THE COMBINATIONAL STRUCTURE OF NON-HOMOGENEOUS MARKOV CHAINS WITH COUNTABLE STATES

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ABSTRACT. Let $P(s,t)$ denote a non-homogeneous continuous parameter Markov chain with countable state space E and parameter space $[a,b]$, $-\infty < a < b < \infty$. Let $R(s,t) = \{(i,j) : P_{ij}(s,t) > 0\}$. It is shown in this paper that $R(s,t)$ is reflexive, transitive, and independent of (s,t) , $s < t$, if a certain weak homogeneity condition holds. It is also shown that the relation $R(s,t)$, unlike in the finite state space case, cannot be expressed even as an infinite (countable) product of reflexive transitive relations for certain non-homogeneous chains in the case when E is infinite. $\square$

KEYWORDS AND PHRASES. Non-homogeneous Markov chains, reflexive and transitive relations, homogeneity condition.

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1. INTRODUCTION AND STATEMENTS OF RESULTS

Throughout this paper, $P(s,t)$ will denote a non-homogeneous continuous parameter Markov chain with countable state space E and parameter space $[a,b]$, $-\infty < a < b < \infty$, such that P is a function from the domain space

$$D = \{(s,t) : a \leq s \leq t \leq b\}$$

into S , the set of countable stochastic matrices with state space E such that the following conditions hold:

- (i) $P_{ij}(s,t)$ is separately continuous in s and in t ;
- (ii) $P(s,t) = P(s,u)P(u,t)$ if $a \leq s \leq u \leq t \leq b$;
- (iii) $P_{ij}(t,t) = \delta_{ij}$.

One important result for homogeneous Markov chains (i.e. when $P(s,t)$ above is a function of $t-s$ alone) is the classical Austin-Ornstein result, namely that if $\bar{P}(u) = P(s, s+u)$, then for $i, j \in E$,

$$\bar{P}_{ij}(u) > 0 \text{ for some } u \Rightarrow \bar{P}_{ij}(u) > 0 \forall u.$$

This means that the relation

$$R = \{(i,j): \bar{P}_{ij}(u) > 0\} \quad (1.1)$$

is independent of u ; it also follows that R is reflexive and transitive.

Conversely, given any reflexive and transitive relation R on E , there exists a standard homogeneous Markov chain $\bar{P}(t)$ on E satisfying (1.1). Thus, it is natural to ask what an analogous result for non-homogeneous Markov chains should be. Kingman and Williams (see Theorem 3, [2]) have shown when E is finite that the relation $R(s,t)$ defined by

$$R(s,t) = \{(i,j): P_{ij}(s,t) > 0\} \quad (1.2)$$

can be expressed as a finite product of reflexive and transitive relations on E . It was also mentioned in [2] that "Our main result is Theorem 3, ... The methods depend heavily on the finiteness of E , and a generalization to infinite state spaces would require new techniques." Our aim in this paper is to tackle the case when E is infinite.

Before we state our main results, let us point out that with no loss of generality, the non-homogeneous chain P defined on D can be considered as defined on the domain

$$D' = \{(s,t): -\infty < s \leq t < \infty\}$$

in the following way. Define

$$\begin{aligned} P(s,t) &= P(s,b) \text{ if } a \leq s \leq b \leq t; \\ &= I \text{ if } b \leq s \leq t; \\ &= P(a,t) \text{ if } s \leq a \leq t; \\ &= I \text{ if } s \leq t \leq a; \\ &= P(a,b) \text{ if } s \leq a \leq b \leq t. \end{aligned}$$

Notice that with this definition P is a non-homogeneous chain on D' satisfying again conditions (i), (ii) and (iii).

