$

Then there exists a nontrivial solution  $X(t)$  of (1.2) for which

$$\limsup_{t \rightarrow \infty} \|X(t)\| > 0.$$

PROOF. Since the null solution of (1.1) is stable, there exists a positive constant  $k$  such that

$$\|\Phi(t) \Phi^{-1}(s)\| \leq k \quad (2.4)$$

for all  $0 \leq t \leq s$  and there exists a nontrivial solution  $Y(t)$  of (1.1) for which (2.2) holds and

$$\|Y(t)\| \leq 1 - \varepsilon \quad (2.5)$$

for  $t \geq t_0$  and for given small positive constant  $\varepsilon (< 1)$ .

Since  $\gamma(t) \in L_1[t_0, \infty)$ , there exists  $T_0 (> t_0)$  such that

$$k \int_t^\infty \gamma(s) ds < \varepsilon \quad \text{for all } t \geq T_0. \quad (2.6)$$

Via the Schauder-Tychonoff Theorem we shall establish the existence of a solution

of the integral equation

$$X(t) = Y(t) - \Phi(t) \int_t^{\infty} \Phi^{-1}(s) f(s, X(s)) ds, \quad t \geq T_0. \quad (2.7)$$

Consider the set

$F = \{U; U(t) = X(t) \text{ is continuous on } J_0 = [T_0, \infty) \text{ and}$

$$\|U(t)\| \leq 1 \text{ for } t \geq T_0\}$$

and define the operator  $T$  by

$$TU(t) = Y(t) - \int_t^{\infty} \Phi(t) \Phi^{-1}(s) f(s, U(s)) ds. \quad (2.8)$$

First, we shall show that  $TF \subset F$ . Taking the norm to both sides of (2.8) and using (2.3), (2.4), (2.5), and (2.6), we obtain for  $U \in F$

$$\begin{aligned} \|TU(t)\| &\leq \|Y(t)\| + \int_t^{\infty} \|\Phi(t) \Phi^{-1}(s) f(s, U(s))\| ds \\ &\leq 1 - \varepsilon + k \int_t^{\infty} \|f(s, U(s))\| ds \\ &\leq 1 - \varepsilon + k \int_t^{\infty} \gamma(s) \|U(s)\|^{\ell} ds \\ &\leq 1 - \varepsilon + k \int_t^{\infty} \gamma(s) ds \\ &\leq 1 - \varepsilon + \varepsilon = 1. \end{aligned}$$

It is clear that  $TU(t)$  is continuous on  $J_0$ . This proves  $TF \subset F$ .

Second, we shall show that  $t$  is continuous. Suppose that the sequence  $\{U_n\}$  in  $F$  converges uniformly to  $U$  in  $F$  on every compact subinterval of  $J_0$ . We claim that  $TU_n$  converges uniformly to  $TU$  on every compact subinterval of  $J_0$ . Let  $\varepsilon_1$  be a small positive number satisfying  $\varepsilon_1 < 1$ . Since  $\gamma(t) \in L_1[t_0, \infty)$ , there exists

$$T_1 > T_0 \text{ so that for } t \geq T_1 \quad k \int_t^{\infty} \gamma(s) ds < \frac{\varepsilon_1}{4}. \quad (2.9)$$

By the uniform convergence, there is an  $N = N(\varepsilon_1, T_1)$  such that if  $n \geq N$ , then

$$\|f(s, U_n(s)) - f(s, U(s))\| < \frac{\varepsilon_1}{2kT_1}, \quad T_0 \leq s \leq T_1. \quad (2.10)$$

Then using (2.8), (2.9), (2.10), (2.3), (2.4), and the fact that  $\|U_n(t)\| \leq 1$  and  $\|U(t)\| \leq 1$  for  $T_0 \leq t < \infty$ , we obtain the following inequalities

$$\begin{aligned}
\|TU_n(t) - TU(t)\| &= \left\| \int_t^\infty \Phi(t)\Phi^{-1}(s)f(s, U_n(s))ds - \int_t^\infty \Phi(t)\Phi^{-1}(s)f(s, U(s))ds \right\| \\
&\leq \int_t^{T_1} \|\Phi(t)\Phi^{-1}(s)\| \|f(s, U_n(s)) - f(s, U(s))\| ds \\
&+ \int_{T_1}^\infty \|\Phi(t)\Phi^{-1}(s)\| \|f(s, U_n(s))\| ds + \int_{T_1}^\infty \|\Phi(t)\Phi^{-1}(s)\| \|f(s, U(s))\| ds \\
&\leq k \int_t^{T_1} \|f(s, U_n(s)) - f(s, U(s))\| ds + 2k \int_{T_1}^\infty \gamma(s) ds \\
&< \frac{\epsilon_1}{2} + \frac{\epsilon_1}{2} = \epsilon_1 \quad \text{for } n \geq N.
\end{aligned}$$

This shows that  $TU_n$  converges uniformly to  $TU$  on every compact subinterval of  $J_0$ . Hence  $T$  is continuous.

Third, we claim that the functions in the image set  $TF$  are equicontinuous and bounded at every point of  $J_0$ . Since  $TF \subset F$ , it is clear that the functions in  $TF$  are uniformly bounded. Now we show that they are equicontinuous at each point of  $J_0$ . For each  $U \in F$ , the function  $z(t) = TU(t)$  is a solution of the linear system

$$\frac{dz}{dt} = A(t)z + f(t, U(t))$$

Since  $\|z(t)\| = \|TU(t)\| \leq 1$  and  $\|f(t, U(t))\|$  is uniformly bounded for  $U \in F$  on any finite  $t$  interval, we see that  $\frac{dz}{dt}$  is uniformly bounded on any finite interval. Therefore, the set of all such  $z$  is equicontinuous at each point of  $J_0$  (see [2, p.6]).

