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ON SOLVING THE PLATEAU PROBLEM IN PARAMETRIC FORM

BARUCH CAHLON

Department of Mathematical Sciences
Oakland University
Rochester, Michigan 48063

ALAN D. SOLOMON

Mathematics and Statistics Research Division
Computer Science Division
Union Carbide Corporation-Nuclear Division
Oak Ridge, Tennessee 37830

LOUIS J. NACHMAN

Department of Mathematical Sciences
Oakland University
Rochester, Michigan 48063

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ABSTRACT. This paper presents a numerical method for finding the solution of Plateau's problem in parametric form. Using the properties of minimal surfaces we succeded in transfering the problem of finding the minimal surface to a problem of minimizing a functional over a class of scalar functions. A numerical method of minimizing a functional using the first variation is presented and convergence is proven. A numerical example is given.

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1. INTRODUCTION.

In this paper, we present a method for the numerical solution of the Plateau problem in parametric form. Specifically, we seek a minimal surface spanning a simple closed curve in 3-dimensional space. If the curve is planar then the problem reduces to that of finding a conformal mapping onto its interior.

To the numerical analyst the Plateau problem presents a formidable challenge. In the non-parametric case, when the surface and bounding curve admit of a single-valued projection onto an x,y plane, the problem reduces to solving the minimal surface equation [1] ,

$$(1 + z_y{}^2)z_{xx} - 2z_x z_y xy + (1 + z_x{}^2)z_{yy} = 0 \tag{1.1}$$

for the height $z(x,y)$ of the surface above the x,y plane, and for boundary values defining the given bounding curve. Finite difference iterative schemes for (1.1) have been examined by Concus [2] and Greenspan [3], [4].

In the parametric case, where the surface is not assumed to admit a single valued planar projection, a vector function representation $\vec{x}(u,v) = (x(u,v), y(u,v), z(u,v))$ is used. Here $\vec{x}(u,v)$ is defined on a domain $\mathcal{D}$ in the (u,v) plane whose structure determines that of the surface. By a theorem of Weierstrass [5] the problem becomes one of finding $\vec{x}(u,v)$ such that

A. $\Delta\vec{x} = 0$, on $\mathcal{D}$

B. $\vec{x}_u{}^2 = \vec{x}_v{}^2$, $\vec{x}_u\vec{x}_v = 0$, on $\mathcal{D}$

C. $\vec{x}$ maps the boundary of $\mathcal{D}$ onto the bounding curve(s) of the surface in a monotonic fashion.

A numerical scheme for simultaneously attaining A,B,C cannot be easily derived, for although B is in fact a boundary condition for A [1], it is not clear how one may work with C .

In the following we introduce a method for computing the solution of this problem. The method depends on our recognizing the solution as a function minimizing the Dirichlet integral over all functions satisfying C . Similar to the method of safe descent of R. Courant [1], we define the Dirichlet integral as a functional $d(g)$ on a class G of scalar functions g which determine the manner in which the surface is "sewed" onto its bounding curve. The percise definition of this functional is given in section 2. In section 3 the derivation of the first variation of $d(g)$ is performed, and as an obvious consequence we see that a stationary value for $d(g)$

defines a minimal surface. In section 4 we define a method for minimizing $d(g)$ based on the use of the first variation; we then prove the convergence of the method and discuss its computational implementation, which is described in section 5.

2. DEFINITION OF d(g).

Let C be a simple closed curve in (x,y,z) space of length 2π, given by

$$C: \vec{x} = \vec{h}(\sigma)\ ,\ 0 \le \sigma \le 2\pi \tag{2.1}$$

for arc length σ. We assume that $\vec{h}$ is twice continuously differentiable with $h'(0) = h'(2\pi)$, $h''(0) = h''(2\pi)$. Let $\vec{t}(\sigma)$, $\vec{b}(\sigma)$, $\vec{n}(\sigma)$, $\tau(\sigma)$ be the tangent, binormal, normal, curvature and torsion of C, respectively. Let $\mathcal{D}$ be the unit circle in the (u,v) plane

$$\mathcal{D}: u^2 + v^2 < 1 \tag{2.2}$$

with boundary

$$\Gamma: u^2 + v^2 = 1 \tag{2.3}$$

and closure $\overline{\mathcal{D}} = \mathcal{D} \cup \Gamma$.

A vector function of u,v on $\mathcal{D}$ is denoted by a lower case letter such as $\vec{x}(u,v)$, while the same function referred to polar coordinates on $\mathcal{D}$, $u = r\cos\theta$, $v = r\sin\theta$, is denoted by the corresponding upper case letter:

$$\vec{x}(u,v) = \vec{X}(r,\theta)\ ,\ 0 \le r \le 1\ ,\ 0 \le \theta \le 2\pi\ . \tag{2.4}$$

For any sufficiently smooth functions $\vec{x},\vec{y}$, the Dirichlet integral $D[\vec{x}]$ of $\vec{x}$ over $\mathcal{D}$ is

$$D[\vec{x}] = \iint_{\mathcal{D}} (\vec{x}_u^{\,2} + \vec{x}_v^{\,2})\,du\,dv \tag{2.5}$$

while the Dirichlet inner product is

$$D[\vec{x},\vec{y}] = \iint_{\mathcal{D}} (\vec{x}_u\vec{y}_u + \vec{x}_v\vec{y}_v)\,du\,dv\ . \tag{2.6}$$

