A NOTE ON ALMOST CONTINUOUS MAPPINGS
AND BAIRE SPACES

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ABSTRACT. We prove the following theorem:

THEOREM. Let Y be a second countable, infinite $R_0$-space. If there are
countably many open sets $0_1, 0_2, \ldots, 0_n, \ldots$ in Y such that $0_1 \not\subset 0_2 \not\subset \ldots \not\subset 0_n \cdots$,
then a topological space X is a Baire space if and only if every mapping $f: X \to Y$
is almost continuous on a dense subset of X. It is an improvement of a theorem due
to Lin and Lin [2].

KEY WORDS AND PHRASES. Separation axiom $R_0$, almost continuous mapping, Baire space.

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54D10.

1. INTRODUCTION.

This note is directed to mathematical specialists or non-specialists familiar
with general topology [1].

Lin and Lin [2] proved the following theorem:

THEOREM 1. Let Y be an arbitrary infinite Hausdorff space. If X is a topo-
logical space such that every mapping $f: X \to Y$ is almost continuous on a dense
subset $D(f)$ of X, then X is a Baire space.

In the theorem above, the almost continuity is in the sense of Husain [3].
The proof of the theorem depends on the following lemma (cf. Long [1, Prob. 14,
p. 147]):

LEMMA 1. Every infinite Hausdorff space contains a countably infinite discrete
subspace.
In this note, we prove a lemma similar to Lemma 1 under weaker conditions, and use it to improve Theorem 1.

2. PRELIMINARIES AND RESULTS.

Before stating the result, we first recall the definition of the separation axiom $R_0$ (cf. [4], [5], [6, p. 49]).

**DEFINITION 1.** A topological space $X$ is $R_0$ if and only if for each $x \in X$ and open subset $U$, $x \in U$ implies $\overline{\{x\}} \in U$.

It is known [1] that $R_0$ is weaker than $T_1$ and is independent of $T_0$, in fact $T_1 = T_0 + R_0$. A Hausdorff space is $R_0$.

**LEMMA 2.** If an infinite space $X$ is $R_0$, and there are countably infinite open sets $0_1, 0_2, \ldots, 0_n, \ldots$ such that $0_1 \not\subseteq 0_2 \not\subseteq \ldots \not\subseteq 0_n \not\subseteq \ldots$, then there is a countably infinite distinct set $S = \{y_1, y_2, \ldots, y_n, \ldots\}$ in $X$ such that for each $n$, there is an open set $V_n$ satisfying $V_n \cap S = \{y_n\}$.

**PROOF.** Without loss of generality we may assume that $0_1$ is not empty. Let $y_1 \in 0_1$ be an arbitrary point. Since $X$ is $R_0$, $\overline{\{y_1\}} \subseteq 0_1$. Let $V_1 = 0_1$. From $0_1 \not\subseteq 0_2$ we can find a $y_2 \in 0_2$ such that $y_2 \not\in 0_1$ and $\overline{\{y_2\}} \subseteq 0_2$. Let $V_2 = 0_2 \cap (0_1 \setminus \{y_1\})$. Then $V_2$ is an open set and $y_2 \in V_2$. If $y_{n-1}$ is chosen and $V_{n-1} = 0_{n-1} \cap (0_{n-2} \setminus \{y_{n-1}\})$ is defined, then since $0_{n-1} \not\subseteq 0_n$, we may choose $y_n \in 0_n$ such that $y_n \not\in 0_{n-1}$ and $\overline{\{y_n\}} \subseteq 0_n$. Let $V_n = 0_n \cap (0_{n-1} \setminus \{y_{n-1}\})$. Then $y_n \in V_n$. Thus we have a countably infinite distinct set $S = \{y_1, y_2, \ldots, y_n, \ldots\}$ and countably infinite distinct open sets $V_1, V_2, \ldots, V_n, \ldots$ such that $y_n \in V_n$ (n = 1, 2, ...). Since $V_n = 0_n \cap (0_{n-1} \setminus \{y_{n-1}\})$, we have $y_i \not\in V_n$ for $i = 1, 2, \ldots, n-1$. Since $y_{n+m} \in 0_{n+m}$ (m > 1), $y_{n+m} \not\in 0_{n+m-1}$, but $0_n \not\subseteq 0_{n+m-1}$, hence $y_{n+m} \not\in 0_n$, $y_{n+m} \not\in V_n$. Therefore, $V_n \cap S = \{y_n\}$.

For convenience we say that a space $X$ has an ascending chain of open sets if there are countably infinite open sets $0_1, 0_2, \ldots, 0_n, \ldots$ such that $0_1 \not\subseteq 0_2 \not\subseteq \ldots \not\subseteq 0_n \not\subseteq \ldots$.

**LEMMA 3.** An infinite Hausdorff space $X$ is an $R_0$-space with an ascending chain of open sets.

**PROOF.** We need only to show that $X$ has an ascending chain of open sets. By Lemma 1, there is a countably infinite discrete subspace $\{y_1, y_2, \ldots, y_n, \ldots\}$, hence
there are disjoint open sets $U_1, U_2, \ldots, U_n, \ldots$ such that $y_n \in U_n$. Let $0_n = \bigcup_{i=1}^{n} U_i$ ($n = 1, 2, \ldots$). Then $0_1, 0_2, \ldots, 0_n, \ldots$ is an ascending chain of open sets.

The converse of Lemma 3 is not true.

**Example 1.** Let $X = [0,1]$ with topology $\tau = \{X\setminus N; N$ is a countable set}. Then $X$ is $R_0$ and $0 = \bigcap_{i=1}^{1, 2, \ldots} \bigcup_{i=1}^{1, 2, \ldots} (i = 1, 2, \ldots)$ is an ascending chain of open sets. $X$ is not Hausdorff.

Now Theorem 1 can be improved as

**Theorem 2.** Let $Y$ be an infinite $R_0$-space with an ascending chain of open sets. If $X$ is a topological space such that every mapping $f: X \to Y$ is almost continuous on a dense subset of $X$, then $X$ is a Baire space.

The proof is all the same as the proof of Theorem 2 in [2].

Similar to Theorem 3 in [2], we have

**Theorem 3.** Let $Y$ be a second countable infinite $R_0$-space with an ascending chain of open sets. Then a topological space $X$ is a Baire space if and only if every mapping $f: X \to Y$ is almost continuous on a dense subset of $X$.

**References**

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