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A NOTE ON SOLUTIONS OF SCHIFFER'S DIFFERENTIAL EQUATION

RONALD J. LEACH
Department of Mathematics
Howard University
Washington, D.C. 20059

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ABSTRACT. We consider Schiffer's differential equation for functions in the class of normalized univalent functions which maximize the $n^{\underline{th}}$ coefficient. By considering a class of functionals converging to the $n^{\underline{th}}$ coefficient functional, we determine some additional symmetries that extremal functions possess.

KEY WORDS AND PHRASES. Univalent function, Schiffer's differential equation, variation, Marty relation.

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1. INTRODUCTION.

Let S denote the class of functions $f(z) = z + a_2z^2 + \ldots$, analytic and univalent in the unit disc D. S is well-known to be compact in the topology of uniform convergence on compact subsets of D. Therefore, variational problems of the form $\operatorname{Re} \phi(f) = \max$ must have solutions in S whenever ϕ is a continuous functional on S. By constructing univalent variations, Schiffer [1, 2] showed that an extremal function f for $\phi(f) = \operatorname{Re} a_n$ must satisfy the differential equation

$$\left(\frac{\zeta f'(\zeta)}{f(\zeta)}\right)^2 p(f(\zeta)) = q(\zeta) \qquad \zeta \in D \tag{1.1}$$

where p and q are rational functions and $q(e^{i\theta}) \geq 0$. The coefficients of p and q depend on the function f and therefore (1.1) is a functional-differential equation. For examples of the uses of the variational method, see [1-10]. For consistency, we follow the notation in Pommerenke [8, p. 183-190]. It is shown there that f is a solution to the problem of maximizing $\operatorname{Re} a_n$; then

$$p(f(\zeta)) = \frac{1}{n}\, \phi_n'\left(\frac{1}{f(\zeta)}\right) - a_n, \tag{1.2}$$

$$q(\zeta) = (n-1)a_n + \sum_{j=1}^{n-1} (ja_j\zeta^{-(n-j)} + j\bar{a}_j\zeta^{n-j}). \tag{1.3}$$

Here $\phi_n(\frac{1}{f(\zeta)})$ is the familiar Faber polynomial of degree n for $\frac{1}{f(\zeta)}$. Equation (1.1) with p and q defined by (1.2) and (1.3) respectively is the Schiffer differential equation which any function maximizing Re a_n must satisfy. (Unfortunately, there are other solutions that are not extremal functions such as $z(1-z^2)^{-1}$ in the case n = 3.)

In this note, we consider a class of functionals $T_r(f)$ that converge to $\Phi(f)$ = Re a_n as $r \to 0$. We compute the Schiffer differential equation for each of these functionals and obtain new conditions that the extremal functions must satisfy. In certain cases, we show that the extremal function must satisfy an infinite system of differential equations. The equations in this system are of the form (1.1) and have the unknown coefficients of the extremal function appear in the equation.

2. THE MAIN THEOREM.

We will need the following result whose proof is an immediate consequence of the formula for the sum of a geometric progression.

<u>Lemma</u>. Let $g(z) = \sum_{n=0}^{\infty} b_n z^n$. Then $\sum_{i=0}^{n-1} g\left(re^{\frac{2\pi ij}{n}}\right) = n \sum_{i=0}^{\infty} b_{kn} r^{kn}$.

<u>THEOREM 1.</u> Let f be a function in S which maximizes $T_r(f) = \text{Re}\, \frac{1}{nr^n} \sum_{j=0}^{n-1} f(z_j)$ where where $z_j = re^{\frac{2\pi ij}{n}}$, $r > 0$. Then

i) with the notation $B_j = f(z_j)$, f must satisfy

$$\left(\frac{zf'(z)}{f(z)}\right)^2 \sum_{j=0}^{n-1} \frac{B_j^2}{f(z)-B_j} = \frac{1}{2} \sum_{j=0}^{n-1} z_j f'(z_j) \frac{z+z_j}{z-z_j} + \frac{1}{2} \sum_{j=0}^{n-1} \overline{z_j}\, \overline{f'(z_j)}\, \frac{1+z_j z}{1-\overline{z_j} z} - \text{Re} \sum_{j=0}^{n-1} f(z_j) \tag{2.1}$$

ii) as $r \to 0$, the functions f (which may depend on r) approach a function in S which maximizes Re a_n.