Let us also point out that Theorem 1 in [2] (with the same proof) remains true

even for E infinite so that for $s \leq t$ and $(s,t) \subseteq (u,v)$,

$$R(s,t) \text{ is reflexive and } R(s,t) \subseteq R(u,v). \quad (1.3)$$

In the rest of this section, we state our results. The state space E is always infinite unless otherwise mentioned. Our main results are Theorems 1 and 2. Theorem 2 shows that the Kingman-Williams result for finite non-homogeneous Markov chains is false in the infinite case. (This problem, though mentioned in [2], was left unsolved in [2].) Theorem 1 presents a necessary and sufficient condition for an Austin-Ornstein type theorem for non-homogeneous Markov chains with countable states in terms of a weak homogeneity condition. It is doubtful to us if this condition can be any further weakened while maintaining the same conclusion. Among other results, there is a proposition in section 2 that holds even in the infinite case and gives a simple proof of the main result in [2]. Finally, in section 3, we present several results for infinite products of reflexive transitive relations on positive integers. Here are our results.

THEOREM 1. (a) Let s be a fixed time parameter. Suppose that for each positive β , there is a h , $0 < h < \beta$, such that for each positive integer m , the following condition holds:

$$R(s+mh, s+(m+1)h) \subseteq R(s+(m-1)h, s+mh). \quad (1.4)$$

Then the relation $R(s,t)$ is reflexive, transitive, and independent of t (for $t > s$).

(b) Consider the following weak homogeneity condition: for every real s and for each positive β , there is a h (depending on s) such that $0 < h < \beta$ and for each positive integer m ,

$$R(s+(m-1)h, s+mh) = R(s+mh, s+(m+1)h). \quad (1.5)$$

Then the relation $R(s,t)$ is reflexive, transitive, and independent of (s,t) , $s < t$, iff condition (1.5) holds. $\square$

THEOREM 2. There are non-homogeneous Markov chains P where the relation $R(s,t)$ cannot be expressed as a finite product of reflexive and transitive relations. $\square$

Our next theorem gives a sufficient condition for $R(s,t)$ to be a product of reflexive, transitive relations. The conditions (i) and (ii) considered in this theorem are natural in the sense that they hold in the finite dimensional situation (see Theorem 5, [2]).

THEOREM 3. Suppose that for each t , there exists $h_t > 0$ such that

$$(i) \quad t - h_t \leq t' < t \Rightarrow R(t', t) = R(t - h_t, t) \quad \text{and}$$

$$(ii) \quad t < t'' \leq t + h_t \Rightarrow R(t, t'') = R(t, t + h_t)$$

hold. Then for $(s, t) \in D$, there exist reflexive, transitive relations $T_1, T_2, \dots, T_m$ such that $R(s, t) = T_1 T_2 \dots T_m$. $\square$

THEOREM 4. Let $t_n < t_{n+1} \rightarrow t$ as $n \rightarrow \infty$. Then for $i \neq j$,

$$\sum_{n=1}^{\infty} P_{ij}(t_n, t_{n+1}) < \infty.$$

Also if $s_n < s_{n+1} \rightarrow s$ as $n \rightarrow \infty$, then for $i \neq j$,

$$\sum_{n=1}^{\infty} P_{ij}(s_{n+1}, s_n) < \infty.$$

If $\lim_{s \rightarrow t} P_{ij}(s, t) = 0$ uniformly in all i different from j (for each j),

then for $t_n < t_{n+1} \rightarrow t$, we have: for each i ,

$$\sum_{n=1}^{\infty} \sum_{k \neq i} P_{ik}(t_n, t_{n+1}) < \infty. \square$$

We remark that though Theorem 4 is not combinatorial in nature and therefore does not blend well in this respect with out other results, we include it here since it uncovers a structural property of a non-homogeneous chain which is by no means obvious and seems to be missing in the literature even in the homogeneous case.

THEOREM 5. There exists a non-homogeneous Markov chain P such that the relation $R(s, t)$ cannot be expressed as an infinite forward product $T_1 T_2 \dots T_n \dots$, of reflexive transitive relations on E . $\square$

THEOREM 6. There exists a non-homogeneous Markov chain P such that the relation $R(s, t)$ cannot be expressed as an infinite backward product $\dots T_{-n+1} \dots T_{-1}$ of reflexive transitive relations on E . $\square$

THEOREM 7. There exists a non-homogeneous Markov chain P such that the relation $R(s, t)$ cannot be expressed as an infinite 2-sided product $\dots T_{-n+1} \dots T_{-1} T_0 T_1 \dots T_{n-1} T_n \dots$ of reflexive transitive relations on E . $\square$