It is known [1] that a vector function $\vec{x} = \vec{X}$ exists on $\mathcal{D}$, for which conditions A,B,C hold. Horeover by the twice continuous differentiability of $\vec{h}$ and the extension of Kellogg's theorem to minimal surfaces [6], [7], $\vec{X}(1,\theta)$ has a Hölder continuous first derivative with respect to θ,

$$|\vec{X}_\theta(1,\theta+\delta) - \vec{X}_\theta(1,\theta)| \le \alpha\delta^\nu \tag{2.7}$$

with α,ν the Hölder constant and Hölder index, respectively. Specifically

THEOREM 1. There exists a function $\vec{x}(u,v)$ satisfying the following conditions:

1.1. $\vec{x}$ is continuous on $\mathcal{D}$;

1.2. $\Delta\vec{x} = 0$ within $\mathcal{D}$;

1.3. $\vec{x}_u^{\,2} = \vec{x}_v^{\,2}$ in $\mathcal{D}$;

1.4. $\vec{x}_u\vec{x}_v = 0$ in $\mathcal{D}$;

1.5. $\vec{X}_\theta\,(1,\theta)$ is Hölder continuous and obeys (2.7) for some values of α , ν .

1.6. $\vec{x}$ maps Γ onto C in a monotonic one-to-one fashion;

1.7. $D[x] < \infty$

1.8. For the function $\vec{x}(u,v)$ the Dirichlet integral (2.5) attains its least value among all functions satisfying 1.1-1.4,1.6,1.7.

LEMMA 1. Condition 1.5 implies 1.7 for any harmonic function $\vec{x}$. Furthermore

$$D[\vec{x}] \le M \tag{2.8}$$

with M a constant dependent only on α , ν .

PROOF: By 1.5, $\vec{X}(1,\theta)$ admits of a uniformly convergent Fourier series expansion

$$\vec{X}(1,\theta) = \frac{\vec{\alpha}_0}{2} + \sum_{j=1}^{\infty} (\vec{\alpha}_j \cos j\theta + \vec{\beta}_j \sin j\theta) \; ;$$

furthermore by a simple calculation

$$\vec{\alpha}_j = \frac{1}{\pi j}\int_0^{2\pi} (\vec{X}_\theta(1,\theta + \tfrac{\pi}{j}) - \vec{X}_\theta(1,\theta))\cos j\theta d\theta$$

$$\vec{\beta}_j = \frac{1}{\pi j}\int_0^{2\pi} (\vec{X}_\theta(1,\theta + \tfrac{\pi}{j}) - \vec{X}_\theta(1,\theta))\sin j\theta d\theta$$

whence

$$|\alpha_j| \; , \; |\beta_j| \le \frac{2\alpha\pi^\nu}{j^{1+\nu}} \tag{2.9}$$

It is known [1] that the Dirichlet integral exists if and only if the series

$$\pi \sum_{j=1}^{\infty} j(\vec{\alpha}_j^{\,2} + \vec{\beta}_j^{\,2}) \tag{2.10}$$

converges, and if so, its value is given by the series. However it is clear that if (2.9) holds then (2.10) does converge, and

$$D[\vec{x}] \le 8M^{2\nu+1}\,\alpha^2 \sum_{j=1}^{\infty} \frac{1}{j^{1+2\nu}} = M \; ;$$

the lemma is proved.

Let Ω be the collection of functions satisfying 1.1, 1.2, 1.5, 1.6. Let $\vec{x} = \vec{X}$ be any function of Ω. By 1.6 there exists a monotonic function $g(\theta)$, $0 \le \theta \le 2\pi$, for which

$$\vec{X}(1,\theta) = \vec{h}(g(\theta)) \tag{2.11}$$

and

$$g(0) = 0,\ g(2\pi) = 2\pi \quad . \tag{2.12}$$

Moreover, by 1.5, $g(\theta)$ has a continuous derivative $g'(\theta)$ satisfying

$$\vec{X}_\theta(1,\theta) = \vec{t}(g(\theta))g'(\theta)$$

with $\vec{t}$ the unit tangent vector. Hence

$$g'(\theta) = |g'(\theta)| = |\vec{X}_\theta(1,\theta)| \tag{2.13}$$

and

$$\begin{aligned} |g'(\theta+\delta) - g'(\theta)| &= \big||\vec{X}_\theta(1,\theta+\delta)| - |\vec{X}_\theta(1,\theta)|\big| \\ &\le |\vec{X}_\theta(1,\theta+\delta) - \vec{X}_\theta(1,\theta)| \\ &\le \alpha\delta^\nu \quad ; \end{aligned}$$

we conclude:

THEOREM 2. Any function $\vec{x}$ of Ω defines a monotonic Hölder continuously differentiable scalar function $g(\theta)$ satisfying (2.11). We will refer to this function as the boundary correspondence function for $\vec{x}$.