<u>PROOF</u>. We follow the outline in Pommerenke [8, p. 183-190]. T_r is a linear functional of degree n and consequently

$$\left(\frac{\zeta f'(\zeta)}{f(\zeta)}\right)^2 p(f(\zeta)) = q(\zeta)$$

where

$$p(w) = \sum_{j=0}^{n-1} \frac{B_j^2}{w - B_j}$$

and

$$q(\zeta) = \frac{1}{2}\sum_{j=0}^{n-1} z_j f'(z_j)\frac{\zeta + z_j}{\zeta - z_j} + \frac{1}{2}\overline{\sum_{j=0}^{n-1} z_j f'(z_j)\frac{1 + \bar{\zeta} z_j}{1 - \bar{\zeta} z_j}} - \operatorname{Re}\sum_{j=0}^{n-1} f(z_j).$$

This proves the first statement of the theorem if we replace ζ by z. To prove the second statement, note that the lemma implies

$$\sum_{j=0}^{n-1} f\left(e^{\frac{2\pi i j}{n}}\right)r = na_n r^n + na_{2n} r^{2n} + \ldots$$

and hence

$$T_r(f) = a_n + O(r^n).$$

If $\varepsilon > 0$ is given, we may choose r so that $O(r^n) < \varepsilon$. Then any function f_0 maximizing $\operatorname{Re} a_n$ has

$$\operatorname{Re} T_t(f) < \operatorname{Re} a_n + \varepsilon$$

and hence f_0 is the limit of functions maximizing $\operatorname{Re} T_r f$ as $r \to 0$.

REMARKS. 1. It is well-known that a function that maximizes $\operatorname{Re} a_n$ actually has $a_n > 0$.

2. If $f(z) = z + a_2 z^2 + \ldots + a_n z^n + \ldots$ maximizes $\operatorname{Re} a_n$, then so do the functions

$$e^{-\frac{2\pi k j}{n-1}} f\left(e^{\frac{2\pi i j}{n-1}} z\right) = z + e^{\frac{2\pi i j}{n-1}} a_2 z^2 + \ldots + a_n z^n + \ldots .$$

Our technique of approximating $\operatorname{Re} a_n$ by $\operatorname{Re} T_r(f)$ will yield only one of the rotations of f; the others can be obtained by considering replacing r by $r_j = e^{\frac{2\pi i j}{n-1}} r$. This observation will explain some later results.

COROLLARY 1. There is a function $f \in S$ which maximizes $\operatorname{Re} a_n$ for which

$$2a_2 a_n = (n+1)a_{n+1} - (n-1)\overline{a_{n-1}}.$$

PROOF. Let $F(z)$ denote the expression appearing in (2.1). We fix $r > 0$ and expand both representations for $F(z)$ about $z = 0$ obtaining, since $f(z) = z + a_2 z^2 + \ldots$,

$$-\sum_{j=0}^{n-1} B_j + (-n - 2a_2\sum_{j=0}^{n-1} B_j)z + O(z^2) = -\frac{1}{2}\sum_{j=0}^{n-1}\left[z_j f'(z_j)\frac{1 + z/z_j}{1 - z/z_j}\right] +$$

$$+\frac{1}{2}\sum_{j=0}^{n-1}\overline{z_j}\,\overline{f'(z_j)}\frac{1 + \overline{z_j} z}{1 - \overline{z_j} z} - \operatorname{Re}\sum_{j=0}^{n-1} B_j = -\frac{1}{2}\sum_{j=0}^{n-1}\left[z_j f'(z_j)(1 + \frac{2z}{z_j} + O(z^2))\right]$$

$$+\frac{1}{2}\sum_{j=0}^{n-1}\left[\overline{z_j}\,\overline{f'(z_j)}\,(1 + 2\overline{z_j} z + O(z^2)\right] - \operatorname{Re}\sum_{j=0}^{n-1} f(z_j) = -i\operatorname{Im}\sum_{j=0}^{n-1} z_j f'(z_j)$$

$$- \operatorname{Re} \sum_{j=0}^{n-1} f(z_j) + \left(\sum_{j=0}^{n-1} [-f'(z_j) + \overline{z_j}^2 f'(z_j)] \right) z + O(z^2).$$