THEOREM 8. Let $(T_n)_{n=1}^{\infty}$ be a sequence of reflexive transitive relations on E . Then there is a non-homogeneous Markov chain P on $[a, b]$, uniformly continuous separately in each variable, such that $R(a, b) = T_1 T_2 \dots T_n \dots$. $\square$

One can also obtain theorems analogous to Theorem 8 for infinite backward as well as two-sided products of reflexive transitive relations. The proofs of Theorems 5, 6 and 7 are contained in the example considered in section 3. The proof of Theorem 8 is also contained in section 3. The proofs of Theorems 1, 2, 3 and 4 are given in section 2.

2. DISCUSSION AND PROOFS (of the first four theorems)

Notice that there are simple examples of non-homogeneous Markov chains where $R(s,t)$ is not independent of t for a given s . For example, consider the two standard homogeneous chains $Q(t)$ and $S(t)$ with state space $\{1,2\}$ such that

$$Q(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S(t) = \begin{pmatrix} e^{-t} & 1-e^{-t} \\ 0 & 1 \end{pmatrix}$$

Let the non-homogeneous chain P be defined by

$$\begin{aligned} P(s,t) &= Q(t-s) \text{ if } s \leq t \leq 1; \\ &= S(t-s) \text{ if } 1 \leq s \leq t; \\ &= Q(1-s)S(t-1) \text{ if } s \leq 1 \leq t. \end{aligned}$$

Let $s < 1$. Then $P_{12}(s,1) = 0$, whereas $P_{12}(s,2) > 0$.

As we can see in this example (see also [2]), in the finite dimensional case the non-homogeneous chains are in principle, formed by taking together several homogeneous chains in a manner shown in the example. In the infinite dimensional case, however, it will appear from our results in this paper that non-homogeneous Markov chains are, in general, results of infinite products (forward, backward or two-sided) of homogeneous Markov chains.

Before we go into the proofs of our main results, let us present a simple proposition. The main result in [2] follows immediately from this proposition.

PROPOSITION 1. Suppose that $R(s,t)$ is not transitive. Then there exists v such that $s < v < t$ and $R(s,t) \neq R(s,v)R(v,t)$ (see [2] for definition of composition of relations), where $R(s,v)$ as well as $R(v,t)$ is properly included in $R(s,t)$. $\square$

PROOF. Write R for $R(s,t)$. There exist i,j,k in E such that $(i,j) \in R$, $(j,k) \in R$, but $(i,k) \notin R$. By separate continuity, there exists u , $s < u < t$, such that $P_{jk}(u,t) > 0$. Since $P(s,t) = P(s,u)P(u,t)$ and $(i,k) \notin R$, we must have $P_{ij}(s,u) = 0$.

Let

$$v = \sup\{u: s < u < t \text{ and } P_{ij}(s,u)=0\}.$$

Then, $P_{ij}(s,v)=0$ and $v < t$. Also, $P_{jk}(v,t)=0$, since otherwise there exists w , $v < w < t$, such that $P_{jk}(w,t) > 0$ and also (because of the supremum property of v) $P_{ij}(s,w) > 0$, which mean that $P_{ik}(s,t) > 0$, contradicting the assumption that $(i,k) \notin R$. This means that $R(s,v)$ and $R(v,t)$ are both properly included in R . $\square$

PROPOSITION 2. Let E be finite. Then given $(s,t) \in D$, there exist

$$s < u_1 < u_2 < \dots < u_m < t$$

such that $R(s,t)=R(s,u_1)R(u_1,u_2)\dots R(u_m,t)$ where each relation on the right side is reflexive and transitive. $\square$

PROOF. This proposition follows by applying Proposition 1 repeatedly.

Now we present proofs of our main results.

PROOF OF THEOREM 1. With no loss of generality, we assume that $P(s,t)$ is defined on $D = \{(s,t): -\infty < s \leq t < \infty\}$. We follow closely the proof given on pages 126-128 of [1] (see Theorem 5, p. 126, [1]). We briefly sketch the proof. By (3), if $x < y < z$, then $R(x,y) \subset R(x,z)$. Therefore, let us suppose, if possible, that there exist $i,j \in E$ such that

$$P_{ij}(s,t)=0 \text{ whenever } s < t \leq t_0,$$

where

$$t_0 = \sup\{t: P_{ij}(s,t)=0\}.$$

We assume that t_0 is finite. The theorem will be proven by reaching a contradiction.