Let G be the set of all functions $g(\theta)$ on $[0,2\pi]$ with Hölder continuous first derivatives, obeying (2.12). Any $g \in G$ defines harmonic function $\vec{x} = \vec{X}$ satisfying (2.11). Moreover by the assumptions on $\vec{h}$, $\vec{X}(1,\theta)$ has a Hölder continuous derivative with respect to θ, whence by lemma 1,

$$D[\vec{x}] < \infty \quad .$$

Since any function $g \in G$ defines in this way a unique $\vec{x} \in \Omega$ and conversely, we can equate the problem of minimizing the Dirichlet functional over Ω, with that of minimizing the scalar functional

$$d(g) = D[\vec{x}] \tag{2.14}$$

over $g \in G$.

3. THE FIRST VARIATION OF d(g).

We now calculate the first variation of $d(g)$. Specifically, let g^* be a function of G for which $d(g)$ assumes a stationary value. Let $\eta(\theta)$ be any Hölder continuously differentiable function for $0 \leq \theta \leq 2\pi$ with $\eta(0) = \eta(2\pi) = 0$ and let ε be any parameter. Then

$$\delta(\varepsilon) = d(g^* + \varepsilon\eta)$$

assumes a stationary value for $\varepsilon = 0$.

THEOREM 3. $\delta(\varepsilon)$ is differentiable for $\varepsilon = 0$. Moreover

$$\delta'(0) = 2\int_0^{2\pi} \vec{X}_r(1,\theta)\vec{t}(g^*(\theta))\eta(\theta)d\theta \tag{3.1}$$

with $\vec{X} \in \Omega$ defined by (10) for $g = g^*$.

PROOF: Clearly

$$\vec{h}(g^* + \varepsilon\eta) = \vec{h}(g^*) + \int_{g^*}^{g^*+\varepsilon\eta} \vec{t}(\sigma)d\sigma . \tag{3.2}$$

Let $\vec{X},\vec{X}^1 \in \Omega$ satisfy $\vec{X}(1,\theta) = \vec{h}(g^*(\theta))$, $\vec{X}^1(1,\theta) = \vec{h}(g^*(\theta) + \varepsilon\eta(\theta))$. Then $\vec{X}^1 = \vec{X} + \varepsilon\vec{Y}^\varepsilon$ for $\vec{Y}^\varepsilon$ the harmonic function on $\mathcal{D}$ for which

$$\vec{Y}^\varepsilon(1,\theta) = \frac{1}{\varepsilon}\int_{g^*(\theta)}^{g^*(\theta)+\varepsilon\eta(\theta)} \vec{t}(\sigma)d\sigma .$$

Clearly

$$\delta(0) = d(g^*)$$

$$\delta(\varepsilon) = d(g^* + \varepsilon\eta)$$

Moreover

$$\delta(\varepsilon) = \delta(0) + 2\varepsilon D[\vec{X},\vec{Y}^\varepsilon] + \varepsilon^2 D[\vec{Y}^\varepsilon] . \tag{3.3}$$

By Gauss's theorem

$$D[\vec{X},\vec{Y}^\varepsilon] = \int_0^{2\pi} \vec{X}_r(1,\theta) \cdot \vec{Y}^\varepsilon(1,\theta)d\theta ;$$

by the continuity of $\vec{t}$,

$$\vec{Y}^\varepsilon(1,\theta) \to \vec{t}(g^*(\theta))\eta(\theta) = \vec{Y}(1,\theta) \tag{3.4}$$

uniformly for $\varepsilon \to 0$, whence

$$D[\vec{X},\vec{Y}^{\varepsilon}] \to \int_0^{2\pi} \vec{X}_r(1,\theta)\vec{t}(g^*(\theta))\eta(\theta)d\theta \quad .$$

LEMMA 2. $D[\vec{Y}^{\varepsilon}]$ is uniformly bounded for all ε as $\varepsilon \to 0$.

PROOF. As $\varepsilon \to 0$ the function $\vec{Y}^{\varepsilon}$ converges uniformly on Γ according to (3.4). Using the Frenet formulas, we see

$$\vec{Y}^{\varepsilon}_{\theta}(1,\theta) = \varkappa\vec{t}(g^* + \varepsilon\eta) + \frac{g^{*\prime}(\theta)}{\varepsilon}\int_{g^*}^{g^*+\varepsilon\eta} \varkappa(\sigma)\vec{n}(\sigma)d\sigma \quad .$$

But then if $g^{*\prime}(\theta)$ is Hölder continuous with index ν , then $\vec{Y}^{\varepsilon}_{\theta}$ is Hölder continuous with Hölder index ν and Hölder constant independent of ε , which implies, by lemma 1,

$$D[\vec{Y}^{\varepsilon}] < M \tag{3.5}$$

with M a constant independent of ε . By the uniform convergence of $\vec{Y}^{\varepsilon}$ to $\vec{Y}$ in Γ , and hence on $\mathcal{D}$, and the lower semi-continuity of the Dirichlet integral [1], (3.5) implies

$$D[\vec{Y}] \leq M \tag{3.6}$$

proving lemma 2.

Rewriting (3.3) as

$$\frac{\delta(\varepsilon) - \delta(0)}{\varepsilon} = 2D[\vec{X},\vec{Y}^{\varepsilon}] + \varepsilon D[\vec{Y}^{\varepsilon}]$$

we see by (3.4), (3.5), that the limit of the left hand side exists for $\varepsilon \to 0$ and (3.1) is proved.

