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ISOMORPHISMS OF SEMIGROUPS OF TRANSFORMATIONS

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ABSTRACT. If M is a centered operand over a semigroup S, the suboperands of M containing zero are characterized in terms of S-homomorphisms of M. Some properties of centered operands over a semigroup with zero are studied.

A Δ-centralizer C of a set M and the semigroup $S(C,\Delta)$ of transformations of M over C are introduced, where Δ is a subset of M. When $\Delta = M$, M is a faithful and irreducible centered operand over $S(C,\Delta)$. Theorems concerning the isomorphisms of semigroups of transformations of sets M_i over Δ_i-centralizers C_i, i = 1,2 are obtained, and the following theorem in ring theory is deduced: Let L_i, i = 1,2 be the rings of linear transformations of vector spaces (M_i,D_i) not necessarily finite dimensional. Then f is an isomorphism of $L_1 \to L_2$ if and only if there exists a 1-1 semilinear transformation h of M_1 onto M_2 such that $fT = hTh^{-1}$ for all $T \in L_1$.

KEY WORDS AND PHRASES. Semigroups of transformations, operand over a semigroup.
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0. INTRODUCTION AND PRELIMINARIES.

In recent times Tully [1], Hoehnke [2], and others have studied the theory of representations of a semigroup by transformations of a set. This paper deals with the study of a certain class of such representations (see Theorem 2.1). In section 1 we define an 0-suboperand of a centered operand M over a (general) semigroup and characterize the same in terms of operand homomorphisms of M. Some properties of centered operands over semigroups with zero are discussed in Section 2. In Section 3 we introduce the concept of a Δ-centralizer C of a set M (with $|M| \geq 2$), for any non-empty subset Δ of M, and define the semigroup $S(C,\Delta)$ of transformations of a set M over C as the set of all self-maps of M which commute with every member of C. We observe that M is a faithful centered operand over $S(C,\Delta)$, and also is irreducible in the case when $\Delta = M$.

In Section 4 we obtain results (Theorems 4.1 and 4.2) which are comparable with Theorem 17.3 of [2], concerning the isomorphisms of semigroups of transformations of sets M_i over centralizers C_i, for i = 1,2, which generalize a similar result concerning the isomorphisms of near-rings of transformations of groups (as also analogous results for loop-near-rings) - Theorem 2.6 of Ramakotaiah [3]; then we thereby deduce the following well-known isomorphism theorem in ring theory (see, for instance, Jacobson [4]): Let L_i, i = 1,2 be the rings of linear transformations of vector spaces (M_i, D_i) not necessarily finite dimensional. Then f is an isomorphism of $L_1 \to L_2$ if and only if there exists a 1-1 semilinear transformation h of M_1 onto M_2 such that $fT = hTh^{-1}$ for all $T \in L_1$.

Throughout this paper, by "an operand over a semigroup" we mean a left operand only. If M is a centered operand over a semigroup with zero, {0} and M are called the trivial suboperands of M. We often write 0 instead of {0}. For the definitions and results on operands, we mostly follow Clifford and Preston [5]. In Weinert [6], the terms "S-set" and "S-mapping" are used to denote "operand over S" and "S-homomorphism" respectively.

The following definitions are taken from Santha Kumari [7].

A system N = (N, +, ., 0) is called a loop-near-ring if the following conditions are satisfied:

(i) (N,+,0) is a loop, which is denoted by N^+,

(ii) (N,.) is a semigroup

(iii) $(a + b).c = a.c + b.c$ for all $a,b,c \in N$

(iv) $a.0 = 0$ for all $a \in N$.

If N is a loop-near-ring, then an additive loop $(G, +, \bar{0})$ is called an N-loop provided there exists a mapping $(n,g) \to ng$ of $N \times G \to G$, such that

(i) $(m + n)g = mg + ng$ and

(ii) $(mn)g = m(ng)$, for all $m,n \in N$ and $g \in G$.

1. 0-SUBOPERANDS OF A CENTERED OPERAND.

In this section, M denotes a centered (left) operand (see [5]) over a (general) semigroup S and 0 denotes the fixed element in M. We observe that, if $\emptyset$ is a S-homomorphism of M into a centered operand M', then $\emptyset(0) = 0$.

DEFINITION. A subset K of M is called an 0-suboperand of M if (if and only if) $S K \subseteq K$ (that is, K is a suboperand of M) and $0 \in K$.

THEOREM 1.1. A subset K of M is an 0-suboperand if and only if $K = \emptyset^{-1}(0)$ for some S-homomorphism $\emptyset$ of M.

PROOF. Suppose a subset K of M is a 0-suboperand of M. Let M/K denote the Rees factor operand corresponding to the suboperand K of M and let π: $M \to M/K$ be the canonical S-homomorphism. Clearly $\pi(x) = K$ if and only if $x \in K$. Thus $K \in M/K$ and, moreover, K is a fixed element of M/K. In fact, K is the only fixed element of M/K. For, if $\pi(t)$ is one such element, then $\pi(t) = s\ \pi(t) = \pi(st)$ for all $s \in S$ and this gives that either t, st, or both belong to K for some $s \in S$ or $t = st$ for all $s \in S$; in any case, we get that $\pi(t) = K$. Hence M/K is a centered operand over S with K as its zero and $\pi^{-1}(K) = K$.