Choose t' such that $t_0 < t' < 2t_0 - s$. Then $P_{ij}(s,t') > 0$.

Let $c > 0$ such that

$$P_{ij}(s,t')=2c. \quad (2.1)$$

Choose $\beta > 0$ such that $P_{ij}(s,t) \geq c$ if $s < t' - \beta < t < t' + \beta$ and such that

$t' + \beta < 2t_0 - s$. Note that for sufficiently large m ,

$$(0, \frac{t'-s}{4m}) \subset \bigcup_{n=m}^{\infty} (\frac{t'-s}{4n}, \frac{t'-s+\beta}{4n}).$$

This means that using compactness of the interval $[s,t']$ and separate continuity of the mapping $t \rightarrow P(s,t)$, it follows that there exists a sufficiently large positive

N such that $\forall t \in [s, t']$,

$$\sum_{k > N} P_{ik}(s, t) < \frac{c}{4} \quad (2.2)$$

and the interval $(\frac{t'-s}{4N}, \frac{t'-s+\beta}{4N})$ contains a positive h that satisfies (1.4) and

$$P_{ij}(s, s+4Nh) \geq c. \quad (2.3)$$

Now we define the set A_m , $\forall m \geq 1$, by

$$A_m = \{k: P_{ik}(s, s+mh) > 0\}. \quad (2.4)$$

Then $A_m \subset A_{m+1}$ and it follows by assumption (1.4) that for $m \geq 2$, $k \in A_{m-1}$ and

$k' \notin A_m$, we have:

$$P_{kk'}(s+mh, s+(m+1)h) = 0 \quad (2.5)$$

since

$$0 = P_{ik'}(s, s+mh) = \sum_{k \in A_{m-1}} P_{ik}(s, s+(m-1)h) P_{kk'}(s+(m-1)h, s+mh).$$

Using (2.5) and noting that $j \in A_{4N}$ while $j \notin A_{2N}$, it follows as in the proof in [1] that the sets

$$B_1 \equiv A_1 \text{ and } B_m \equiv A_m - A_{m-1} \quad (2 \leq m \leq 2N)$$

are all nonempty and pairwise disjoint. One can then reach a contradiction by following the same procedure as given in the proof in [1]. This completes the proof of part (a).

To prove part (b) of the theorem, let us first show that under condition (1.5) we have: for $s < u < t$, $R(s, t) = R(u, t)$. To this end, let $0 < \beta < \min\{u-s, t-u\}$, where $s < u < t$. By assumption then there exists a h , $0 < h < \beta$, satisfying (1.5). Let m be the smallest positive integer such that

$$s < s+mh < u \leq s+(m+1)h < t.$$

Then we have:

$$P_{ij}(s, t) > 0$$

$$\Rightarrow P_{ij}(s, s+h) > 0 \text{ (by part (a) of this theorem)}$$

$$\begin{aligned}
&\Rightarrow P_{ij}(s+(m+1)h, s+(m+2)h) > 0 \quad (\text{by condition (1.5)}) \\
&\Rightarrow P_{ij}(s+(m+1)h, t) > 0 \quad (\text{the reason being that part (a) applies if} \\
&\quad t < s+(m+2)h, \text{ and (1.3) applies otherwise}) \\
&\Rightarrow P_{ij}(u, t) > 0 \quad (\text{by (1.3)}).
\end{aligned}$$

This proves that $R(s, t) = R(u, t)$ whenever $s < u < t$ and condition (1.5) holds. This result along with part (a) and the reflexivity property in (1.3) implies immediately the conclusion in part (b). $\square$

Now we prove the following lemma which will be needed in the proof of Theorem 2.