LEMMA 3. If a stationary value for $d(g)$ is attained for some $g^* \in G$, then g^* defines a minimal surface.

PROOF. If

$$D[\vec{X},\vec{Y}] = 0$$

for all $\eta(\theta)$, then by the fundamental theorem of the calculus of variations

$$\vec{X}_r(1,\theta)\vec{t}(g(\theta)) = 0 \tag{3.7}$$

which by (2.13) implies

$$\vec{X}_r(1,\theta)\vec{X}_\theta(1,\theta) = 0 \ . \tag{3.8}$$

However, by a standard argument [8], (3.8) implies that $\vec{x}$ defines a minimal surface on $\mathcal{D}$, and the lemma is proved.

4. AN ALGORITHM FOR MINIMIZING d(g)

We now derive an algorithm for solving the problem

$$\min_{g \in G} . d(g) \tag{4.1}$$

numerically. The algorithm rests upon a Rayleigh-Ritz type of approach, in which we solve (4.1) over a sequence of finite dimensional subsets of G , yielding a sequence of functions converging to the solution. At some stage in the algorithm we will need to require that a "three-points condition" in which three given points of Γ are mapped into three given points of C , is obeyed. Since $\overline{\mathcal{D}}$ can be mapped conformally onto itself by a Mobius transformation in which the images of three given points can be preassigned, while a function $g \in G$ can be considered as mapping Γ onto itself, a three points condition can always be attained through the composition of two elements of G . In addition the Dirichlet integral is invariant under the Mobius transformation. In order to guarantee convergence of the algorithm we impose an additional smoothness assumption on the curve C and as a consequence, on the functions of G . We also assume that a minimal surface solving (4.1) has no branch points on the boundary.

For the purposes of this section, we will assume that the function $\vec{h}$ of (2.1) has a Hölder continuous second derivative. Again using the extension of Kellogg's theorem to minimal surfaces, we see that a solution g^* to (4.1) has a Hölder continuous second derivative. We now redefine the collection G as the set of all monotonic twice differentiable functions g obeying (2.12) and having a Hölder continuous derivative of second order. Let

$$\mu = \inf_{g \in G} d(g) = d(g^*) . \tag{4.2}$$

Let g_0 be any function of G. Then $g^* - g_0$ vanishes for $\theta = 0, 2\pi$, and has a Hölder continuous second derivative. Hence $g^* - g_0$ has a uniformly convergent Fourier series expansion

$$g^* - g_0 = \sum_{j=1}^{\infty} (a_j \cos j\theta + b_j \sin j\theta) + \frac{a_0}{2} ; \tag{4.3}$$

moreover by calculations identical to those used in deriving (2.9).

$$|a_j| , |b_j| \leq \frac{\alpha\pi^{\gamma-1}}{j^{2+\gamma}} = \frac{\beta}{j^{2+\gamma}} \tag{4.4}$$

while

$$|a_0| \leq 8\pi^2 \tag{4.5}$$

where α, γ are the Hölder constant and Hölder exponent, respectively. Clearly now the series in the relation

$$g^* = g_0 + \frac{a_0}{2} + \sum_{j=1}^{\infty} (a_j \cos j\theta + b_j \sin j\theta) \tag{4.6}$$

can be differentiated termwise, since the resulting series itself converges uniformly, obtaining

$$g^{*\prime}(\theta) = g_0'(\theta) + \sum_{j=1}^{\infty} (-ja_j \sin j\theta + jb_j \cos j\theta) . \tag{4.7}$$

If we make the (reasonable) assumption that the minimal surface defined by g^* has no branch points at the boundary, then for some possitive constant ω_0

$$g^{*\prime}(\theta) \geq \omega_0 > 0 , \quad 0 \leq \theta \leq 2\pi . \tag{4.8}$$

Let

$$S_n(\theta) = \frac{a_0}{2} + \sum_{j=1}^{n} (a_j \cos j\theta + b_j \sin j\theta) .$$

Then by (4.8) and the uniform convergence of the series n (4.7),

LEMMA 4. The sequences $[g_0 + S_n]$, $[g_0' + S_n']$ converge uniformly to g^*, $g^{*\prime}$ respectively, on $[0, 2\pi]$; for n sufficiently large, $g_0 + S_n$ is monotonically increasing.

Finally, using the methods of section 2, LEMMA 5. $d(g_0 + S_n) \to d(g^*)$ as $n \to \infty$.

PROOF. This assertion is easily proved by obtaining estimates of the form (2.9) which under the heightened smoothness assumption for $\vec{h}$ attain one higher power of l/j . Clearly the Fourier coefficients of $\vec{h}$ depend continuously (in the L_2-norm) on the argument of $\vec{h}$, while by these estimates the series (2.10) converges uniformly; hence this series, which is equal to the Dirichlet integral, depends continuously upon the argument of $\vec{h}$, thus proving our assertion.

Before describing our algorithm we will turn to some properties of functions of the form $g_0 + S_n$.

