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THE n-DIMENSIONAL DISTRIBUTIONAL MELLIN TRANSFORMATION

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ABSTRACT. The n-dimensional distributional Mellin transformation is developed using the testing function space $M_{c,d}$ and its dual $M'_{c,d}$. The standard theorems on analyticity, uniqueness and continuity are proved. A necessary and sufficient condition for a function to be an n-dimensional Mellin transformation is proved by the help of a boundedness property for distribution in $M'_{c,d}$. Some operational transform formulas are also introduced.

KEY WORDS AND PHRASES. *Distributional Mellin Transformation, Distributions, and Test function spaces.*

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1. INTRODUCTION.

The Mellin transformation was previously extended to certain generalized functions by Zemanian [1] and Fung Kang [2]. In the present paper, we develop the n-dimensional distributional Mellin transformation.

For the sake of brevity, we shall use the following notations. R^n and C^n are respectively real and complex n-dimensional euclidean spaces. The symbols z and s stand for elements of C^n representing the n-triples $(z_1,z_2,\dots,z_n)$ and $(s_1,s_2,\dots,s_n)$ respectively. We take $x \in R^n$, $t \in R^n$, $\sigma \in R^n$, $\omega \in R^n$ and $s = \sigma + i\omega \in C^n$. A function on a subset of R^n shall be denoted by $h(x) = h(x_1,x_2,\dots,x_n)$. By $[x]$ we mean the product $x_1,x_2,\dots,x_n$. Thus, $[x^s] = x_1^{s_1},x_2^{s_2},\dots,x_n^{s_n}$ where $s = \{s_1,s_2,\dots,s_n\}$ and $[e^{-st}] = \exp(-s_1t_1-\dots-s_nt_n)$. By log x we mean

$\{\log x_1,\ldots,\log x_n\}$ and, by xt, we mean $\{x_1t_1, x_2t_2,\ldots,x_nt_n\}$. Also, $x^s = \{x_1^{s_1},\ldots,x_n^{s_n}\}$ and $e^{-st} = \{e^{-s_1t_1},\ldots,e^{-s_nt_n}\}$. The notation $x \le y$ and $x < y$ mean respectively $x_\nu \le y_\nu$ and $x_\nu < y_\nu$ ($\nu = 1,2,\ldots,n$). The letters k and m shall denote non-negative integers in R^n, i.e., k_ν and m_ν are non-negative integers. Letting $k = k_1 + k_2 + \ldots + k_n$, D_x^k shall denote $\dfrac{\sigma^k}{\sigma x_1^{k_1} \sigma x_2^{k_2} \ldots \sigma x_n^{k_n}}$.

By a smooth function we mean a function that possesses partial derivatives of all orders at all points of its domain.

2. THE TESTING FUNCTION SPACE $M_{c,d}$.

Let R_+^n denote the open domain $0 < x < \infty$. We define $\eta_{c,d}(x)$ as the product function $\prod_{\nu=1}^{n} \eta_{c_\nu,d_\nu}(x_\nu)$

$$\text{where } \eta_{c_\nu,d_\nu}(x_\nu) = \begin{cases} x_\nu^{-c_\nu} & \text{if } 0 < x_\nu < 1/e \\ x_\nu^{-d_\nu} & \text{if } e < x_\nu < \infty \end{cases} .$$

In fact, $M_{c,d}$ is the linear space of all smooth functions $f(x)$ defined on R_+^n with values in C^1, which satisfy the following set of inequalities.

For each non-negative integer k,

$$\left|\eta_{c,d}(x)\ [x^{k+1}]\ D_x^k\ f(x)\right| \le Q_k\ , \quad 0 < x < \infty. \tag{2.1}$$

Q_k denotes constants which depend upon the choices of k and f.

Any smooth function, whose support is contained in R_+^n, is in $M_{c,d}$. Other members of $M_{c,d}$ are $[x^{s-1}]$ for $c \le \text{Re } s \le d$ and $[(\log x)^k\ x^{s-1}]$ for $c < \text{Re } s < d$.