LEMMA. Let n be a given positive integer. Then there exists a non-homogeneous Markov chain P with state space $\{1, 2, \dots, n\}$ such that for some $s < t$, $R(s, t)$ cannot be written as a product $T_1 T_2 \dots T_m$, where each T_i is reflexive and transitive and $m < n-1$. $\square$

PROOF. Let us define the relations $R_1, R_2, \dots, R_{n-1}$ as follows:

$$R_k = \{(i, i) : 1 \leq i \leq n\} \cup \{(n-k, n-k+1)\}, \quad 1 \leq k \leq n-1. \quad (2.6)$$

Then each R_k is reflexive and transitive, and the product

$$R = R_1 R_2 \dots R_{n-1} = \{(i, i) : 1 \leq i \leq n\} \cup \{(i, i+1) : 1 \leq i \leq n-1\}. \quad (2.7)$$

By line 21, p. 82 in [2], R is embeddable; that is, there exists a non-homogeneous Markov chain P such that $R(a, b) = R$.

Now suppose that there are m , $m < n-1$, reflexive transitive relations

$T_1, T_2, \dots, T_m$ such that

$$R = T_1 T_2 \dots T_m. \quad (2.8)$$

We claim that

$$(i, i+1) \notin T_1 \text{ if } i+1 < n; \text{ also } T_1 \subseteq R. \quad (2.9)$$

Notice that $T_1 T_1 = T_1$ so that $T_1 R \subseteq R$. If $(i, i+1) \in T_1$ for some $i < n-1$, then since

$(i+1, i+2) \in R$ for $i \leq n-2$, $(i, i+2) \in R$ for some $i < n-1$. This contradicts (2.6).

Thus (2.9) is verified. Now it follows from (2.7), (2.8) and (2.9) that the relation $R^{(1)}$ defined by

$$\begin{aligned}
R^{(1)} &= \{(i, i+1) : 1 \leq i \leq n-2\} \\
&\subseteq T_2 T_3 \dots T_m \subseteq R.
\end{aligned} \quad (2.10)$$

It again follows as in (2.9) that

$$(i, i+1) \notin T_2 \text{ if } i+1 < n-1$$

so that the relation $R^{(2)}$ defined by

$$R^{(2)} = \{(i, i+1) : 1 \leq i \leq n-3\} \subseteq T_3 \dots T_m \subseteq R.$$

Continuing, we see that if $m < n-1$, then

$$\{(1, 2), (2, 3)\} \subseteq T_m \subseteq R.$$

Since T_m is transitive, $(1, 3) \in T_m \subseteq R$. This contradicts (2.7) and the lemma follows. $\square$

PROOF OF THEOREM 2. Define the sets A_i , $i \geq 1$, as follows:

$$A_i = \{2^{i-1}, 2^{i-1}+1, \dots, 2^i-1\}$$

Consider the state space $E = \bigcup_{i=1}^{\infty} A_i$. For each positive integer k , let $P^{(k)}$ be a non-homogeneous Markov chain with state space A_k defined as in the lemma. Define the non-homogeneous Markov chain P with E as state space as follows:

$$\begin{aligned} P_{ij}(s, t) &= P_{ij}^{(k)}(s, t) \text{ if } (i, j) \in A_k \times A_k; \\ &= 0 \text{ if } (i, j) \in A_{k_1} \times A_{k_2} \text{ } (k_1 \neq k_2). \end{aligned}$$

Note that by construction, the relation $R^{(k)}(a, b)$, a and b remaining the same for all k , cannot be expressed as a product of fewer than $2^{k-1}-1$ reflexive transitive relations on A_k . Now if we write

$$R(a, b) = T_1 T_2 \dots T_m \text{ (the } T_i\text{'s are reflexive transitive), then since}$$

each $T_i \subseteq R(a, b)$, $(i, j) \notin T_i$ if $(i, j) \in A_{k_1} \times A_{k_2}$ ($k_1 \neq k_2$).