For any constants A_j , B_j , let

$$T_n(\theta) = g_0(\theta) + \frac{A_0}{2} + \sum_{j=1}^{n} (A_j \cos j\theta + B_j \sin j\theta) . \tag{4.9}$$

LEMMA 6. Suppose that for $0 \le \theta \le 2\pi$, the function $T_n(\theta)$ is monotonically increasing. Then

$$|A_j| \ , \ |B_j| \le \frac{4}{j} . \tag{4.10}$$

PROOF. If T_n is monotonically increasing, then

$$g_0'(\theta) + \sum_{j=1}^{n} (-jA_j \sin j\theta + jB_j \cos j\theta) \ge 0 . \tag{4.11}$$

For all $k = 1,2,\dots,n$ $1 + \cos k\theta \ge 0$. Multiplying (4.11) by this function and integrating over $[0,2\pi]$ we obtain

$$2\pi + \int_0^{2\pi} g_0'(\theta)\cos k\theta \, d\theta + k\pi B_k \ge 0$$

or

$$B_k \ge -4/k .$$

Similarly, multiplying by $-1 + \cos k\theta \le 0$, we find

$$B_k \le 4/k$$

or

$$|B_k| \le 4/k . \tag{4.12}$$

In the same way, multiplying by $(\pm 1 + \sin k\theta)$ and integrating over $[0, 2\pi]$, we obtain

$$|A_k| \leq 4/k \tag{4.13}$$

and the lemma is proved.

Let M_n denote the collection of functions $T_n(\theta)$ for which (4.10) is satisfied. Let C_n denote the subset of M_n consisting of those T_n for which $g_0 + T_n$ is monotonic. Let $\hat{M}_n$ denote the subset of the $2n + 1$ dimensional Euclidean space of vectors

$$\vec{\varepsilon}_{2n+1} = (A_0, A_1, B_1, \ldots, A_n, B_n) \tag{4.14}$$

satisfying (4.5), (4.12), (4.13). Then $\hat{M}_n$ is closed and bounded. Let $\hat{C}_n$ denote the set of vectors (4.14) for which $g_0 + T_n$ is monotonic, or lies in C_n. Then $\hat{C}_n$ is closed and convex.

For any $n = 1, 2, \ldots,$ let

$$\delta(\vec{\varepsilon}_{2n+1}) = d(g_0 + T_n) \tag{4.15}$$

with $\vec{\varepsilon}_{2n+1}$, T_n defined by (4.9), (4.14). Using the methods of theorem 3 we find

THEOREM 4. The function $\delta(\vec{\varepsilon}_{2n+1})$ is lower semi-continuous, and has partial derivatives with respect to each of its independent variables; moreover

$$\frac{\partial \delta}{\partial A_j} = 2 \int_0^{2\pi} \vec{X}_r(1,\theta)\vec{t}(g_0 + T_n)\cos j\theta \, d\theta \tag{4.16}$$

$$\frac{\partial \delta}{\partial B_j} = 2 \int_0^{2\pi} \vec{X}_r(1,\theta)\vec{t}(g_0 + T_n)\sin j\theta \, d\theta \tag{4.17}$$

where $\vec{X}(1,\theta) = \vec{h}(g_0 + T_n)$.

By the lower semi-continuity, $\delta(\vec{\varepsilon}_{2n+1})$ attains a least value on the closed bounded set $\hat{C}_n$. Assume this is attained at a point ε^*_{2n+1} , whence

$$\mu_n = \inf_{\hat{C}_n}. \delta(\varepsilon_{2n+1}) = \delta(\varepsilon^*_{2n+1}) = \delta^*_{2n+1} \ . \tag{4.18}$$

An algorithm for the solution of (4.1) can now be defined in the following steps:

I. Using a gradient search type method [9] which rests on (4.16, 4.17) find a value δ^*_{2n+1} solving (4.18).

II. This value defines a monotonic function $g_0 + T^*_n$ (by (4.9)) which minimizes d on C_n :

$$\mu_n = \inf_{C_n} d(g_0 + T_n) = d(g_0 + T^*_n) \; . \tag{4.19}$$

III. Using a Mcbius transformation of $\overline{R}$ onto itself, derive a monotonic function g^*_n satisfying the three-points condition; clearly

$$\mu_n = d(g^*_n) \; . \tag{4.20}$$

IV. g^*_n defines a function $\vec{X}^*_n$ such that

$$\vec{X}^*_n(1,\theta) = \vec{h}(g^*_n(\theta))$$

and

$$\mu_n = D[\vec{X}^*_n] \tag{4.21}$$

V. Clearly

$$\mu_{n+1} \leq \mu_n \; , \quad n = 1,2,\ldots \; . \tag{4.22}$$

VI. By the monotonicity of the g^*_n , the three points condition and (4.22), there is a subsequence of the functions $\{\vec{x}^*_n\}$ which converges uniformly to a function $\vec{z}$. ([1]).

THEOREM 5. $D[\vec{z}] = \mu$ for μ defined by (4.2).

PROOF. For n sufficiently large, $g_0 + S_n$ belongs to C_n . Hence

$$\mu \leq D[\vec{z}] \leq \mu_n \leq d(g_0 + S_n)$$

and by lemma 5, our claim is proved.

5. NUMERICAL EXAMPLE

As an example of an application of the results in the previous sections we consider the following.