All of the hypotheses of the Schauder-Tychonoff Theorem are satisfied. Thus there exists a  $U \in F$  such that  $U(t) = TU(t)$ ; that is, there exists a solution  $X(t)$  of

$$X(t) = Y(t) - \Phi(t) \int_t^\infty \Phi^{-1}(s) f(s, X(s)) ds$$

Thus, from the hypotheses and the above equality, we obtain .

$$\limsup_{t \rightarrow \infty} \|X(t) - Y(t)\| = 0 \quad (2.11)$$

Since  $\limsup_{t \rightarrow \infty} \|Y(t)\| > 0$ , (2.11) implies that  $\limsup_{t \rightarrow \infty} \|X(t)\| > 0$ . This proves the theorem.

It is clear that (1.4) can be written as the form (1.2) with

$$A(t) = \begin{pmatrix} 0 & 1 \\ -a(t) & 0 \end{pmatrix} \quad \text{and} \quad F(t, X) = \begin{pmatrix} 0 \\ g(t, x, x') \end{pmatrix}$$

where  $X = \text{colum}(x, x')$ . Thus we can apply Theorem 2.2 to (1.4) to obtain the following corollary which is a generalization of Theorem 2 in [1].

COROLLARY 2.2. Suppose that the null solution of (1.3) is stable and that there exists  $\gamma(t) \in L_1[t_0, \infty)$  such that for some positive constant  $\ell$

$$||g(t, x, x')|| \leq \gamma(t) (|x| + |x'|)^\ell .$$

Then there exists a nontrivial solution  $x(t)$  of (1.4) for which

$$\limsup_{t \rightarrow \infty} (|x| + |x'|) > 0 .$$

PROOF. Since  $t_r A(t) = 0$  for Corollary 2.1, we know that there exists a solution  $Y(t)$  of (1.1) for which

$$\limsup_{t \rightarrow \infty} ||Y(t)|| > 0 .$$

If we take  $||X|| = |x| + |x'|$ , then the corollary follows from Theorem 2.2.

### 3. ASYMPTOTIC BEHAVIOR FOR (1.6).

In this section we shall show that if there exists a solution  $Y(t)$  of (1.1) for which  $\limsup_{t \rightarrow \infty} ||Y(t)|| > 0 (= \infty)$ , then there exists a solution  $X(t)$  of (1.6) for which  $\limsup_{t \rightarrow \infty} ||X(t)|| > 0 (= \infty)$ .

THEOREM 3.1. Suppose that there exists a solution  $Y(t)$  of (1.1) for which

$$\limsup_{t \rightarrow \infty} ||Y(t)|| > 0. \tag{3.1}$$

Then there exists a solution  $X(t)$  of (1.6) for which

$$\limsup_{t \rightarrow \infty} ||X(t)|| > 0.$$

PROOF. From the variation of constants formula we know that any solution  $X(t)$  of (1.6) can be written as the form below

$$X(t) = \Phi(t)c + \Phi(t) \int_0^t \Phi^{-1}(s) B(s) ds \tag{3.3}$$

Hence we shall choose  $c$  so that  $Y(t) = \Phi(t)c$  satisfies (3.1).

First, let us suppose

$$\limsup_{t \rightarrow \infty} ||\Phi(t) \int_0^t \Phi^{-1}(s) B(s) ds|| > 0. \tag{3.4}$$

Let  $X_1(t) = X(t) - Y(t)$ . It is clear that  $X_1(t)$  is a solution of (1.6). Thus from (3.3) and (3.4) we obtain

$$\begin{aligned} \limsup_{t \rightarrow \infty} ||X_1(t)|| &= \limsup_{t \rightarrow \infty} ||X(t) - Y(t)|| \\ &= \limsup_{t \rightarrow \infty} ||\phi(t) \int_0^t \psi^{-1}(s) B(s) ds|| > 0 . \end{aligned}$$

Thus there exists a solution  $X_1(t)$  of (1.6) for which (3.2) holds.

Second, suppose that

$$\lim_{t \rightarrow \infty} ||\phi(t) \int_0^t \psi^{-1}(s) B(s) ds|| = 0 . \quad (3.5)$$

Taking the norm to both sides of (3.3) and using (3.1) and (3.5) we obtain

$$\begin{aligned} \limsup_{t \rightarrow \infty} ||X(t)|| &\geq \limsup_{t \rightarrow \infty} (||Y(t)|| - ||\phi(t) \int_0^t \psi^{-1}(s) B(s) ds||) \\ &\geq \limsup_{t \rightarrow \infty} ||Y(t)|| - \limsup_{t \rightarrow \infty} ||\phi(t) \int_0^t \psi^{-1}(s) B(s) ds|| \\ &\geq \limsup_{t \rightarrow \infty} ||Y(t)|| > 0 . \end{aligned}$$

This shows that  $X(t)$  satisfies (3.2). The theorem then follows.

Using the same argument as Theorem 3.1 we also can obtain the following theorem.

**THEOREM 3.2.** Suppose that there exists a solution  $Y(t)$  of (1.1) for which

$$\limsup_{t \rightarrow \infty} ||Y(t)|| = \infty .$$

Then there exists a solution  $X(t)$  of (1.6) for which

$$\limsup_{t \rightarrow \infty} ||X(t)|| = \infty .$$

**PROOF.** Since the proof is almost the same as Theorem 3.1, we shall omit the detail.

**REMARKS.** It is interesting to note that Hatvani and Pintér [3] have studied this type of problem for equation (1.4).

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