This means that if $T_i^{(k)}$ is the restriction of T_i on $A_k \times A_k$, then

$$R^{(k)}(a, b) = T_1^{(k)} T_2^{(k)} \dots T_m^{(k)}.$$

This is a contradiction since then m cannot be finite. $\square$

PROOF OF THEOREM 3. Observe that if h_t is as in conditions (i) and (ii), then the relations $R(t', t)$ as well as $R(t, t')$ is reflexive and transitive, where t' and t'' are as described in the theorem. Reflexivity follows from (1.3). For the transitive property, suppose that $P_{ij}(t', t) > 0$ and $P_{jk}(t', t) > 0$. By separate continuity, there exists, u , $t' < u < t$, such that $P_{ij}(t', u) = 0$. Since by condition (i),

$R(t',t)=R(u,t)$, $P_{jk}(u,t) > 0$. Therefore, $P_{ik}(t',t) \geq P_{ij}(t',u)P_{jk}(u,t) > 0$. Thus, $R(t',t)$ is transitive. Similarly, $R(t,t')$ is also transitive. The theorem now follows easily by using compactness of the interval $[s,t]$. $\square$

PROOF OF THEOREM 4. Write $P_n = P(t_n, t_{n+1})$. Then $Q_k = \lim_{n \rightarrow \infty} P_{k+1} P_{k+2} \dots P_n$ exists and equals $P(t_{k+1}, t)$ so that $\lim_{k \rightarrow \infty} Q_k = I$. We can also make similar observations using backward products. The theorem now follows immediately from related results proven in detail in [3]. $\square$

3. INFINITE PRODUCTS OF REFLEXIVE AND TRANSITIVE RELATIONS

In this section, we will consider a second example to show that for a non-homogeneous Markov chain P with countable states, the relation $R(s,t)$ need not be even of the form

$$T_1 T_2 \dots T_n \dots$$

where each T_i is reflexive and transitive. Here an infinite product simply means a relation that is the union of all the finite partial products of the T_i 's in the same order. It will be clear that the same result remains true even if we consider products of the form

$$T_{-n} T_{-n+1} \dots T_{-1} \text{ or } \dots T_{-n} T_{-n+1} \dots T_{-1} T_0 T_1 \dots T_n T_{n+1} \dots$$

where each T_i is reflexive and transitive. We will also show that given any such infinite product, there is always a non-homogeneous Markov chain P on $[a,b]$ such that $R(a,b)$ is the given infinite product.

First, the example. Let $E = \{1, 2, 3, \dots\}$ be the state space and let

$s < s_{n+1} < s_n < \dots < s_2 < s_1 = t$ and $\lim_{n \rightarrow \infty} s_n = s$. We know that there exists on each interval $[s_{n+1}, s_n]$ a homogeneous Markov chain $P^{(n)}$ with state space E such that

$$R^{(n)}(s_n - s_{n+1}) = \{(n+1, n+2)\} \cup U_0,$$

where $U_0 = \{(i, i) : 1 \leq i < \infty\}$

(Notice that above one could actually define $P^{(n)}(u)$ as follows:

$$\begin{aligned} P_{ij}^{(n)}(u) &= 1 \text{ if } i=j \neq n+1, \\ &= \exp(-u) \text{ if } i=j=n+1, \end{aligned}$$

$$= 1 - \exp(-u) \text{ if } i=n+1 \text{ and } j=n+2.$$

Let us now define P on subintervals of $[s, t]$ as follows:

Case 1. Let $s < u \leq v \leq t$. In this case, there exist positive integers n_1, n_2 such that

$$s_{n_1+1} < u \leq s_{n_1} \leq \dots \leq s_{n_2} \leq v < s_{n_2-1}.$$

$$\text{We define } P(u, v) = P^{(n_1)}_{(s_{n_1}-u)} P^{(n_1-1)}_{(s_{n_1-1}-s_{n_1})} \dots P^{(n_2-1)}_{(v-s_{n_2})}.$$

Case 2. $s=u < v$. Suppose that $s_{m+1} < v \leq s_m$. Note that because of our definition in Case 1, we have for $n > m$:

$$\begin{aligned} P_{ij}(s_n, v) &= 1 \text{ if either } i=j < m+1 \text{ or } i=j > n, \\ &= \exp(-(v-s_{m+1})) \text{ if } i=j=m+1, \\ &= \exp(-(s_k-s_{k+1})) \text{ if } m+1 < i=j=k+1 \leq n. \end{aligned}$$