The converse part can be easily proved by direct verification.

REMARKS 1.2. Clearly $\{0\}$ is the smallest 0-suboperand and M is the largest, under set inclusion. Also, the family F of all 0-suboperands of M is closed under arbitrary unions and intersections. Hence, F is a complete lattice under set inclusion, with set union and set intersection as the lattice operations.

It is a straightforward verification to see that

PROPOSITION 1.3. Let M' be a centered operand over S and let $\emptyset$: $M \to M'$ be a S-homomorphism. Then, (a) for every 0-suboperand K of M, $\emptyset(K)$ is a 0-suboperand of M' and (b) for every 0-suboperand K' of M', $\emptyset^{-1}(K')$ is a 0-suboperand of M.

PROPOSITION 1.4. Let K be a 0-suboperand of M and let M/K denote the Rees factor operand corresponding to K. Let π: $M \to M/K$ be the canonical homomorphism. Then, (a) a subset B of M/K is a 0-suboperand of M/K if and only if $\pi^{-1}(B)$ is a 0-suboperand of M and (b) $A \to \pi(A)$ is a one-to-one correspondence between the suboperands of M containing K and the 0-suboperands of M/K.

PROOF. (a) follows from Proposition 1.3, and the proof of (b) is routine.

2. ALMOST IRREDUCIBLE SUBOPERANDS AND ANNIHILATORS.

In this section we concentrate on centered operands over semigroups with zero, and our study is motivated by the following:

THEOREM 2.1. Let S be a semigroup with zero. Then, there exists a one-to-one correspondence between the representations $\emptyset$ of S by transformations of a set such that $\emptyset(0)$ is a constant map and the centered (left) operands over S.

PROOF. Let T_M denote the full transformation semigroup of a set M and let $\emptyset$: $S \to T_M$ be a representation of S such that $\emptyset(0)$ is a constant map. Now M is an operand over S with multiplication defined by $a.x = \emptyset(a)(x)$ for all $a \in S$, $x \in M$. Let $\emptyset(0)(M) = \{t\}$. For any $a \in S$, $a.t = \emptyset(a)(t) = \emptyset(a)(\emptyset(0)(t)) = (\emptyset(a)\emptyset(0))(t) = \emptyset(a0)(t) = \emptyset(0)(t) = t$ and so t is a fixed element of M. On the other hand, if y is a fixed element of M, then we have $y = \emptyset(0)(y) = t$. Hence M is a centered operand over S. Conversely, if M is a centered operand over S, then the map $\emptyset$: $S \to T_M$ given by $\emptyset(a)(x) = a.x$ for all $a \in S$, $x \in M$ is a representation of S by transformations of M such that $\emptyset(0)(x) = 0$ for all $x \in M$. Hence the result.

Throughout the rest of this section, S denotes a semigroup with zero and $M \neq 0$ denotes a centered (left) operand over S. For any centered operand N over S and a suboperand K of N, N/K denotes the Rees factor operand corresponding to K.

DEFINITION. M is said to be almost irreducible (a.irreducible) if M has no nontrivial suboperands.

REMARKS 2.2. Clearly, irreducibility (see [5]) implies a.irreducibility. Also, a.irreducibility implies irreducibility except possibly in the case when M has exactly two elements (also see Proposition 2.4 below). We use the term 'monogenic' synonymous to 'strictly cyclic'. We say that M is monogenic by t (or, equivalently, t is an S-generator of M) if and only if $St = M$.

DEFINITION. M is said to be strongly monogenic if to each $t \in M$, $St = 0$ or M.

We note that M can be strongly monogenic without being monogenic. But in the presence of $SM \neq 0$, 'M is strongly monogenic' implies 'M is monogenic'. The following results are easy consequences of the above definitions.

PROPOSITION 2.3. If K is a suboperand of M and $k \in K$ is an S-generator of M, then $K = M$.

PROPOSITION 2.4. M is irreducible if and only if M is a.irreducible and monogenic.

DEFINITION. M is said to be faithful if the representation associated with M is faithful (see [1]).

DEFINITION. Let C be a nonempty subset of M. Then $\{s \in S \mid sC = 0\}$ is called the annihilator of C and is denoted by A(C). For any $t \in M$, A({t}) is denoted by A(t).

PROPOSITION 2.5. For any nonempty subset C of M, A(C) is a left ideal of S. In particular, A(t) is a left ideal of S for each $t \in M$.

PROOF. It can be directly verified that $S.A(C) \subseteq A(C)$.

PROPOSITION 2.6. If M is faithful, then A(M) = 0.

PROOF. Let $s \in A(M)$. Then, for $t \in M$ we have $st = 0 = 0t$ and this gives $s = 0$ since M is faithful.

PROPOSITION 2.7. Suppose M is a.irreducible. Then the following hold.

(a) M is strongly monogenic

(b) If L is a left ideal of S, then, for any $t \in M$, $Lt = 0$ or M.

(c) If A(M) = 0 and $0 \neq L$ is a left ideal of S, then there exists $t \in M$ such that $Lt = M$.

PROOF. (a) is obvious, since Sx is a suboperand for each $x \in M$. (b) is clear if we observe that Lt is a suboperand of M. Now we prove (c). Since A(M) = 0 and $L \neq 0$, it follows that $L \nsubseteq A(M)$. Therefore, there exists $t \in M$ such that $Lt \neq 0$; hence, $Lt = M$.