Equating coefficients, since $B_j = f(z_j)$, we obtain

$$- \sum_{j=0}^{n-1} f(z_j) = -i\operatorname{Im} \sum_{j=0}^{n-1} z_j f'(z_j) - \operatorname{Re} \sum_{j=0}^{n-1} f(z_j) \tag{2.2}$$

$$-n - 2a_2 \sum_{j=0}^{n-1} f(z_j) = - \sum_{j=0}^{n-1} f'(z_j) + \sum_{j=0}^{n-1} \overline{z_j}^2 \overline{f'(z_j)} \tag{2.3}$$

Applying the argument of the lemma to (2.2) and (2.3), we obtain

$$-na_n r^n + O(r^{2n}) = -i\operatorname{Im} n(n+1)a_{n+1} r^n - \operatorname{Re} na_n r^n + O(r^{2n}) \tag{2.4}$$

$$-n - 2a_2 na_n r^n + O(r^{2n}) = -n - n(n+1)a_{n+1} r^n + n(n-1)\overline{a_{n-1}}\, r^n + O(r^{2n}). \tag{2.5}$$

Upon dividing (2.5) by r^n and letting $r \to 0$, we obtain

$$2a_2 a_n = (n+1)a_{n+1} - (n-1)\overline{a_{n-1}}\ .$$

REMARK. The conclusion of the corollary is the well-known Marty relation. It was originally derived by very elementary methods. Hummel [6, p. 77] observed that the Marty relation can also be obtained by considering the Schiffer differential equation for the functional $\operatorname{Re} a_n$.

3. CONSEQUENCE OF THE MAIN THEOREM.

THEOREM 2. Suppose that a fixed function f maximizes $\operatorname{Re} T_r(f)$ for some sequence of r's converging to 0. Then f satisfies the system of functional-differential equations

$$\left(\frac{zf'(z)}{f(z)}\right)^2 \frac{1}{kn} \phi'_{kn}\left(\frac{1}{f(z)}\right) - a_n = (kn-1)a_{kn} + \sum_{n=1}^{kn-1} (ja_j z^{-(kn-j)} + j\bar{a}_j z^{kn-j}),\ k = 1,2,\ldots \tag{3.1}$$

where $\phi_{kn}(w)$ is the knth Faber polynomial for W.

PROOF. By Theorem 1, f must satisfy the functional-differential equation

$$\left(\frac{zf'(z)}{f(z)}\right)^2 \sum_{j=0}^{n-1} \frac{B_j^2}{f(z) - B_j} = \frac{1}{2} \sum_{j=0}^{n-1} z_j f'(z_j) \frac{z + z_j}{z - z_j} + \frac{1}{2} \sum_{j=0}^{n-1} \overline{z_j}\, \overline{f'(z_j)} \frac{1 + \overline{z_j} z}{1 - \overline{z_j} z} - \operatorname{Re} \sum_{j=0}^{n-1} f(z_j) \tag{3.2}$$

where $B_j = f(z_j) = f\left(re^{\frac{2\pi ij}{n}}\right)$. For fixed z, the expression F(z) defined by (3.2)

is an analytic function of r for r in some small interval about 0. We expand (3.2) in powers of r noting that the lemma insures that only powers of r^{kn} can appear. We show the argument only for powers of r^n since the computation for higher powers is similar.