This clearly shows that $\lim_{n \rightarrow \infty} P(s_n, v)$ exists and we define

$$P(s, v) = \lim_{n \rightarrow \infty} P(s_n, v).$$

It is now easy to see that P as defined above is a separately continuous non-homogeneous Markov chain and $R(s, t)$ is given by

$$R(s, t) = U_0 \cup \{(i, i+1) : i \in E\}.$$

We claim that $R(s, t)$ cannot be written in the form $T_1 T_2$, where T_1 is reflexive and transitive, $T_1 \neq U_0$, and T_2 any reflexive relation. To see this, notice that if $R = T_1 T_2$ as above, then $T_1 R = R$ since $T_1 = T_1^2$. Since $T_1 \neq U_0$ and $T_1 \subset R$, there is a $i \in E$ such that $(i, i+1) \in T_1$. Since $(i+1, i+2) \in R$, this means that $(i, i+2) \in R$ and this is a contradiction. Thus, $R(s, t)$ cannot be written as an infinite product of reflexive transitive relations. We also claim that this $R(s, t)$ cannot be written even in the form

$$\dots T_{-n} T_{-n+1} \dots T_{-1} T_0 T_1 \dots T_n T_{n+1} \dots$$

where the T_i 's are reflexive and transitive and different from U_0 . To prove this claim, let us suppose that R has this form. Since $T_{-1} \neq U_0$ and $T_{-1} \subset R(s, t)$, there is an $i_0 \in E$ such that $(i_0, i_0+1) \in T_{-1}$. The element (i_0+1, i_0+2) , being an element of R , must be in T_m for some m , but this m must be less than -1 since otherwise the ele-

ment (i_0, i_0+2) would be in R . Repeating this argument, it follows that

$$\{(j, j+1): j \geq i_0\} \subset \bigcup_{i=-\infty}^{-1} T_i.$$

Since for each $i \in E$, $(i, i+1) \in T_m$ for some m , this means that there is a positive integer N such that

$$R(s, t) \subset \bigcup_{i=-\infty}^N T_i.$$

Since we can assume with no loss of generality that any two consecutive T_i 's are distinct, there is a $m > N$ such that $(i, i+1) \in T_m$ for some $i > 1$. But since $(i-1, i) \in T_n$ for some $n \leq N$, this means that $(i-1, i+2) \in R$, and this is a contradiction. Let us point out that in the above example the relation $R(s, t)$ can be written, however, in the form $\dots T_{-n} T_{-n+1} \dots T_{-1}$, where $T_{-n} = U_0 \cup \{(n, n+1)\}$ is reflexive and transitive for each n . However, if we modify the above example so as to have

$$R^{(n)}(s_{n+1}, s_n) = \{(n+1, n)\} \cup U_0,$$

then again we have a non-homogeneous Markov chain P such that $R(s, t)$ cannot be expressed in the form $\dots T_{-n} T_{-n+1} \dots T_{-1}$, where each T_i is reflexive and transitive.

In what follows, we show that given an infinite product (forward, backward or two-sided) of reflexive and transitive relations on E , we can always construct a non-homogeneous Markov chain P on a given interval $[a, b]$ such that $R(a, b)$ is the given infinite product. First, a useful proposition.

PROPOSITION 3. Let $0 < c < 1$. Suppose that T is any given reflexive and transitive relation on E . Then there is a homogeneous Markov chain $P(t)$ on $[0, \infty)$ such that

$\lim_{t \rightarrow 0+} P(t) = P(0)$, $R(t) = T$, and

$$0 < t < \frac{1}{2c} \Rightarrow \|P(t) - I\| \leq 2ct. \square$$

(Here, $\|A\| = \sup_{i,j} |A_{ij}|$.)