PROPOSITION 2.8. Let $0 \neq L$ be a left ideal of S. If L is a.irreducible as an operand over S (in the natural way), then L is a 0-minimal left ideal of S.

PROOF. Let J be a left ideal of S with $0 \subseteq J \subseteq L$. Then J is a suboperand of the operand L over S. Since L is a.irreducible, we have J = 0 or L. Hence the result.

DEFINITION. M is said to be smooth if any S-homomorphism $\emptyset$ of M satisfying $\emptyset^{-1}(0) = 0$ is injective.

DEFINITION. If $\emptyset$ is an S-homomorphism of M, then the congruence $\emptyset^{-1} \circ \emptyset$ is called the kernel of $\emptyset$ and is denoted by ker $\emptyset$.

PROPOSITION 2.9. The following are equivalent:

(a) M is a primitive operand (see [1]) over S.

(b) For any S-homomorphism $\emptyset$ of M, ker $\emptyset = \Delta_M$ (the diagonal of M x M) or M x M.

(c) M is smooth and a.irreducible.

PROOF. (a) $\rightarrow$ (b) is trivial. Assume (b). Let $\emptyset$ be an S-homomorphism of M with $\emptyset^{-1}(0) = 0$. Therefore, ker $\emptyset = \Delta_M$ and so $\emptyset$ is injective. Hence M is smooth. To show that m is a.irreducible, let K be a suboperand of M. Then $K = \emptyset^{-1}(0)$ for some S-homomorphism $\emptyset$ of M. But from hypothesis, if follows that $\emptyset^{-1}(0) = 0$ or M. Thus (c) is proved. Finally, assume (c). To prove (a), it is enough to prove (b), since every congruence in M is the kernel of some S-homorphism of M. Now, let $\emptyset$ be an S-homomorphism of M. Then $\emptyset^{-1}(0) = 0$ or M (since M is a.irreducible) and hence $\emptyset$ is injective or $\emptyset$ is the zero map. Therefore, ker $\emptyset = \Delta_M$ or M x M, proving (b).

THEOREM 2.10. Let M,M' be centered operands over S. Let $\emptyset$: $M \rightarrow M'$ be an S-epimorphism. Let $K = \emptyset^{-1}(0)$. If M/K is smooth over S, then M' is S-isomorphic to M/K.

PROOF. Let π: $M \rightarrow M/K$ be the canonical homomorphism. Since $K = \emptyset^{-1}(0)$, we get that ker $\pi \subseteq$ ker $\emptyset$. Therefore, "$h(\pi(x)) = \emptyset(x)$ for all $x \in M$" defines an S-epimorphism h of M/K onto M'. Further, $h(\pi(x)) = 0$ if and only if $\pi(x) = K$, which is the zero of M/K, and, since M/K is smooth, it follows that h is injective. Thus h is an isomorphism.

THEOREM 2.11. Suppose M is irreducible. For any non-zero $t \in M$, if S/A(t) is smooth over S, then A(t) is a maximal left ideal of S.

PROOF. Let $0 \neq t \in M$ and assume that S/A(t) is smooth. Since M is irreducible, t is an S-generator of M, by Lemma 11.16(B) of [5]. Therefore $A(t) \neq S$. Also, the map $\emptyset_t$: $s \rightarrow st$ from S into M is an S-epimorphism, and $\emptyset^{-1}(0) = A(t)$. Now, by Theorem 2.10, S/A(t) is isomorphic to M and therefore S/A(t) is a.irreducible. If $A(t) \subseteq L$ is a left ideal of S, then by Proposition 1.4 it follows that $\pi(L)$ is a suboperand of S/A(t) where π: $S \rightarrow S/A(t)$ is the canonical S-homomorphism. But then, $\pi(L) = A(t)$ or S/A(t) which gives that L = A(t) or S. Hence the result.

THEOREM 2.12. Suppose M is a.irreducible. Let L be a 0-minimal left ideal of S such that (i) $A \not\subseteq A(C)$ for some $C \subseteq M$ and (ii) for any S-somomorphism α of L into M, $\alpha^{-1}(0) = 0$ implies α is injective. Then L is S-isomorphic to M.

PROOF. Since $L \not\subseteq A(C)$ there exists $m \in M$ such that $Im \neq 0$. Therefore $Lm = M$ by Proposition 2.7(b). Therefore the map $\emptyset$: $\ell \to \ell m$ from L onto M is an S-epimorphism. Moreover, $\emptyset^{-1}(0)$ is a left ideal of S and is properly contained in L, and hence is 0. Therefore $\emptyset$ is injective, by (ii) of hypothesis. Hence the result.

DEFINITION. S is said to be primitive if S admits a faithful and irreducible centered operand. If M is one such operand, we say that S acts primitively on M.

Now, Theorem 2.12 yields the following, by taking M for C.

COROLLARY 2.13. Let S act primitively on M and let L be a 0-minimal left ideal of S such that, for any S-homomorphism α of L into M, $\alpha^{-1}(0) = 0$ implies α is injective. Then L is S-isomorphic to M.