$$\left(\frac{zf'(z)}{f(z)}\right)^2 \sum_{j=0}^{n-1} \left[\frac{B_j^2}{f(z)}\left(1 + \frac{B_j}{f(z)} + \frac{B_j^2}{f(z)^2} + \dots\right)\right]$$

$$= \frac{1}{2}\sum_{j=0}^{n-1} z_j f'(z_j)\left(1 + \frac{2z_j}{z} + \dots\right)$$

$$+ \frac{1}{2}\sum_{j=0}^{n-1} \overline{z_j}\, f'(z_j)(1 + 2\overline{z_j}z + \dots) - \operatorname{Re}\sum_{j=0}^{n-1} f(z_j).$$

$$\left(\frac{zf'(z)}{f(z)}\right)^2 \left[nr^n\left(\frac{C_1(n)}{f(z)} + \frac{C_2(n)}{f(z)^2} + \dots + \frac{1}{f(z)^{n-1}}\right) + O(r^{2n})\right]$$

$$= \frac{1}{2}n^2 a_n r^n + O(r^{2n}) + \sum_{j=0}^{n-1}\left[z_j^2\frac{f'(z_j)}{z} + z_j^2\frac{f'(z_j)}{z^2} + \dots\right] + \frac{1}{2}n^2\overline{a_n}r^n$$

$$+ O(r^{2n}) + \sum_{j=0}^{n-1}\left[\overline{z_j}^2\,\overline{f'(z_j)}z + \overline{z_j}^3\,\overline{f'(z_j)}z^2 + \dots\right] - \operatorname{Re} na_n r^n + O(r^{2n})$$

$$= [(n^2 - n)a_n + n\sum_{j=1}^{n-1}(ja_j z^{-(n-j)} + j\,\bar{a}_j z^{n-j})]r^n + O(r^{2n}).$$

The coefficients $C_m(n)$ are obtained in the following manner:

$$nr^n C_1(n) = \sum_{j=0}^{n-1} B_j^2$$

$$= \sum_{j=0}^{n-1} (f(z_j))^2$$

$$= \sum_{j=0}^{n-1} (z_j + a_2 z_j^2 + \dots)^2$$

$$= \sum_{j=0}^{n-1}\sum_{m=1}^{\infty}\sum_{m_1+m_2=m} a_{m_1}a_{m_2}\left(re^{\frac{2\pi ij}{n}}\right)^m$$

$$= nr^n \sum_{m_1+m_2=n} a_{m_1}a_{m_2} + O(r^{2n}).$$

$$nr^n C_2(n) = \sum_{j=0}^{n-1} B_j^3$$

$$= \sum_{j=0}^{n-1} (z_j + a_2 z_j^2 + \ldots)^3$$

$$= \sum_{j=0}^{n-1} \sum_{m=1}^{\infty} \Sigma a_{m_1} a_{m_2} a_{m_3} \left(re^{\frac{2\pi ij}{n}}\right)^m \tag{3.3}$$

$$= nr^n \Sigma a_{m_1} a_{m_2} a_{m_3} + O(r^{2n}). \tag{3.4}$$

The sum $\Sigma a_{m_1} a_{m_2} a_{m_3}$ is taken over all positive integers m_1, m_2, m_3 with $m_1 + m_2 + m_3 = n$. This procedure yields in general

$$nr^n C_\ell(n) = nr^n \Sigma a_{m_1} \ldots a_{m_{\ell+1}}$$

where $m_1 + m_2 + \ldots + m_\ell = n$, $m_i > 0$. We recognize $\Sigma_{\ell=1}^{n-1} C_\ell(n) f^{-\ell}$ as $\frac{1}{n}\phi_n'(\frac{1}{f(z)}) - a_n$ [8, p. 57]. This proves the result if $k = 1$. The other equations for $k = 2, \ldots$ are obtained in a similar manner by equating coefficients of higher powers of r^n.

REMARKS. 1. A result of Pfluger [7] shows that a Koebe function $k(z) = z(1 - e^{i\theta}z)^{-2}$ always satisfies (1.1).

2. The assumption that f is essentially the only extremal function for the problem of maximizing Re a_n is used quite strongly in this proof. If there were more than one function, the coefficients of the extremal function for T_r would depend upon r, making the functional-differential equations even more complicated. It seems reasonable to suppose that there is essentially one extremal function (apart from rotations) for each n but we are unable to prove this.