Let C be a simple closed curve in (x,y,z) space of length 2π, given by

$$C\ ;\quad \vec{X} = \vec{h}(\theta)\ ,\quad 0 \le \theta \le 2\pi$$

where

$$\vec{h}(\theta) = \begin{cases} (\theta,\ 0,\ 0) & \theta \in [0,\ \frac{\pi}{3}] \\ (\frac{\pi}{3},\ \theta - \frac{\pi}{3},\ 0) & \theta \in [\frac{\pi}{3},\ \frac{2}{3}\pi] \\ (\pi - \theta,\ \frac{\pi}{3},\ 0) & \theta \in [\frac{2}{3}\pi,\ \pi] \\ (0,\ \frac{\pi}{3},\ \theta - \pi) & \theta \in [\pi,\ \frac{4}{3}\pi] \\ (0,\ \frac{5}{3}\pi - \theta,\ \frac{\pi}{3}) & \theta \in [\frac{4}{3}\pi,\ \frac{5}{3}\pi] \\ (0,\ 0,\ 2\pi - \theta) & \theta \in [\frac{5}{3}\pi,\ 2\pi] \end{cases} \tag{5.1}$$

$$\vec{h}'(\theta) = T(\theta) = \begin{cases} (1,\ 0,\ 0) & \theta \in [0,\ \frac{\pi}{3}] \\ (0,\ 1,\ 0) & \theta \in [\frac{\pi}{3},\ \frac{2}{3}\pi] \\ (-1,\ 0,\ 0) & \theta \in [\frac{2}{3}\pi,\ \pi] \\ (0,\ 0,\ 1) & \theta \in [\pi,\ \frac{4}{3}\pi] \\ (0,\ -1,\ 0) & \theta \in [\frac{4}{3}\pi,\ \frac{5}{3}\pi] \\ (0,\ 0,\ -1) & \theta \in [\frac{5}{3}\pi,\ 2\pi] \end{cases} \tag{5.2}$$

$$T(\theta) = (T^1(\theta)\ ,\ T^2(\theta)\ ,\ T^3(\theta))\ .$$

Our problem is to find a minimal surface spanned by a curve C.

Let k, $A_1,\dots,A_k$ $B_1,\dots,B_k$ be given. The function $g(\theta)$ will be

$$g(\theta) = \theta + \frac{A_0}{2} + \sum_{j=1}^{k} (A_j \cos j\theta + B_j \sin j\theta)\ .$$

The monotonicity of g (lemma 6) demands

$$-\frac{4}{j} \le A_j\ ,\ B_j \le \frac{4}{j}$$

while we can guarantee

$$g(\theta) = 0 , \quad g(2\pi) = 2\pi$$

by choosing

$$A_0 = -2 \sum_{j=1}^{k} A_j . \tag{5.3}$$

Let the function $\vec{H}(\theta) = \vec{h}(g(\theta))$ and $\vec{H}(\theta) = (H^1(\theta) , H^2(\theta) , H^3(\theta))$.

Now we solve the Laplace equation

$$\Delta\vec{X} = 0 \tag{5.4}$$

on the domain $\mathcal{D}$ with the boundary condition $\vec{X}(1,\theta) = \vec{H}(\theta)$ (see Eq. (2.11)).

Define the mesh points in the $r - \theta$ plane by the points of intersection of the circles $r = ih$ $(i = 0,1,2,\ldots, i_0, i_0 + 1,\ldots,N)$ and the straight lines $\theta = j\delta\theta$ $(j = 0,1,\ldots,M)$.

Let $\vec{X}(ih,j\delta\theta) = (X^1_{i,j}, X^2_{i,j}, X^3_{i,j})$. The value of $X^\tau_{i,j}$ for $0 \le i \le i_0 - 1$, $1 \le j \le M$ and $\tau = 1,2,3$ are obtained from Poisson's integral

$$X^\tau_{i,j} = \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - (ih)^2)H^\tau(\alpha)}{1 + i^2h^2 - 2(ih)\cos(\alpha - j\delta\theta)} \, d\alpha \quad . \tag{5.5}$$

We compute the integral of Eq. (5.5) by the compound Simpson's rule.

To obtain the value of $X^\tau_{i,j}$ for $N \ge i \ge i_0$, $1 \le j \le M$, $\tau = 1,2,3$ we use the following.

Consider Laplace's equation in polar coordinates

$$\frac{\partial^2 X^\tau}{\partial r^2} + \frac{1}{r}\frac{\partial X^\tau}{\partial r} + \frac{1}{r^2}\frac{\partial^2 X^\tau}{\partial\theta^2} = 0 , \quad \tau = 1,2,3 \quad . \tag{5.6}$$

Then Laplace's equation at the point (i,j) may then be approximated by

$$\frac{X^\tau_{i+i,j} - 2X^\tau_{i,j} + X^\tau_{i-1,j}}{h^2} + \frac{1}{ih}\,\frac{(X^\tau_{i+1,j} - X^\tau_{i-1,j})}{2h} +$$

$$\frac{1}{(ih)^2}\,\frac{(X^\tau_{i,j+1} - 2X^\tau_{i,j} + X^\tau_{i,j-1})}{(\delta\theta)^2} = 0 \tag{5.7}$$

giving

$$(1 - \frac{1}{2i})X^{\tau}_{i-1,j} + (1 + \frac{1}{2i})X^{\tau}_{i+1,j} - 2(1 + \frac{1}{(i\delta\theta)^2})X^{\tau}_{i,j} + \frac{1}{(i\delta\theta)^2} X^{\tau}_{i,j-1} + \frac{1}{(i\delta\theta)^2} X^{\tau}_{i,j+1} = 0 . \tag{5.8}$$