3. SEMIGROUPS OF TRANSFORMATIONS OVER A CENTRALIZER.

Here we mainly introduce two concepts, namely (1) a centralizer C of a non-empty set M in a generalized form and (2) the semigroup S(C) of transformations (of M) over a centralizer C of M, and study some preliminary properties of the centered operand M over S(C). Theorem 3.7 plays the key role in deducing the corresponding results for near-rings, of [3], and loop-near-rings from some of our main results.

Throughout this section, M denotes a set with $|M| \geq 2$ and such that $0 \in M$ is a distinguished element. I denotes the identity mapping on M and $\overline{0}$, the constant map on M with range $\{0\}$.

DEFINITION. By an endomorphism of M, we mean a mapping of M into itself fixing 0. A bijective endomorphism of M is called an automorphism of M.

DEFINITION. Let be a non-empty subset of M. A set C of endomorphisms of M is called a Δ-centralizer of M if

(i) $\overline{0} \in C$

(ii) $C - \overline{0}$ is a group of automorphisms of M

(iii) $\alpha(\Delta) \subseteq \Delta$ for all $\alpha \in C - \overline{0}$

(iv) $\alpha, \beta \in C$, $0 \neq w \in \Delta$ and $\alpha(w) = \beta(w)$ imply $\alpha = \beta$.

If $\Delta = M$, then a Δ-centralizer of M is referred to as a centralizer of M.

The set $\{I,\overline{0}\}$ is a Δ-centralizer of M for any $\Delta \subseteq M$. To get a non-trivial example, take $M = \{0,a,b,c\}$ and let $C = \{I, \overline{0}, \alpha\}$ where α interchanges a and b keeping the other elements fixed. Then C is a Δ-centralizer of M where $\Delta = \{a,b\}$.

Evidently, any centralizer of a group G (see Ramakotaiah [8], Definition 2) is a centralizer of the set G (with the identity element of G acting as the distinguished element). We notice that M is a vector set in the sense of [2], over any centralizer of M.

LEMMA 3.1 Let C be a set of endomorphisms of M containing $\overline{0}$ such that $C - \overline{0}$ is a group of automorphisms of M. Then C is a Δ-centralizer of M for some subset of M containing non-zero elements of M if and only if $\bigcup_{\substack{\alpha \in C-\overline{0} \\ \alpha \neq I}} \{x \in M \mid \alpha(x)=x\} \neq M$.

PROOF. Write $F_\alpha = \{x \in M \mid \alpha(x) = x\}$ for each $\alpha \in C$, and put $\bigcup_{\substack{\alpha \in C-\overline{0} \\ \alpha \neq I}} F_\alpha = M_1$.

Suppose $M_1 \neq M$. Put $\Delta = M - M_1$. Then Δ contains a non-zero element, and we shall show that C is a Δ-centralizer of M. Let $w \in \Delta$ and $\beta \in C-\overline{0}$, with $\beta(w) \notin \Delta$. Then $\beta(w) \in M_1$ which implies that there exists $\alpha \in C-\overline{0}$, $\alpha \neq I$ such that $\alpha(\beta(w)) = \beta(w)$. Now $I \neq \beta^{-1}\alpha\beta \in C-\overline{0}$ and $\beta^{-1}\alpha\beta(w) = w$ which says that $w \notin \Delta$, a contradiction. Hence $\beta(\Delta) \in \Delta$. The rest is also similar.

Conversely, if C is a Δ-centralizer of M such that Δ contains a non-zero element say w, then it can be easily verified that $w \notin M_1$; hence $M \neq M_1$, and the proof is complete.

In the rest of this section, C denotes a non-trivial Δ-centralizer of M with $0 \in \Delta$.

DEFINITION. A mapping T of M into M is called a transformation of M over C if $T\alpha = \alpha T$ for all $\alpha \in C$.

REMARK 3.2. Any transformation of M over C fixes 0. The set of all transformations of M over C, denoted by $S(C,\Delta)$, is a semigroup with zero and unity element (under composition of mappings) and M is a centered operand over $S(C,\Delta)$ in a natural way. Moreover M is faithful. In case $\Delta = M$, we shall denote $S(C,\Delta)$ by $S(C)$. By a straightforward verification, one can see that:

PROPOSITION 3.3. For any $\alpha \in C$, $\{x \in M \mid \alpha(x) = x\}$ is a suboperand of M.

The relation $\sim$ in Δ defined by $x \sim y$ if and only if there exists $\alpha \in C - \overline{0}$ such that $\alpha(x) = y$ is clearly an equivalence relation on Δ and the equivalence classes are called the orbits of C on Δ. The following lemma can be proved on the same lines as in Lemma 8 of [8] and is a generalization of the latter.

LEMMA 3.4. Let $0 \neq w \in \Delta$ and $w' \in M$. Then there exists $T \in S(C,\Delta)$ such that (i) $T(w) = w'$ and (ii) T maps elements of M which do not belong to the orbit of w onto 0.