Proof. Let $0 < c_n < 1$ such that $\sum_{n=1}^{\infty} c_n = \frac{c}{2} < \frac{1}{2}$. Define the matrix D with state space E such that

$$D_{ij} = c_j \text{ if } (i, j) \in T \text{ and } i \neq j;$$

$$= 0 \text{ if } (i,j) \notin T;$$

$$= - \sum_{k \neq i} c_k \text{ if } i=j.$$

Notice that it follows easily by induction that

$$(1) \quad \sum_{j=1}^{\infty} (D^n)_{ij} = 0 \text{ for each positive integer } n;$$

$$(2) \quad ||D^n|| \leq c^n \text{ for each positive integer } n.$$

Define $P(t) = \exp(tD)$. Then $P(t)$ is well-defined and a stochastic matrix for each t

It is also verified easily that $T=R(t)$. Notice that if $0 < 2ct < 1$, then

$$\begin{aligned} ||P(t)-I|| &\leq ct.[1+ct+c^2t^2+\dots] \\ &= ct/[1-ct] < 2ct. \square \end{aligned}$$

PROOF OF THEOREM 7. Choose a sequence (s_n) such that

$a=s_1 < s_2 < \dots < s_{n+1} \rightarrow b$. For each T_n , define $P^{(n)}$, a homogeneous Markov chain as constructed in Prop. 3, such that $R^{(n)}(s_{n+1}-s_n) = T_n$. We now define a non-homogeneous Markov chain P on subintervals of $[a,b)$ as follows:

If u, v are such that

$$s_m \leq u < s_{m+1} < \dots < s_{m+p} \leq v < s_{m+p+1},$$

then define

$$P(u,v)=P^{(m)}(s_{m+1}-u)P^{(m+1)}(s_{m+2}-s_{m+1})\dots P^{(m+p)}(v-s_{m+p}).$$

We claim that $\lim_{v \rightarrow b-} P(u,v)$ exists and is a stochastic matrix. Once we prove

this, we will define $P(a,b)$ as the limit of $P(a,v)$ as $v \rightarrow b-$.

It will be sufficient to show that for each $u \in [a,b)$,

(i) $\lim_{v \rightarrow b-} P_{ij}(u,v)$ exists (for i,j in E) and (ii) given $\epsilon > 0$ and $i \in E$, there

is a positive integer N and a $\delta > 0$ such that

$$\sum_{j=1}^N P_{ij}(u,v) > 1 - \epsilon$$

whenever $u \leq v$ and $b - \delta < v < b$.

To establish the above results, we choose m_0 such that

$$m \geq m_0 \Rightarrow b - s_m < \frac{1}{2c}.$$

Let $n \geq m_0$ and $s_n \leq v \leq v' \leq s_{n+1}$. Then for $u \leq v$,

$$\begin{aligned} & |P_{ij}(u, v) - P_{ij}(u, v')| \\ &= |P_{ij}(u, v) - \sum_k P_{ik}(u, v) P_{kj}(v, v')| \\ &\leq P_{ij}(u, v) \cdot |1 - P_{ij}(v, v')| + \sum_{k \neq j} P_{ik}(u, v) P_{kj}(v, v') \\ &\leq 2c \cdot (v' - v) \end{aligned}$$

so that

$$||P(u, v) - P(u, v')|| \leq 2c(v' - v). \quad (3.1)$$

Now if $s_n \leq v \leq s_{n+1}$ and $s_{n+p} \leq v' \leq s_{n+p+1}$, then

$$\begin{aligned} & ||P(u, v) - P(u, v')|| \\ &\leq ||P(u, v) - P(u, s_{n+1})|| \\ &+ ||P(u, s_{n+1}) - P(u, s_{n+2})|| \\ &\quad + \dots + ||P(u, s_{n+p}) - P(u, v')|| \\ &\leq 2c(v' - v), \text{ by (3.1).} \end{aligned}$$

Taking $u = v$, we have

$$||I - P(v, v')|| < \epsilon \quad (3.2)$$

whenever $v' - v < \epsilon/2c$.

Choose $\delta > 0$ such that $b - \delta \leq v \leq v' < b \Rightarrow v' - v < \epsilon/2c$.

Then for $v = b - \delta$, let N be such that

$$\sum_{j=1}^N P_{ij}(u, v) > 1 - \epsilon. \quad (3.3)$$

Since

$$\sum_{j=1}^N P_{ij}(u, v') \geq \sum_{j=1}^N P_{ij}(u, v) P_{jj}(v, v')$$

$$> 1 - 2\varepsilon, \text{ by (3.2) and (3.3)}$$

the theorem follows. $\square$

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