DOUBLY STOCHASTIC RIGHT MULTIPLIERS

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ABSTRACT. Let $P(G)$ be the set of normalized regular Borel measures on a compact group G . Let D_r be the set of doubly stochastic (d.s.) measures λ on $G \times G$ such that $\lambda(As \times Bs) = \lambda(A \times B)$, where $s \in G$, and A and B are Borel subsets of G . We show that there exists a bijection $\mu \leftrightarrow \lambda$ between $P(G)$ and D_r such that $\lambda \phi^{-1} = \mu \otimes \mu$, where μ is normalized Haar measure on G , and $\phi(x, y) = (x, xy^{-1})$ for $x, y \in G$. Further, we show that there exists a bijection between D_r and M_r , the set of d.s. right multipliers of $L_1(G)$. It follows from these results that the mapping $\mu \mapsto T_\mu$ defined by $T_\mu f = \mu * f$ is a topological isomorphism of the compact convex semigroups $P(G)$ and M_r . It is shown that M_r is the closed convex hull of left translation operators in the strong operator topology of $B[L_2(G)]$.

KEY WORDS AND PHRASES. *Compact group, regular Borel measures, doubly stochastic measures, multipliers.*

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1. INTRODUCTION.

Let G be an arbitrary compact Hausdorff group, let $B(G)$ be the σ -algebra of Borel subsets of G , and let m be normalized Haar measure on G . Let $P(G)$ be the set of regular Borel measures μ on G such that $\mu(G) = 1$. Assume that $1 \leq p \leq \infty$. Let $L_p(G)$ denote the complex Banach space $L_p(G, B(G), m)$, and let $B[L_p(G)]$ denote the complex Banach space of bounded linear operators from $L_p(G)$ into itself. An operator T in $B[L_p(G)]$ is called a positive contraction on $L_p(G)$ if $Tf \geq 0$ for each nonnegative f in $L_p(G)$ and $\|T\|_p \leq 1$.

For each positive contraction T on $L_1(G)$, the adjoint T^* determined by the equation $\int_G (Tf)g \, dm = \int_G f(T^*g) \, dm$ for $f \in L_1(G)$ and $g \in L_\infty(G)$ is a positive contraction on $L_\infty(G)$. A positive contraction T on $L_1(G)$ such that $T1 = 1$, or equivalently, $\int_G Tg \, dm = \int_G g \, dm$ for $g \in L_\infty(G)$, is called a doubly stochastic (d.s.)

operator. Let D be the set of d.s. operators. Note that $T \in D$ if and only if $T^* \in D$. By the Riesz convexity theorem, each d.s. operator T is also a positive contraction on $L_p(G)$, $1 < p < \infty$, with $\|T\|_p = 1$. Let Φ be the class of measure-preserving maps ϕ from $(G, B(G), m)$ onto itself, and let Φ_1 be the class of maps ϕ in Φ that are invertible and $\phi^{-1} \in \Phi$. Then each ϕ in Φ gives rise to a d.s. operator T_ϕ that is defined by $T_\phi f(x) = f(\phi(x))$. For brevity we also write Φ for $\{T_\phi : \phi \in \Phi\}$ and Φ_1 for $\{T_\phi : \phi \in \Phi_1\}$. For $s \in G$, let L_s and R_s be the left and the right translation operators defined by $L_s f(x) = f(s^{-1}x)$ and $R_s f(x) = f(xs^{-1})$. Then both L_s and R_s are in Φ_1 .

An operator T in D is called a right multiplier (centralizer) of $L_1(G)$ if T commutes with right translation operators, that is, $TR_s = R_s T$ for each $s \in G$. Let M_r be the set of d.s. right multipliers of $L_1(G)$. The set M_ℓ of d.s. left multipliers of $L_1(G)$ is defined in an analogous fashion. We see readily that $L_s \in M_r$ and $R_s \in M_\ell$ for each $s \in G$. Let $M = M_r \cap M_\ell$. It is plain that $L_s = R_s \in M$ for each $s \in G^Z$, the center of G . Note that if G is Abelian, then $M = M_r = M_\ell$.