REMARK 3.5. It follows from Lemma 3.4 that every non-zero element of Δ is a $S(C,\Delta)$-generator of M. Hence, if C is a centralizer of M, then S(C) acts primitively on M. If M is a group (respectively loop) and Δ is a non-empty subset of M, then we can analogously define (1) a Δ-centralizer C of the group (loop) M - so that it reduces to the centralizer of the group (loop) M when $\Delta = M$ - and (2) the near-ring $N(C,\Delta)$ (loop-near-ring $L(C,\Delta)$) of all transformations of M over C. Then M is a faithful $N(C,\Delta)$-group ($L(C,\Delta)$-loop). Also, as sets, $N(C,\Delta)$ and $L(C,\Delta)$ both coincide with our $S(C,\Delta)$. Thus we have:

COROLLARY 3.6. Let M be a group (loop), and $0 \neq \Delta$, a subset of M containing 0 and C, a Δ-centralizer of the group (loop) M. Then every non-zero element of Δ is a $N(C,\Delta)$-generator ($L(C,\Delta)$-generator) of M. Hence, if $\Delta = M$, then M is a N(C)-group (L(C)-loop) of type 2 and N(C) (L(C)) acts 2-primitively on M.

Using Lemma 3.4, we obtain the following theorem which is crucial in extending some of our main results to near-rings (loop-near-rings) of transformations of a group (loop) M over a centralizer of the group (loop) M.

THEOREM 3.7. Let M,C be as in Corollary 3.6. Let M' be a $N(C,\Delta)$-group ($L(C,\Delta)$-loop). Then any $S(C,\Delta)$-(operand) homomorphism $\emptyset$ of M into M' is a $N(C,\Delta)$-group ($L(C,\Delta)$-loop)homomorphism (that is, preserves addition also); hence, if $\emptyset^{-1}(0) = 0$, then $\emptyset$ is injective.

PROOF. Let $\emptyset$: $M \to M'$ be an $S(C,\Delta)$-homomorphism. Fix a non-zero element w of Δ. Let $x,y \in M$. Then, by Lemma 3.4, there exist $T_1,T_2 \in N(C,\Delta)(=S(C,\Delta))$ such that $T_1(w) = x$, $T_2(w) = y$. Now $\emptyset(x + y) = \emptyset(T_1(w) + T_2(w)) = \emptyset(T_1 + T_2)(w) = (T_1 + T_2)\emptyset(w) = T_1\,\emptyset(w) + T_2\,\emptyset(w) = \emptyset\,T_1(w) + \emptyset\,T_2(w) = \emptyset(x) + \emptyset(y)$. Hence the result.

Theorem 3.7 can be generalized to the case of Universal Algebras, as follows. We assume that (A,Ω) is a Universal algebra such that A has a distinguished element 0 (that is, some $f \in \Omega$ is nullary) and $0 \in \Delta \subseteq A$.

DEFINITION. A set C of endomorphisms of the Ω-algebra A is called a Δ-centralizer of the Ω-algebra A if (i) $\overline{0} \in C$ (ii) $C - \overline{0}$ is a group of automorphisms of A (iii) $\alpha(\Delta) \subseteq \Delta$ for all $\alpha \in C$ and (iv) $\alpha,\beta \in C$, $0 \neq w \in \Delta$, $\alpha(w) = \beta(w)$ imply $\alpha = \beta$.

Let C be a Δ-centralizer of the Ω-algebra A. We denote by $U(C,\Delta)$, the set of all transformations of A which commute with every member of C. Defining operations pointwise, and adding the binary operation "o" of composition of mappings, we get a Universal algebra $(U(C,\Delta), \Omega \cup \{o\})$. Now A is a centered operand over $U(C,\Delta)$ and we have the following theorem whose proof is similar to that of Theorem 3.7.

THEOREM 3.8. Let A, $U(C,\Delta)$ be as above. Let B be a Ω-algebra such that there is a (left) multiplication of the elements of B by the elements of $U(C,\Delta)$, satisfying (i) $f(T_1, \ldots, T_n).b = f(T_1 \cdot b, \ldots, T_n \cdot b)$ for all $f \in \Omega$, $T_1, \ldots, T_n \in U(C,\Delta)$ and $b \in B$ (ii) $(T_1T_2).b = T_1 \cdot (T_2 \cdot b)$ for all T_1, $T_2 \in U(C,\Delta)$ and $b \in B$. Then any $U(C,\Delta)$-(operand) homomorphism of A into B is a Ω-algebra homomorphism.

With the usual notation, we have:

LEMMA 3.9. Let $0 \neq w \in \Delta$. Then M is $S(C,\Delta)$-isomorphic to $A(M-\Gamma)$ where Γ is the orbit of w.

PROOF. Consider the $S(C,\Delta)$-homomorphism $\emptyset$: $T \to T(w)$ from $A(M-\Gamma)$ into M. That $\emptyset$ is surjective follows from Lemma 3.4 and $\emptyset$ can be shown to be injective using the definition of $S(C,\Delta)$. Hence the result.

THEOREM 3.10. Suppose M is a.irreducible over $S(C,\Delta)$. Let Γ be a non-zero orbit. Then $A(M-\Gamma)$ is an irreducible operand over $S(C,\Delta)$ and hence $A(M-\Gamma)$ is a 0-minimal left ideal of $S(C,\Delta)$; further, $A(M-\Gamma) \not\subseteq A(\Delta)$.

PROOF. The first part is an easy consequence of Lemma 3.9. To prove the last part, consider the map T: $M \to M$ which is identity on Γ and 0 elsewhere. Now $T \in S(C,\Delta)$ and thereby $T \in A(M-\Gamma) - A(\Delta)$.