μ_ν represents a seminorm defined by

$$\mu_\nu = \mu_\nu(f) = \max_{0\le|k|\le\nu}\ \sup_x \left|\eta_{c,d}(x)\ [x^{k-1}]\ D_x^k\ f(x)\right|. \tag{2.2}$$

Of course, the collection $\{\mu_\nu\}$ is a multinorm, being a separating collection of seminorms. Thus we can assign to $M_{c,d}$ the topology generated by $\{\mu_\nu\}$.

A sequence $\{f_\nu\}_{\nu=1}^\infty$ is a Cauchy sequence in $M_{c,d}$ if and only if each $f_\nu \in M_{c,d}$ and, for each fixed k, the functions $\eta_{c,d}(x)\ [x^{k+1}]\ D_x^k\ f_\nu(x)$ converges uniformly on R_+^n as $\nu \to \infty$. Hence, $M_{c,d}$ is sequentially complete.

THEOREM 2.1. The mapping

$$f(x) \to [e^{-p}]\ f(e^{-p}) = g(p) \tag{2.3}$$

is an isomorphism from $M_{c,d}$ into $L_{c,d}$ where $L_{c,d}$ denotes the testing function space defined by Sinha [3].

The inverse mapping is given by

$$g(p) \to [x^{-1}]\ f(-\log x) = f(x) \tag{2.4}$$

PROOF. The proof of this theorem is easy and is therefore omitted.

3. THE DUAL SPACE $M'_{c,d}$.

$M'_{c,d}$ is the dual space of $M_{c,d}$. Multiplication by a complex number, equality, and addition are defined in the usual way. In fact, $M'_{c,d}$ is a linear space over C^1. By $\langle h,f\rangle$ we mean a number that $h \in M'_{c,d}$ assigns to $f \in M_{c,d}$. If the support (Miller [4], §1.6) of a distribution h is contained in a compact subset of R^n_+, then $h \in M'_{c,d}$, $c,d \in R^n$ with $c < d$. Also, every member of $M'_{c,d}$ is a distribution on R^n_+.

Let us define a (weak) topology for $M'_{c,d}$ by using the following separating set of seminorms. For every $f \in M_{c,d}$, we define a seminorm $\zeta_f(h)$ on $M'_{c,d}$ by

$$\zeta_f(h) = |\langle h,f\rangle|, \quad (h \in M'_{c,d}).$$

In fact, a sequence $\{h_\nu\}_{\nu=1}^\infty (h \in M'_{c,d})$ is a Cauchy sequence in $M'_{c,d}$ if and only if, for all $f \in M_{c,d}$, the numerical sequence $\{\langle h_\nu, f\rangle\}_{\nu=1}^\infty$ converges.

We can easily prove that $M'_{c,d}$ is sequentially complete.

In view of Theorem 1, we can relate to each $h(x) \in M'_{c,d}$ a distribution $h(e^{-p}) \in L'_{c,d}$ (see [3]) by

$$\langle h(e^{-p}),\ g(p)\rangle = \langle h(x),\ f(x)\rangle. \tag{3.1}$$

Conversly, if $\psi(p) \in L'_{c,d}$, then $\psi(-\log x) \in M'_{c,d}$ is given by

$$\langle \psi(-\log x),\ f(x)\rangle = \langle \psi(p),\ g(p)\rangle. \tag{3.2}$$

Using (3.1) and (3.2), we can easily have the following theorem:

THEOREM 3.1. The mapping $h(x) \to h(e^{-p})$ defined by (3.1), is an isomorphism from $M'_{c,d}$ onto $L'_{c,d}$. The inverse mapping is given by (3.2).

4. THE n-DIMENSIONAL DISTRIBUTIONAL MELLIN TRANSFORMATION M.

DEFINITION. We define the n-dimensional distributional Mellin transformation Mh as the function H(s) on Ω_h into C^1 by

$$(Mh)(s) = H(s) = \langle h(x),\ [x^{s-1}]\rangle \quad \text{for } s \in \Omega_h, \tag{4.1}$$

where Ω_h is the tube of definition of the n-dimensional distributional Laplace transformation (see [3]).

In fact, the R.H.S. of (4.1) has a meaning because the application of $h \in M'_{c,d}$ to $[x^{s-1}] \in M_{c,d}$.