For a semigroup S , the set of elements x in S such that $xy = yx$ for each y in S is called the center of S and is denoted by S^Z . For topological semigroups S_1 and S_2 , an (algebraic) isomorphism of S_1 into S_2 which is also a homeomorphism is called a topological isomorphism of S_1 into S_2 . All functions on G are Borel measurable and will always be considered up to m -equivalence. For two functions f and g on G , $f = g$, $f \neq g$ mean that the equality and the inequality, respectively, are satisfied in the almost everywhere (a.e.) sense with respect to m .

It follows from Theorem 1 of Wendel [1] (see also Edwards [2]; Hewitt and Ross [3], 35.5) that for each T in M_r , there exists a unique μ in $P(G)$ such that $Tf = \mu * f$ for each $f \in L_1(G)$. Using a d.s. measure, Brown [4] gave an alternate proof of the above result when the underlying group G is Abelian and compact. The purpose of this paper is to extend Brown's work [4] for an arbitrary compact group.

Certain preliminary results on $P(G)$ and d.s. measures are given in Section 2. In Section 3 we show that there exists a bijection between M_r and D_r (Proposition 5), and that there exists a bijection between D_r and $P(G)$ (Proposition 6). Using Propositions 5 and 6, we prove (Theorem 1) that the mapping $\mu \rightarrow T_\mu$ defined by $T_\mu f = \mu * f$ is a topological isomorphism of the compact convex semigroups $P(G)$ and M_r . In Section 4 we show that $T \in M_r$ is an isometry of $L_1(G)$ if and only if T is a left translation operator (Theorem 3), and that the set of extreme points of M_r is the group $\underline{G}$ of left translation operators, and M_r is the closed convex hull of $\underline{G}$ in the strong operator topology of $B[L_2(G)]$ (Theorem 4). By minor modifications of our arguments we obtain analogous results for M_ℓ .

2. PRELIMINARIES.

Let $C(G)$ be the Banach space of complex continuous functions on G . For $\mu, \nu \in P(G)$, there exists, by the Riesz representation theorem, a unique measure $\mu * \nu$

in $P(G)$ such that $\int f(z) d\mu * \nu(z) = \iint f(xy) d\mu(x) d\nu(y)$ for each $f \in C(G)$. Thus $P(G)$ is a semigroup under the convolution operation. It follows from Theorem 2 of Stromberg [5] that $\mu * \nu(A) = \int_G \mu(Ay^{-1}) d\nu(y) = \int_G \nu(x^{-1}A) d\mu(x)$ for each $A \in B(G)$. As usual we shall identify $P(G)$ with a subset of $C(G)^*$, the dual space of $C(G)$. We show readily that $P(G)$ is convex and is compact in the weak* topology. It is well-known (Rosenblatt [6]) that for μ, ν in $P(G)$, the convolution operation $\mu * \nu$ is jointly continuous in μ and ν with respect to the weak* topology. Therefore $P(G)$ with the convolution operation and the weak* topology, is a compact, convex, Hausdorff semigroup.

We state without proof the following result of Stromberg [5].

LEMMA 1. (Stromberg). Let X be a compact Hausdorff space and f a continuous mapping of X into itself. If μ is a regular Borel measure on X , then so is the measure μf^{-1} .

For $\mu \in P(G)$, the adjoint μ' defined by $\mu'(A) = \mu(A^{-1})$ is an element of $P(G)$ and $(\mu')' = \mu$. For each $x \in G$, ϵ_x be the probability measure such that $\epsilon_x(A) = \chi_A(x)$ for $A \subset G$, where χ_A is the characteristic function of A . Note that $\epsilon_x^{-1} = \epsilon_{x^{-1}}$. It is easily seen that the mapping $x \rightarrow \epsilon_x$ is a topological isomorphism of G into the compact semigroup $P(G)$.

A characterization of the center $P^Z(G)$ of the semigroup $P(G)$ is given by Stromberg [7]. It is straightforward to prove the following proposition.

PROPOSITION 1. $P^Z(G)$ is a compact, convex, Abelian subsemigroup of $P(G)$.