The bracketed statement of the following corollary is due to [3], Lemma 2.1.

COROLLARY 3.11. Let $M \neq 0$ be a group (loop) and C, a centralizer of M. Then N(C)(L(C)) contains a left ideal K which is N(C)-group isomorphic (L(C)-loop isomorphic) to M and hence is a N(C)-group (L(C)-loop) of type 2. Afortiori, K is a minimal left ideal of N(C)(L(C)).

PROOF. C is a centralizer of also the set M and N(C), L(C) are both equal to S(C), as sets. Let Γ be a non-zero orbit of C on M. A simple verification shows that $A(M-\Gamma)$ is a left ideal of the near-ring (loop-near-ring) N(C). By Lemma 3.9, M is S(C)-operand isomorphic to $A(M-\Gamma)$ and so by Theorem 3.7, M is N(C)-group isomorphic (L(C)-loop isomorphic) to $A(M-\Gamma)$. Since M is a N(C)-group (L(C)-loop) of type 2 (see Corollary 3.6), so is $A(M-\Gamma)$. The rest follows from Theorem 3.10.

As an immediate consequence of Corollary 2.13 we have:

PROPOSITION 3.12. Suppose M is a.irreducible and let L be an 0-minimal left ideal of $S(C,\Delta)$ such that for any $S(C,\Delta)$-homomorphism α of L into M, $\alpha^{-1}(0) = 0$ implies α is injective. Then L is $S(C,\Delta)$-isomorphic to M.

THEOREM 3.13. Let C' be a Δ-centralizer of M such that $C \subseteq C'$. Then $S(C,\Delta)= S(C',\Delta)$ if and only if $C = C'$.

PROOF. One way is clear. To prove the converse, let $S(C,\Delta) = S(C',\Delta)$ and assume that $C \neq C'$. Then there exists $\alpha' \in C' - C$. Let $0 \neq w \in \Delta$. It can be seen that $\alpha'(w) \notin \Gamma$, the orbit of w with respect to C. Now by Lemma 3.4, there exists $T \in S(C,\Delta)$ such that $T(w) = \alpha'(w)$ and T maps $M-\Gamma$ onto 0. But then $T \in S(C',\Delta)$ and so $\alpha'T(w) = T\alpha'(w) = 0$. Therefore $w = 0$, a contradiction.

COROLLARY 3.14. ([3], Theorem 1.2.) Let M be a group and $C \subseteq C'$, centralizers of M. Then $N(C) = N(C')$ if and only if $C = C'$.

The following result generalizes Corollary 1.3 of [3] (as also the analogous result for loops) and the proof is analogous to that of the latter.

PROPOSITION 3.15. Suppose (a) M is a.irreducible (b) for any $S(C,\Delta)$-endomorphism α of M, $\alpha^{-1}(0) = 0$ implies α is injective and (c) Δ-0 is the set of all $S(C,\Delta)$-generators of M. Then the set of all endomorphisms of M satisfying (i) $\alpha T = T\alpha$ for all $T \in S(C,\Delta)$ and (ii) $\alpha(\Delta) \subseteq \Delta$, is C itself.

PROPOSITION 3.16. If Δ' is the set of all $S(C,\Delta)$-generators of M together with zero, then C is a Δ'-centralizer of M and $S(C,\Delta) = S(C,\Delta')$.

PROOF. Let $\alpha \in C$. Then for any $0 \neq w \in \Delta'$, $S(C,\Delta)(\alpha(w)) = \alpha(S(C,\Delta)(w)) = \alpha(M) = M$. Therefore, $\alpha(\Delta') \subseteq \Delta'$ for all $\alpha \in C$ and, similarly, the other conditions can be verified to show that C is Δ'-centralizer of M. The rest is obvious.

In Proposition 3.16, there is no harm in taking M as a group (or a loop) and C as a Δ-centralizer of the group M (loop M).

4. ISOMORPHISMS OF SEMIGROUPS OF TRANSFORMATIONS.

We introduce here the concept of a generalized semi-space as a generalization of semi-space introduced by [3].

DEFINITION. A generalized semi-space is a triple (M,Δ,C) where M is a set with $0 \in M$, $0 \in \Delta \subseteq M$ and C is a Δ-centralizer of M. If $\Delta = M$, we omit Δ and write simply as (M,C).

DEFINITION. Let (M_i, Δ_i, C_i) be generalized semi-spaces for $i = 1,2$. A map σ: $C_1 \to C_2$ is called an isomorphism of C_1 onto C_2 if $\sigma(\overline{0}) = \overline{0}$ and σ is a group isomorphism of $C_1 - \overline{0}$ onto $C_2 - \overline{0}$.

Throughout the rest of this paper, unless otherwise stated, (M_i, Δ_i, C_i) denotes a generalized semi-space for $i = 1,2$.

DEFINITION. A map h: $M_1 \to M_2$ is called a semi-linear transformation of M_1 into M_2 if (i) h fixes 0 and $h(\Delta_1) \subseteq \Delta_2$ and (ii) there exists an isomorphism σ of C_1 onto C_2 such that $h\alpha = \sigma(\alpha)h$ for all $\alpha \in C_1$.