If these equations are written out in detail for $i = i_0$, $i_0 + 1,\ldots,N$ and $j = 1,2,\ldots,M$ and by using the relation

$$X^{\tau}_{i,j} = X^{\tau}_{i,j+M} \tag{5.9}$$

then it will be found that their matrix form is

$$AX^{\tau} = d \tag{5.10}$$

where X^{τ} , d are column vectors whose transposed are

$$X^{\tau} = (X^{\tau}_{i_0}, X^{\tau}_{i_0+1}, \ldots, X^{\tau}_{n}) , \tag{5.11}$$

where

$$X^{\tau}_{k} = (X^{\tau}_{k,1}, X^{\tau}_{k,2}, \ldots, X^{\tau}_{k,M}) , \quad i_0 \leq k \leq N , \tag{5.12}$$

$$d_1 = (1 - \frac{1}{2(i_0 - 1)})X_{i_0 - 1} \tag{5.13}$$

$$d_2 = d_3 = \ldots = d_{N-1} = 0 \tag{5.14}$$

$$d_N = (H^{\tau}_1, \ldots, H^{\tau}_M) , \quad H^{\tau}_j = H^{\tau}(j\delta\theta) . \tag{5.15}$$

The matrix A is given by

$$A = \begin{bmatrix} D_1 & (1 + \frac{1}{2})I & & & \\ (1 - \frac{1}{2\cdot 2})I & D_2 & (1 + \frac{1}{2\cdot 2})I & & \\ & & & & \\ & (1 - \frac{1}{2(N-1)})I & D_{N-1} & (1 + \frac{1}{2(N-1)})I \\ & & (1 - \frac{1}{2\cdot N})I & D_N \end{bmatrix} \tag{5.16}$$

where each D_{ℓ} and I are $M \times M$ matrices and

$$D_\ell = \begin{bmatrix} -2-2a_\ell & a_\ell & & a_\ell \\ a_\ell & -2-2a_\ell & a_\ell & \\ & a_\ell & -2-2a_\ell & a_\ell \\ a_\ell & & & -2-2a_\ell \end{bmatrix} \tag{5.17}$$

$a_\ell = \dfrac{1}{(\ell\delta\theta)^2}$.

To solve Eq. (5.10) we use the same method as in [10].

We factorized the supermatrices in the form

$$A = LU \tag{5.18}$$

$$L = \begin{bmatrix} I & & \\ L_2 & I & \\ & L_N & I \end{bmatrix} \quad U = \begin{bmatrix} U_1 & V_1 & & & \\ & U_2 & V_2 & & \\ & & & & \\ & & & U_{N-1} & V_{N-1} \\ & & & & U_N \end{bmatrix} \tag{5.19}$$

and

$$U_1 = D_1 \ , \quad V_1 = (1 + \tfrac{1}{2})I$$

$$m > 1 \begin{cases} L_m = (1 - \frac{1}{2m})D_{m-1}^{-1} \ , \quad V_m = D_m - (1 - \frac{1}{2\cdot m})D_{m-1}^{-1}(1 + \frac{1}{2(m-1)}) \\ V_m = (1 + \frac{1}{2\cdot m})I \end{cases} \tag{5.20}$$

This method has been described by Wilson [11] though not actually for elliptic difference equations but for equation of a similar form.

To solve Eq. (5.10) we first solve the following equation

$$LY = d \ , \quad Y = (Y_1, \ldots, Y_N) \ . \tag{5.21}$$

The solution of Eq. (5.21) is given by

$$Y_1 = d_1 \qquad \text{(see Eqs. 5.13-5.14)} \tag{5.22}$$

$$Y_k = -L_k Y_{k-1} \,, \quad N-1 \geq k \geq 2 \tag{5.23}$$

$$Y_N = d_N - L_N Y_{N-1} \,. \tag{5.24}$$

Now we solve the equation

$$UX^{\tau} = Y \,, \tag{5.25}$$

by solving the following equations

$$U_N X_N^{\tau} = Y_N \tag{5.26}$$

$$U_{N-j} X_{N-j}^{\tau} + V_{N-j} X_{N-j+1}^{\tau} = Y_{N-j} \,, \quad N-1 \geq j \geq 1 \,. \tag{5.27}$$

Thus we obtain the values of $X_{i,j}^{\tau}$ for all $1 \leq i \leq N$, $1 \leq j \leq M$. We do these calculations for $\tau = 1,2,3$.

In the next step we calculate the values of $\frac{\partial X^{\tau}}{\partial r}$, $\frac{\partial X^{\tau}}{\partial \theta}$ at the points (i,j) , $i = 1,\ldots,N+1$, $j = 0,1,\ldots,M$ by standard difference equation method, then we compute the integral

$$D = \iint_{\mathcal{D}} \left\{ \sum_{\tau=1}^{3} \left[\left(\frac{\partial X^{\tau}}{\partial r}\right)^2 r + \frac{1}{r}\left(\frac{\partial X^{\tau}}{\partial \theta}\right)^2 \right] \right\} dr\, d\theta, \tag{5.28}$$

by approximating it by a generalization of Simpson rule [12].