Let $P^1(G)$ be the set of idempotents μ of $P(G)$, that is, $\mu * \mu = \mu$. For any compact subgroup H of G , let m_H be normalized Haar measure on H . We shall always extend the measure m_H in $P(H)$ to a unique measure in $P(G)$, denoted also by m_H , as follows: $m_H(A) = m_H(A \cap H)$ for $A \in B(G)$. Then $m_H \in P^1(G)$. Theorem 1 of Wendel [8] states that $\mu \in P^1(G)$ if and only if there exists a unique compact subgroup H of G such that $\mu = m_H$. It is routine to verify that $P^1(G)$ is a compact subset of $P(G)$. Since, for arbitrary compact subgroups H and K of G , the set HK is not always a subgroup of G and $m_H * m_K = m_{HK}$, the set $P^1(G)$ is not necessarily a subsemigroup of $P(G)$. Let e denote the identity of G . Observe that

$$\frac{1}{2}(m + \epsilon_e) * \frac{1}{2}(m + \epsilon_e) = (3m + \epsilon_e)/4 \neq \frac{1}{2}(m + \epsilon_e).$$

Therefore $P^1(G)$ is not a convex set.

The set $P^1(G) \cap P^Z(G)$ contains Haar measure m and the point mass ϵ_e and is a compact subset of $P(G)$. For $\mu \in P(G)$, let $S(\mu)$ be the support of μ .

PROPOSITION 2. Let μ be in $P(G)$. The following assertions are equivalent:

- (i) μ is in $P^1(G) \cap P^Z(G)$;
- (ii) there exists a unique compact subgroup H of G such that $\mu = m_H$ and $\epsilon_s * m_H * \epsilon_s^{-1} = m_H$ for each $s \in G$;
- (iii) there exists a unique compact normal subgroup H of G such that $\mu = m_H$.

PROOF. It is known (Stromberg [7]; Wendel [8]) that (i) and (ii) are equivalent.

Suppose that (ii) holds. Then we have, for each $s \in G$,

$H = S(m_H) = S(\varepsilon_s * m_H * \varepsilon'_s) = S(\varepsilon_s)S(m_H)S(\varepsilon'_s) = sHs^{-1}$, so that H is a compact normal subgroup of G . Thus (ii) implies (iii).

Suppose that $\mu = m_H$, where H is a unique compact normal subgroup of G . We shall show that $\varepsilon_s * m_H * \varepsilon'_s = m_H$ for each $s \in G$, or equivalently, $m_H(s^{-1}Es) = m_H(E)$ for $s \in G$ and $E \in B(G)$. Let $\phi_s(x) = sxs^{-1}$. Since the inner automorphism ϕ_s of G is a homeomorphism of G onto itself, and $\phi_s(H) = H$, we have, from Lemma 1, $m_H|_s^{-1} \in P(G)$, and $S(m_H\phi_s^{-1}) = H$. For $s \in G$, $a \in H$, and $E \in B(G) \cap H$, we have $h = s^{-1}a^{-1}s \in H$, $\phi_s(x)a^{-1} = \phi_s(xh)$, and

$$m_H\phi_s^{-1}(Ea) = \int_H \chi_E(\phi_s(x)a^{-1}) dm_H(x) = \int_H \chi_E(\phi_s(xh)) dm_H(x) = \int_H \chi_E(\phi_s(x)) dm_H(x) = m_H\phi_s^{-1}(E),$$

so that by the uniqueness of Haar measure on H , $m_H = m_H\phi_s^{-1}$. That is,

$$m_H(E) = m_H\phi_s^{-1}(E) = m(s^{-1}Es). \text{ Therefore (iii) implies (ii). } \square$$

It is easy to verify that for $\mu \in P^i(G)$ and $\nu \in P^i(G) \cap P^Z(G)$, $\mu * \nu = \nu * \mu \in P^i(G)$.

We also have

PROPOSITION 3. $P^i(G) \cap P^Z(G)$ is a compact, non-convex subsemigroup of $P(G)$.

We omit the elementary proof of this proposition.