If we wish to indicate σ also, we shall denote the semilinear transformation by (h,σ). We notice that, if (G_1,C_1) and (G_2,C_2) are semi-spaces, then any semi-linear transformation of the semi-spaces (G_1,C_1) and (G_2,C_2) is a semi-linear transformation of the generalized semi-spaces (G_1,C_1) and (G_2,C_2).

DEFINITION. A semilinear transformation h: $M_1 \to M_2$ is called a 1-1 semilinear transformation if h is bijective and $h(\Delta_1) = \Delta_2$.

If (h,σ) is a 1-1 semilinear transformation of M_1 onto M_2 (h^{-1}, σ^{-1}) is one such from M_2 onto M_1. The proof of the following theorem is analogous to that of Lemma 2.7 of [3].

THEOREM 4.1. Let (h,σ) be a 1-1 semilinear transformation of M_1 onto M_2. Then, $\emptyset(T) = hTh^{-1}$ for all $T \in S(C_1, \Delta_1)$ defines an isomorphism of $S(C_1, \Delta_1)$ onto $S(C_2, \Delta_2)$.

Conversely,

THEOREM 4.2. Let $\emptyset$ be an isomorphism of $S(C_1, \Delta_1)$ onto $S(C_2, \Delta_2)$ and suppose that M_i is a.irreducible over $S(C_i, \Delta_i)$ for $i = 1,2$. Then M_2 can be regarded as a faithful, irreducible operand over $S(C_1, \Delta_1)$. Further, suppose that

(i) for any $S(C_1, \Delta_1)$-homomorphism α of M_1 into M_2, $\alpha^{-1}(0) = 0$ implies α is injective.

(ii) for $i = 1,2$, $\Delta_i - 0$ is the set of all $S(C_i, \Delta_i)$-generators of M_i.

(iii) for any $S(C_i, \Delta_i)$-endomorphism α of M_i, $\alpha^{-1}(0) = 0$ implies α is injective, for $i = 1,2$.

Then there exists a 1-1 semilinear transformation (h,σ) of M_1 onto M_2 such that h is an $S(C_1, \Delta_1)$-isomorphism of M_1 onto M_2 and $\emptyset(T) = hTh^{-1}$ for all $T \in S(C_1, \Delta_1)$.

Before proving this theorem, we give the following three lemmas in each of which it is assumed that $\emptyset$ is an isomorphism of $S(C_1, \Delta_1)$ onto $S(C_2, \Delta_2)$ and that for $i = 1,2$, M_i is a.irreducible over $S(C_i, \Delta_i)$.

LEMMA 4.3. M_2 can be regarded as a faithful and irreducible operand over $S(C_1, \Delta_1)$.

PROOF. The left multiplication '.' given by $T\cdot m = \emptyset(T)(m)$ for each $T \in S(C_1,\Delta_1)$ and $m \in M_2$ serves the purpose.

LEMMA 4.4. Suppose conditions (i) and (ii) of Theorem 4.2 are also satisfied. Then there exists an $S(C_1, \Delta_1)$-isomorphism h of M_1 onto M_2 such that $h(\Delta_1) = \Delta_2$ and $\emptyset(T) = hTh^{-1}$ for all $T \in S(C_1, \Delta_1)$.

PROOF. Let Γ be a non-zero orbit of C_1 over Δ_1. Lemma 3.9 says that M_1 is $S(C_1, \Delta_1)$-isomorphic to $A(M_1-\Gamma)$. Using Theorem 3.10 and condition (i) of the hypothesis, we get from Proposition 3.12 that $A(M_1-\Gamma)$ is $S(C_1, \Delta_1)$-isomorphic to M_2. So, there exists an $S(C_1, \Delta_1)$-isomorphism $h\colon M_1 \to M_2$. Now, let $T \in S(C_1, \Delta_1)$ and $m_1 \in M_1$. Then $h\,T(m_1) = T\cdot h(m_1) = \emptyset(T)\,h(m_1)$ and hence $hT = \emptyset(T)h$, which means $\emptyset(T) = hTh^{-1}$. It remains to show that $h(\Delta_1) = \Delta_2$. Let $0 \neq w_1 \in \Delta_1$. Then $S(C_2, \Delta_2)\,h(w_1) = \emptyset(S(C_1, \Delta_1))h(w_1) = (h\,S(C_1, \Delta_1)h^{-1})h(w_1) = h\,S(C_1, \Delta_1)(w_1) = h(M_1) = M_2$. Therefore, $h(w_1) \in \Delta_2$, by condition (ii). Thus $h(\Delta_1) \subseteq \Delta_2$. To prove the reverse inclusion, let $0 \neq w_2 \in \Delta_2$. Then w_2 is an $S(C_1, \Delta_1)$-generator of M_2, and so $S(C_1, \Delta_1)\,h^{-1}(w_2) = h^{-1}(S(C_1, \Delta_1)w_2) = h^{-1}(M_2) = M_1$. Therefore, $h^{-1}(\Delta_2) \subseteq \Delta_1$, and this completes the proof.

LEMMA 4.5. Assume all the hypothesis of Theorem 4.2, and let h be an $S(C_1, \Delta_1)$-isomorphism of M_1 onto M_2 such that $h(\Delta_1) = \Delta_2$ and $\emptyset(T) = hTh^{-1}$ for all $T \in S(C_1,\Delta_1)$ (the existence of h being ensured by Lemma 4.4). Then $h\,\alpha_1\,h^{-1} \in C_2$ for each $\alpha_1 \in C_1$ and $\sigma\colon\ \alpha_1 \to h\alpha_1h^{-1}$ is an isomorphism of C_1 onto C_2.

