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RESEARCH NOTES

DUAL CHARACTERIZATION OF THE DIEUDONNE-SCHWARTZ THEOREM ON BOUNDED SETS

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ABSTRACT. The Dieudonné-Schwartz Theorem on bounded sets in a strict inductive limit is investigated for non-strict inductive limits. Its validity is shown to be closely connected with the problem of whether the projective limit of the strong duals is a strong dual itself. A counter-example is given to show that the Dieudonné-Schwartz Theorem is not in general valid for an inductive limit of a sequence of reflexive, Fréchet spaces.

KEY WORDS AND PHRASES. Locally convex space, inductive and projective limit, barrelled space, bounded set.

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1. INTRODUCTION

This paper is written for those with at least an elementary knowledge of the theory of locally convex spaces. A good reference is the book of Schaeffer [1]. Let $E_1 \subset E_2 \subset \dots$ be a sequence of locally convex, Hausdorff, linear topological spaces such that each E_n is continuously contained in E_{n+1}, and such that the union $E = \bigcup_{m=1}^{\infty} E_m$ is Hausdorff as a locally convex inductive limit. It is obvious that any bounded subset of a space E_n is also bounded in E. If <u>each</u> bounded subset of E arises in this way, we shall say that the DSP (Dieudonné-Schwartz Property) holds. A well-known theorem of Dieudonné-Schwartz states that the DSP holds provided each E_n is closed in E_{n+1} and has the topology inherited from E_{n+1} (see [1] or [4] II.6.5).

In duality theory an increasing sequence $E_1 \subset E_2 \subset \ldots$ corresponds to a decreasing sequence $F_1 \supset F_2 \supset \ldots$ where each F_n is dual to E_n. The intersection $F = \bigcap_{m=1}^{\infty} F_m$, endowed with the projective limit topology induced from the weak topologies $\sigma(F_n,E_n)$, may be identified with the dual of E relative to the weak topology $\sigma(F,E)$ ([1] IV.4.5). In applications the strong topologies $\beta(F_n,E_n)$ are often of interest, along with the projective limit $\pi(F)$ induced by these on F. The strong topology $\beta(F,E)$ is always at least as fine as $\pi(F)$. The problem of determining when $\pi(F)$ equals $\beta(F,E)$ turns out to be closely connected with determination of the validity of the DSP. A precise statement of this is given in the theorem below.

2. THE QIU PROPERTY

Recent work by Qiu [2] suggests a slight relaxation of the DSP. Let $\mathcal{B}$ be the set of all subsets B of E such that B is bounded in some E_n. We say that the QP (Qiu Property) holds if each bounded subset of E is contained in the <u>closure</u> of some $B \in \mathcal{B}$.

For spaces V and W dual to one another, and a subset S of V, we write $\overline{S}^V$ for the $\sigma(V,W)$-closure of S in V and $S^{\circ W}$ for the polar of S in W. Thus, if $\langle S\rangle$ denotes the convex hull of S, the Bipolar Theorem ([1] IV.1.5) states that $\overline{\langle S\rangle}^V = (S^{\circ W})^{\circ V}$. We note for use below that the polars of the closed, radial, convex bounded subsets of V are just the barrels of W, and vice versa.

THEOREM. A necessary and sufficient condition for the QP to hold is that $\pi(F) = \beta(F,E)$.

PROOF. Suppose first that $\pi(F) = \beta(F,E)$, and let B be an arbitrary bounded subset of E. Then $B^{\circ F}$ is a barrel and so contains a $\beta(F,E)$-open neighborhood of 0. From $\pi(F) = \beta(F,E)$ now follows that there is a barrel A in some F_n such that $A \cap F \subset B^{\circ F}$. Letting $D = A^{\circ E_n}$, we see that D is bounded in E_n and $D^{\circ F_n} = A$. Because $D^{\circ F}$ is just $D^{\circ F_n} \cap F = A \cap F$, we have $D^{\circ F} \subset B^{\circ F}$. Consequently, $B \subset \overline{\langle B\rangle}^E = (B^{\circ F})^{\circ E} \subset (D^{\circ F})^{\circ E}$. Since the Bipolar Theorem guarantees that $(D^{\circ F})^{\circ E}$ is just the closure of D in E, we have shown that the QP holds.