For $\mu \in P(G)$, let $P: G \times B(G) \rightarrow [0,1]$ be such that

$$P(x,A) = \mu' * \varepsilon_x(A) = \mu(xA^{-1}).$$

Then $P(x,A)$ is a transition (probability) function which is a regular probability measure on $B(G)$ for each $x \in G$ and a Borel function of x for each $A \in B(G)$. It follows easily that

$$\int_G P(x,A) dm(x) = m(A) \quad (2.1)$$

for $A \in B(G)$, and

$$P(xs,As) = P(x,A) \quad (2.2)$$

for $s, x \in G$ and $A \in B(G)$. Note that the transition function $P(x,A)$ has an invariant measure m and is invariant under right translations. The transition function $P(x,A) = \mu(xA^{-1})$ gives rise to the Markov operator $P = P_\mu$ from $C(G)$ into itself by the formula $Pf(x) = \int_G P(x,dy)f(y) = \int_G f(y^{-1}x)d\mu(y)$. It follows from (2.1) and (2.2) that $\int_G Pfdm = \int_G fdm$ for $f \in C(G)$, $P1 = 1$, and $PR_s = R_sP$ for $s \in G$. See Rosenblatt [6] for details. By the usual argument the Markov operator P_μ is uniquely extended to an operator T_μ in M_r . We shall see in Proposition 7 that each $T \in M_r$ is induced by a unique Markov operator P_μ on $C(G)$.

Let $P(G \times G)$ be the set of regular probability measures on $(G \times G, B(G \times G))$, where $B(G \times G)$ denotes the σ -algebra of Borel subsets of $G \times G$. A measure λ in $P(G \times G)$ is called a d.s. measure on $G \times G$ if

$$\lambda(A \times G) = \lambda(G \times A) = m(A) \quad (2.3)$$

for each $A \in B(G)$. Let $\mathcal{D}$ be the set of d.s. measures on $G \times G$. A probability measure λ on the product measurable space $(G \times G, B(G) \times B(G))$ satisfying (2.3) is called doubly stochastic by Brown [4]. Let $\{X_n : n \geq 1\}$ be the right random walk on G generated by m and $\mu \in P(G)$. That is, $\{X_n\}$ is the Markov process with state space G , initial distribution m , and stationary transition function $P(x, A) = \mu(xA^{-1})$. If λ denotes the joint distribution of X_1 and X_2 , then λ is a d.s. measure on $(G \times G, B(G) \times B(G))$. If G is metrizable, then $B(G) \times B(G) = B(G \times G)$, so that these two definitions are equivalent. However we have $B(G) \times B(G) \subsetneq B(G \times G)$ in general, so that a d.s. measure in the sense of Brown [4] is not an element of $\mathcal{D}$. Let $B_0(G)$ be the σ -algebra of Baire subsets of G , and let $B_0(G \times G)$ be the σ -algebra of Baire subsets of $G \times G$. Note that $B_0(G \times G) = B_0(G) \times B_0(G)$.

LEMMA 2. For each probability measure σ on $(G \times G, B(G) \times B(G))$ satisfying (2.3), there exists a unique λ in $\mathcal{D}$ such that $\lambda(E) = \sigma(E)$ for each E in $B(G) \times B(G)$.

PROOF. Let σ_0 be the Baire restriction of σ , that is, $\sigma_0(E) = \sigma(E)$ for each $E \in B_0(G \times G)$. Then σ_0 is a Baire measure on $(G \times G, B_0(G \times G))$ such that $\sigma_0(A \times G) = \sigma_0(G \times A) = m(A)$ for each $A \in B_0(G)$. Let λ be the unique, regular Borel measure on $(G \times G, B(G \times G))$ which extends σ_0 (see Halmos [9], 54.D). We shall show that λ also extends σ . It is enough to show that $\lambda(C_1 \times C_2) = \sigma(C_1 \times C_2)$ for all compact sets C_1 and C_2 in G . By the regularity of Haar measure m there exist compact G'_0 's, A_1 and A_2 , such that $C_j \subset A_j$ and $m(A_j - C_j) = 0$ for $j = 1, 2$. Since m is completion regular by a theorem of Kakutani-Kodaira [10] (see also, Halmos [9], 64.H), there exist B_1 and B_2 in $B_0(G)$ such that $A_j - C_j \subset B_j$ and $m(B_j) = 0$ for $j = 1, 2$. Note that

$$A_1 \times A_2 - C_1 \times C_2 = [(A_1 - C_1) \times A_2] \cup [C_1 \times (A_2 - C_2)] \subset (B_1 \times G) \cup (G \times B_2).$$