PROOF. Let $\alpha_1 \in C_1$. Write $\alpha_2 = h\alpha_1 h^{-1}$. In view of Proposition 3.15, it suffices to show that (a) α_2 is an endomorphism of M_2 and (b) $\alpha_2(\Delta_2) \subseteq \Delta_2$ and (c) $\alpha_2 T_2 = T_2\alpha_2$ for all $T_2 \in S(C_2, \Delta_2)$.

(a) is obvious. Since $h(\Delta_1) = \Delta_2$ and h is an isomorphism, we have $\alpha_2(\Delta_2) = h\alpha_1 h^{-1}(\Delta_2) = h\alpha_1(\Delta_1) \subseteq h(\Delta_1) = \Delta_2$, proving (b). Finally, let $T_2 \in S(C_2, \Delta_2)$ and $m_2 \in M_2$. Then there exist $T_1 \in S(C_1, \Delta_1)$ and $m_1 \in M_1$ such that $\emptyset(T_1) = T_2$ and $h(m_1) = m_2$. Now $\alpha_2 T_2(m_2) = h\alpha_1 h^{-1}\emptyset(T_1)h(m_1) = h\alpha_1 h^{-1}hT_1h^{-1}h(m_1) = h\alpha_1 T_1(m_1) = hT_1\alpha_1(m_1) = hT_1h^{-1}h\alpha_1 h^{-1}h(m_1) = T_2\alpha_2(m_2)$, which proves (c).

PROOF OF THEOREM 4.2. In view of Lemmas 4.3, 4.4 and 4.5, it remains to show that $\alpha_1 \in C_1$ implies $h\alpha_1 = \sigma(\alpha_1)h$, which is clear from the definition of σ. Hence the theorem.

REMARK. The particular case of Theorem 4.2 when $\Delta = M$ can also be deduced from [2], Theorem 17.3.

COROLLARY 4.6. (Isomorphism Theorem for Near-rings of Transformations, Theorem 2.6 of [3]). Let (G_i, C_i), i = 1,2, be semi-spaces (G_1, G_2 are groups). (a) If there exists a 1-1 semi-linear transformation h of G_1 onto G_2, then $f(A) = hAh^{-1}$ for all $A \in N(C_1)$ defines an isomorphism of $N(C_1)$ onto $N(C_2)$. (b) If f is an isomorphism of $N(C_1)$ onto $N(C_2)$, then there exists a 1-1 semi-linear transformation h of G_1 onto G_2 such that $f(A) = hAh^{-1}$ for all $A \in N(C_1)$.

PROOF. (a): Clearly h is a 1-1 semilinear transformation of the generalized semi-spaces (G_1, C_1) and (G_2, C_2). Now by Theorem 4.1, $f(A) = hAh^{-1}$ for all $A \in N(C_1)$ defines a multiplicative semigroup isomorphism of $N(C_1)$ onto $N(C_2)$. We show that f preserves addition also. Let $A_1, A_2 \in N(C_1)$ and $g_2 \in G_2$. We have $f(A_1 + A_2)(g_2) = h(A_1 + A_2)h^{-1}(g_2) = h(A_1h^{-1}(g_2) + A_2h^{-1}(g_2)) = hA_1h^{-1}(g_2) + hA_2h^{-1}(g_2) = f(A_1)(g_2) + f(A_2)(g_2) = (f(A_1) + f(A_2))(g_2)$; hence, $f(A_1 + A_2) = f(A_1) + f(A_2)$. Thus f is a near-ring isomorphism.

(b): We deduce this part from Theorem 4.2. From Lemma 3.4 we get that every non-zero element of G_i is an $N(C_i)$-generator of G_i for i = 1,2 and so G_i is an a.irreducible operand over $N(C_i)$ for i = 1,2 and condition (ii) of Theorem 4.2 is satisfied here. That conditions (i) and (iii) are also satisfied here, follows from Theorem 3.7. Hence there exists a 1-1 semilinear transformation h of the generalized semi-space (G_1, C_1) onto (G_2, C_2) such that h is an $N(C_1)$-isomorphism of G_1 onto G_2 and $f(A) = hAh^{-1}$ for all $A \in N(C_1)$. By Theorem 3.7, h is a group isomorphism too, and hence h is a 1-1 semilinear transformation of the semispaces (G_1, C_1) and (G_2, C_2). Hence the result.

We now get the following Isomorphism Theorem for rings of linear transformations of vector spaces over division rings, the proof being the same as that given in [3], Corollary 2.13.

COROLLARY 4.7. Let L_i, i = 1,2, be the rings of linear transformations of vector spaces (M_i, D_i) not necessarily finite dimensional. Then f is an isomorphism of $L_1 \to L_2$ if and only if there exists a 1-1 semilinear transformation h of M_1 onto M_2 such that $fT = hTh^{-1}$ for all $T \in L_1$.

REMARK 4.8. In the case of loops also, we can define semi-spaces and their semilinear transformations analogously, and all the corollaries obtained in this section for groups hold for loops as well, with 'near-ring of transformations' replaced by 'loop-near-ring of transformations'.

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