In the third step we compute the value

$$E = \max_{1 \leq j \leq M} \sum_{\tau=1}^{3} \left(\frac{\partial X^{\tau}}{\partial r}\right)_{N+1,j} \left(\frac{\partial X^{\tau}}{\partial \theta}\right)_{N+1,j} \quad \text{(see Eq. (3.8))}. \tag{5.29}$$

In our example we first compute the value of the integral D and of E . We do this by choosing of $\{A_j\}$, $\{B_j\}$ $(j = 1,\ldots,k)$ in a random way and such that $\{A_j\}$, $\{B_j\}$ satisfies Eqs. (4.12), (4.13). For $|E|$ not sufficiently big we stop the random process and then we use gradient method [9].

To use the gradient method we calculate the value of the gradient by approximating the integrals

$$\frac{\partial\delta}{\partial A_j} = 2 \sum_{\tau=1}^{3} \int_0^{2\pi} \frac{\partial X^\tau}{\partial r} (1,\theta)T^\tau(g(\theta))\cos j\theta \ d\theta \tag{5.30}$$

$$\frac{\partial\delta}{\partial B_j} = 2 \sum_{\tau=1}^{3} \int_0^{2\pi} \frac{\partial X^\tau}{\partial \theta} (1,\theta)T^\tau(g(\theta))\sin j\theta \ d\theta \tag{5.31}$$

As before we approximate the integrals (5.30), (5.31) by the compound Simpson's rule.

We halted our process when the values of $|E|$ were smaller than ε .

In the following table we see the numerical results for $\varepsilon = 2 \cdot 10^{-3}$, $k = 10$ and $N = 21$, $M = 31$, $i_0 = 10$.

In Table I we present a selected result that was obtained by random choices of A_j , B_j . In Table II we see selected results that were obtained by using gradient method. The intial value of $\{A_j\}$, $\{B_j\}$ for the gradient method are the best results obtained by random selection. In Figure 1, we show the minimal surface drawn from the values of $X^\tau_{i,j}$, and using the closed curve C given in (5.1).

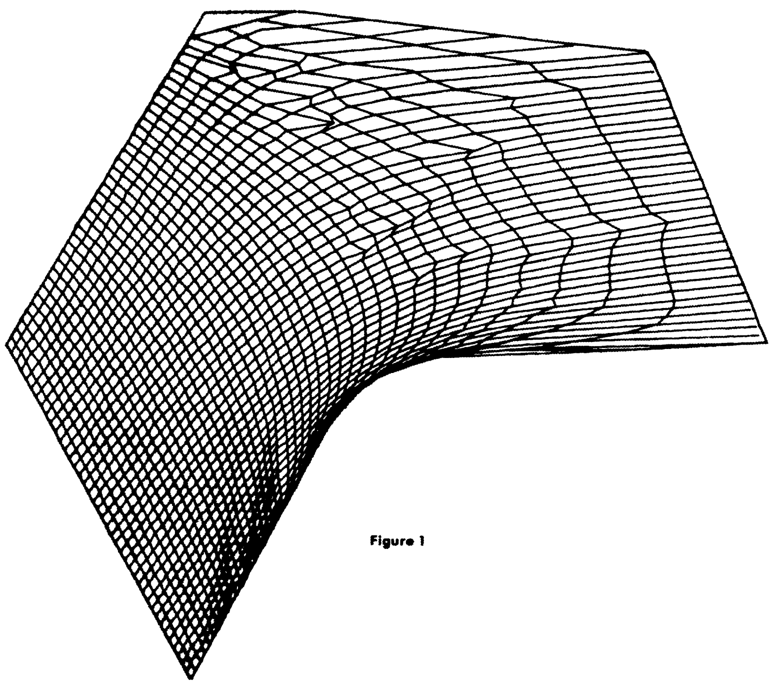

Figure 1

Table I

Calculations by Random Choice for the Coefficients of (A_j, B_j)

Value of Integral D	Value of $\|E\|$
134598.84615	30000.70486
12036.91971	246.44340
1811.33271	1238.02080
711.49210	18.52716
272.28744	78.04352
256.88041	42.43274
181.60385	43.24106
101.73822	59.30201
73.75771	20.54170
53.35582	9.87578
49.01988	13.00610
33.56237	16.20803
24.76756	9.08349
14.99357	12.11406
9.34689	6.42093
8.00960	9.95630
7.97180	4.27323
7.00766	6.51285
6.67786	3.15938
5.42465	3.54083

Table II

Calculations by Gradient Method for the Coefficients of (A_j, B_j)

Value of Integral D	Value of $\lvert E \rvert$
5.42465	3.54083
5.33672	2.86892
5.11811	2.57031
4.96691	1.05302
4.87354	0.47714
4.77961	0.34489
4.75412	0.68156
4.703091	0.81391
4.59962	0.46404
4.58185	0.45846
4.50000	0.39333
4.41726	0.32073
4.38645	0.22813
4.32602	0.19501
4.30832	0.24215
4.28768	0.36908
4.24679	0.22170
4.23814	0.13596
4.13554	0.02041
4.085048	0.00174

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