Then we have

$$\lambda(A_1 \times A_2 - C_1 \times C_2) \leq \lambda(B_1 \times G) + \lambda(G \times B_2) = \sigma_0(B_1 \times G) + \sigma_0(G \times B_2) = m(B_1) + m(B_2) = 0,$$

so that $\lambda(C_1 \times C_2) = \lambda(A_1 \times A_2) = \sigma_0(A_1 \times A_2)$. Similarly we also have $\sigma(C_1 \times C_2) = \sigma(A_1 \times A_2) = \sigma_0(A_1 \times A_2)$, and so our assertion is proved.

If λ_1 and λ_2 are any two measures in $\mathcal{D}$ both of which extend σ , then they also extend the Baire measure σ_0 , so that $\lambda_1 = \lambda_2$. $\square$

We obtain readily from Corollary of Brown [11], together with Lemma 2, the following proposition.

PROPOSITION 4. There exists a bijection $T \leftrightarrow \lambda$ between $\mathcal{D}$ and $\mathcal{D}$ such that

$$\lambda(A \times B) = \int_G \chi_A T \chi_B dm \quad (2.4)$$

for all $A, B \in B(G)$.

3. DOUBLY STOCHASTIC RIGHT MULTIPLIERS.

Let $\mathbf{D}_r$ be the set of measures λ in $\mathbf{D}$ such that $\lambda(As \times Bs) = \lambda(A \times B)$ for all $A, B \in B(G)$, $s \in G$. For each $s \in G$, let $\tau_s(x, y) = (xs^{-1}, ys^{-1})$ for $x, y \in G$. It is easy to see that for each $\lambda \in \mathbf{D}$, $\lambda \in \mathbf{D}_r$ if and only if $\lambda \tau_s^{-1} = \lambda$ for each $s \in G$.

PROPOSITION 5. There exists a bijection $T \leftrightarrow \lambda$ between M_r and $\mathbf{D}_r$ satisfying relation (2.4).

PROOF. Let T and λ be the associated d.s. operator and d.s. measure as in Proposition 4. Then $T \in M_r$ iff $T = R_{s^{-1}} T R_s$ for all $s \in G$ iff $T \chi_B = R_{s^{-1}} T R_s \chi_B$ for all $B \in B(G)$, $s \in G$ iff $\lambda(A \times B) = \lambda(As \times Bs)$ for all $A, B \in B(G)$, $s \in G$. $\square$

For $\mu, \nu \in P(G)$, there exists, by the Riesz representation theorem, a unique regular Borel measure $\mu \times \nu$ in $P(G \times G)$ such that

$$\int_{G \times G} f(x, y) d\mu \times \nu(x, y) = \int_G (\int_G f(x, y) d\mu(x)) d\nu(y) = \int_G (\int_G f(x, y) d\nu(y)) d\mu(x)$$

for all continuous functions $f(x, y)$ on $G \times G$. Note that $\mu \times \nu$ is the unique regular Borel measure on $G \times G$ which extends the product measure $\mu \times \nu$ on $(G \times G, B(G) \times B(G))$. Note also that Haar measure $m \times m$ on the compact group $G \times G$ is an element of $\mathbf{D}_r$. Let ξ and η be the mappings from $G \times G$ onto G defined by $\xi(x, y) = y^{-1}x$ and $\eta(x, y) = xy^{-1}$. Then both ξ and η are continuous surjections. Define ϕ and ψ on $G \times G$ by

$$\phi(x, y) = (x, xy^{-1}), \quad \psi(x, y) = (x, y^{-1}x).$$

Then ϕ is a homeomorphism of $G \times G$ onto itself with $\phi^{-1} = \psi$ and is $m \times m$ measure-preserving.

PROPOSITION 6. There exists a bijection $\lambda \leftrightarrow \mu$ between $\mathbf{D}_r$ and $P(G)$ such that $\lambda \phi^{-1} = m \times \mu$.