Now suppose that the QP holds, and let A be an arbitrary barrel of F. Then $A^{\circ E}$ is bounded and so there exists a bounded set B of some E_n such that $A^{\circ E}$ is in the closure $\overline{B}^E$ of B in E. Since $B^{\circ F} = B^{\circ F_n} \cap F$ and $B^{\circ F_n}$ is a barrel (being the polar of a bounded set), it follows that $B^{\circ F}$ is a $\pi(F)$-neighborhood of 0. But we have

$A^{\circ E} \subset \overline{B}^E \subset \overline{\langle B\rangle}^E$ so

$$B^{\circ F} = (\overline{\langle B\rangle}^E)^{\circ F} \subset (A^{\circ E})^{\circ F} = A.$$

We have shown that $\beta(F,E) \subset \pi(F)$. The reverse inequality is evident. Q.E.D.

3. COUNTER-EXAMPLE

It was demonstrated in [3] that the DSP holds when all the E_n are reflexive Banach spaces. The following example shows that, for reflexive Fréchet spaces, even the QP may fail to hold.

For each $n \in \mathbb{N}$, let D_n be the region $\mathbb{R} \setminus \{1,2,\ldots,n\}$ and let E_n be the linear space of functions infinitely differentiable on D_n. For $n,m \in \mathbb{N}$ let $K_{n,m}$ be the compact set $\{x \in D_n\colon |x| \le m,\ |x - j| \ge \frac{1}{m}$ for all $j = 1,2,\ldots,n\}$ and, for each $f \in E_n$, let

$$||f||_{n,m} = \sup\{m|f^{(i)}(x)|\colon x \in K_{n,m},\ i = 0,1,\ldots,m\}.$$

Then each E_n, equipped with the locally convex topology generated by the family $\{||\ ||_{n,m}\colon m = 0,1,\ldots\}$, is a nuclear Fréchet space ([4] III.8.3). Hence each E_n is a Montel space ([4] III.7.2, Corollary 2) and thus reflexive. We proceed to show that $E = \bigcup_{m=1}^{\infty} E_n$ does not have the QP.

For each $n \in \mathbb{N}$, and $x \in \mathbb{R}$, let $f_n(x) = (x - n)^{n-\frac{1}{2}} e^{-(x-n)^2}$ and let $c_n = \sup\{|f_n^{(i)}\colon x \in D_n\setminus[n - 1, n + 1],\ i = 0,1,\ldots,n - 1\}$. Clearly, each f_n is in E_n. Let V be any neighborhood of 0 in E. Then, for some $m \in \mathbb{N}$, the $||\ ||_{1,m}$-unit ball W of E_1 is contained by V. Evidently $\frac{1}{nc_n} f_n$ is in W for $n = m + 1, m + 2, \ldots$. Consequently there exists $k > 0$ such that $h_n = \frac{1}{nc_n} f_n \in kV$ for all $n \in \mathbb{N}$—that is, then set $B = \{h_n\colon n \in \mathbb{N}\}$ is bounded in E.

Let D be a bounded subset of one of the spaces E_n. Then the number

$$M = \sup\{|h^{(n+1)}(x)|\colon x \in [n + \tfrac{1}{2},\ n + \tfrac{3}{2}],\ h \in D\} \tag{3.1}$$

is finite. Let p be the polynomial (with non-vanishing constant term) such that

$$h_{n+1}^{(n+1)}(x) = (x - n - 1)^{-\frac{1}{2}} p(x)\, e^{-(x-n-1)^2} \text{ for all } x \in D_{n+1}.$$

Evidently there exists some $r > 2$ such that

$$\inf\{|h_{n+1}^{(n+1)}(x)|\colon\ x \in [n + 1 - \tfrac{1}{r},\ n + 1 - \tfrac{1}{2r}]\} \ge M + 2. \tag{3.2}$$

Let K be any integer larger than $\frac{1}{2r}$ and n + 1. For each $m \in \mathbb{N}$, let S_m be the $\| \|_{m,k}$-unit ball of E_n. Then, for each $g \in S_m$, we have

$$\sup\{|g^{(n+1)}(x)|: x \in [n + 1 - \frac{1}{k},\ n + 1 - \frac{1}{2k}]\} \leq 1. \tag{3.3}$$

Evidently this last inequality also holds for all g in the convex hull H of the union $\bigcup_{m=1}^{\infty} S$. From (3.2) and (3.3) follows that the set $h_{n+1} + H$ is a neighborhood of h_{n+1} in E such that, for each $g \in h_{n+1} + H$,

$$\inf\{|g^{(n+1)}(x)|: x \in [n + 1 - \frac{1}{k},\ n + 1 - \frac{1}{2k}]\} \geq M + 1. \tag{3.4}$$

Hence, (3.1) and (3.4) imply that $D \cap (h_{n+1} + H) = \phi$. Thus the QP does not hold.

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