3. The equation for $k = 1$ is of course the familiar Schiffer differential equation for a function f maximizing Re a_n. The nature of this family of equations suggests that, if $f(z) = z + a_2 z^2 + \ldots$ is a function which maximizes Re a_n, f also maximizes Re a_{2n}, Re $a_{3n}, \ldots$. If so, the Bieberbach conjecture would follow from a result of Hayman [11, p. 104]. He showed that, if $f \in S$, $\lim_{k\to\infty} \frac{|a_k|}{k} = \alpha \leq 1$, with $\alpha = 1$ only if $f(z) = z(1 - e^{i\theta}z)^{-2}$.

THEOREM 3. Suppose that f satisfies the hypothesis of Theorem 2. Then f satisfies the functional equation

$$\frac{\frac{1}{kn}\phi'_{kn}(\frac{1}{f(z)}) - a_{kn}}{\frac{1}{\ell n}\phi'_{\ell n}(\frac{1}{f(z)}) - a_{\ell n}} = \frac{(kn-1)a_{kn} + \sum_{n=1}^{kn-1}(ja_j z^{-(kn-j)} + j\overline{a_j} z^{kn-j})}{(\ell n-1)a_{\ell n} + \sum_{j=1}^{n-1}(ja_j z^{-(\ell n-j)} + j\overline{a_j} z^{\ell n-j})} \tag{3.5}$$

and hence f is algebraic.

PROOF. Divide the kth equation in the system (3.1) by the ℓth equation.

The following result is of interest only if the Bieberbach conjecture is false.

THEOREM 4. Suppose there is a function f not of the form $z(1 - e^{i\theta}z)^{-2}$ and that f satisfies the hypothesis of Theorem 2. Then there is a number θ_0 such that $e^{i\theta_0}$ is simultaneously a zero of

$$q_k(z) = (kn-1)a_{nk} + \sum_{j=0}^{kn-1}(ja_j e^{-(kn-j)i\theta} + j\overline{a_j} e^{(kn-j)i\theta}) \quad k = 1,2,\ldots$$

PROOF. Pfluger [7] has shown that if f is a function that maximizes Re a_n, then $\mathrm{Re}[\frac{1}{n}\phi'_n(\frac{1}{f(z)}) - a_n] < 0$ unless f is a rotation of the Koebe function. (He actually proved this theorem for any linear functional and the rational function p related to it by (1.2).) It is well-known that $\frac{1}{n}\phi'_n(\frac{1}{f(z)}) - a_n = 0$ if and only if f is a Koebe function. (See [8, p. 194], [6, Theorem 13.6].)

We consider equation (3.5) with $\ell = 1$. It is well-known that since f maximizes Re a_n, the function q defined by (1.3) must have at least one zero on $z = 1$. Since the left-handed side of (3.5) is analytic by assumption, each zero $e^{i\theta}$ of

$$q_1(e^{i\theta}) = (n-1)a_n + \sum_{j=1}^{n-1}(ja_j e^{-(n-j)i\theta} + j\overline{a_j} e^{-(n-j)i\theta}$$

must also be a zero of $q_k(e^{i\theta})$, $k = 1,2,\ldots$. This completes the proof.

THEOREM 5. Suppose that f satisfies the hypothesis of Theorem 2. Then

i) a_{kn} is real $k = 1,2,\ldots$

ii) $2a_2 a_{kn} = (kn-1)a_{kn+1} - (kn-1)\overline{a_{kn-1}}$.

PROOF. Since f is essentially the unique function maximizing Re a_n, the equations (2.2) and (2.3) are valid for all $r = |z_j|$ in some neighborhood of 0. Equating coefficients of r^{kn} in (2.2) yields, after an application of the lemma,

$$-na_{kn} = in \operatorname{Im} kn\, a_{kn} - n \operatorname{Re} a_{kn}$$

or

$$-i \operatorname{Im} a_{kn} = -i \operatorname{Im} k\, na_{kn}$$

which implies that a_{kn} is real.

Equating coefficients in (1.6) and applying the lemma, we obtain

$$-2a_2 n\, a_{kn} = -n(kn + 1)a_{kn+1} + n(nk - 1)\overline{a_{nk-1}}$$

and the result follows after division by -n.

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