Setting $g(p) = [e^{-sp}]$ and $f(x) = [x^{-1}]\ g(-\log x) = [x^{s-1}]$ and using Theorem 2.1, we can have the following theorem:

THEOREM 4.1. The distribution h(x) is n-dimensional Mellin transformable if $h(e^{-p})$ is n-dimensional Laplace transformable. In such a case, $Mh(x) = H(x) = Lh(e^{-p})$ for ever $s \in \Omega_h$.

Using Theorems 2.1 and 3.1, we can have the following theorems:

THEOREM 4.2. (The Analiticity Theorem). If $Mh = H(s)$ for $s \in \Omega_h$, then $H(s)$ is analytic on Ω_h and

$$\frac{\partial H}{\partial s_\nu} = \langle(\log x_\nu)\ h(p),\ [x^s]\rangle, \quad s \in \Omega_h. \tag{4.2}$$

The proof is analogous to that given in [3].

THEOREM 4.3. (The Uniqueness Theorem). If $Mh = H(s)$ for $s \in \Omega_h$ and $Mg = G(s)$ for $s \in \Omega_g$, if $\Omega_h \cap \Omega_g$ is non-void, and if $H(s) = G(s)$ for $s \in \Omega_h \cap \Omega_g$, then $h=g$ in the same sense of equality in $M'_{c,d}$ where $c,d \in \Omega_h$ and $c < d$. The proof is analogous to that given in [3].

THEOREM 4.4. (The Continuity Theorem). If $\{h_\nu\}_{\nu=1}^{\infty}$ converges in $M'_{c,d}$ to h for some $c,d \in R^n_+$ $(c < d)$ and if $Mh_\nu = H_\nu(s)$, then $Lh = H(s)$ exists for at least $c < \mathrm{Re}\ s < d$ and $\{H_\nu(s)\}_{\nu=1}^{\infty}$ converges pointwise in the tube of definition $c < \mathrm{Re}\ s < d$ to $H(s)$.

PROOF. Since $[x^s]$ is in $M_{c,d}$ for each s satisfying $c < \mathrm{Re}\ s < d$, the theorem follows from the definition of convergence in $M'_{c,d}$ and the fact that $M'_{c,d}$ is sequentially complete.

5. A BOUNDEDNESS PROPERTY FOR DISTRIBUTIONS IN $M'_{c,d}$.

For each $h \in M'_{c,d}$, there exists a non-negative integer $r \in R^1_+$ and a positive constant $c \in R^1_+$ such that, for all ψ in $M_{c,d}$,

$$|\langle h,\psi\rangle| \le c\mu\ (\psi). \tag{5.1}$$

6. A NECESSARY AND SUFFICIENT CONDITION FOR M(s) TO BE AN n-DIMENSIONAL MELLIN TRANSFORM.

A necessary and sufficient condition for a function $M(s)$ to be the n-dimensional Mellin transform of a distribution h is that there be a tube $c < \text{Re } s < d$ $(c < d)$ on which $M(s)$ is analytic and bounded when

$$|M(s)| \leq P(|s|) \tag{6.1}$$

where $P(|s|)$ is a polynomial in $|s|$.

It can be easily proved by using the boundedness property of Section 5 and (Bochner [6], Theorem 60, p. 242 and §4, p. 244).

7. SOME OPERATIONAL TRANSFORM FORMULAS FOR THE n-DIMENSIONAL DISTRIBUTIONAL MELLIN TRANSFORMATION.

Let us suppose that $\mathfrak{M}h(p) = H(s)$ for $s \in \Omega_h$ and $p \in R^n_+$, $\alpha \in C^n$. We can easily have the following operational transform formulas (Using Theorem 4):

(11) $M\, D^k_p\, h(p) = s^k\, H(s)$, $s \in \Omega_h$,

(12) $M\{[x^\alpha]\}h = H(s + \alpha)$, $s + \alpha \in \Omega_h$,

(13) $\mathfrak{M}h\ (\log x) = H(-s)$, $-s \in \Omega_h$,

(14) $\mathfrak{M}h\{\tau(-\log x)\} = [\tau^{-1}]\ H(s/\tau)$, $s/\tau \in \Omega_h$, $\tau > 0$.

Also, by using Theorem 5, we can have

(15) $M\{(-\log x)^k\ h(-\log x)\} = (-)^{|k|}\ D^k_s\ H(s)$, $s \in \Omega_h$.

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