LEMMA 3. There exists an injection $\lambda \rightarrow \mu$ from $\mathbf{D}_r$ into $P(G)$ such that $\lambda \phi^{-1} = m \times \mu$, and in this case we have $\mu = \lambda \eta^{-1}$.

PROOF. For each $\lambda \in \mathbf{D}_r$ and for each $B \in B(G)$, let θ_B be the Borel measure on G defined by $\theta_B(A) = \lambda \phi^{-1}(A \times B)$. It follows from the regularity of Haar measure m that, for $\varepsilon > 0$ and $A \in B(G)$, there exist a compact set C and an open set U such that $C \subset A \subset U$ and $m(U - C) < \varepsilon$. Then we have $\theta_B(U - C) \leq m(U - C) < \varepsilon$, so that θ_B is a regular Borel measure. Since, for each $s \in G$, the mapping $\tau_s(x, y) = (xs^{-1}, ys^{-1})$ is λ measure-preserving,

$$\theta_B(A) = \lambda \phi^{-1}(A \times B) = \lambda \tau_s^{-1} \phi^{-1}(A \times B) = \lambda \phi^{-1}(As \times B) = \theta_B(As)$$

for all $s \in G$, $A \in B(G)$, so that by the uniqueness of Haar measure there exists $c \geq 0$ such that $\theta_B = cm$. It follows that $c = \theta_B(G) = \lambda \eta^{-1}(B)$ and $\lambda \phi^{-1}(A \times B) = m(A) \lambda \eta^{-1}(B)$ for all $A, B \in B(G)$. By Lemma 1 we have $\lambda \phi^{-1} \in P(G \times G)$ and $\lambda \eta^{-1} \in P(G)$. Since two regular Borel measures $\lambda \phi^{-1}$ and $m \times \lambda \eta^{-1}$ agree on all Baire sets of $G \times G$, we have $\lambda \phi^{-1} = m \times \lambda \eta^{-1}$. If $\lambda \phi^{-1} = m \times \mu$ for some $\mu \in P(G)$, then $\mu(A) = m \times \mu(G \times A) = \lambda \phi^{-1}(G \times A) = \lambda \eta^{-1}(A)$ for $A \in B(G)$, so that $\mu = \lambda \eta^{-1}$. It is clear

that the mapping $\lambda \rightarrow \mu$ is injective. $\square$

PROOF OF PROPOSITION 6. It remains to show that the injection defined in Lemma 3 is surjective. For each $\mu \in P(G)$, let $\lambda = (m\mu)\psi^{-1}$. By Lemma 1 we have $\lambda \in P(G \times G)$. It follows that, for A and B in $B(G)$,

$$\lambda(A \times B) = (m\mu)\psi^{-1}(A \times B) = \int_G m(A \cap yB) d\mu(y),$$

so that $\lambda(A \times G) = \lambda(G \times A) = m(A)$. Since $m(As \cap yBs) = m(A \cap yB)$ for all $A, B \in B(G)$, $s, y \in G$, we obtain $\lambda(A \times B) = \lambda(As \times Bs)$. Therefore $\lambda \in \mathbf{D}_r$. Using $\phi = \psi^{-1}$ we also have $\lambda\phi^{-1} = m\mu$. $\square$

PROPOSITION 7. There exists a bijection $\mu \leftrightarrow T_{\mu}$ between $P(G)$ and M_r such that $T_{\mu}f = \mu * f$ for each $f \in L_1(G)$.

PROOF. Let $\mu \in P(G)$, $\lambda \in \mathbf{D}_r$, and $T \in M_r$ be the associated elements determined by Propositions 5 and 6. Clearly the mapping $\mu \rightarrow T_{\mu} = T$ is a bijection from $p(G)$ onto M_r . Then we have

$$\int_G \chi_A T \chi_B d\mu = \lambda(A \times B) = (m\mu)\psi^{-1}(A \times B) = \int_G \chi_A(x) \left(\int_G \chi_B(y^{-1}x) d\mu(y) \right) d\mu(x)$$

for A and B in $B(G)$, so that $T \chi_B(x) = \int_G \chi_B(y^{-1}x) d\mu(y)$. Therefore we have $Tf = \mu * f$ for each $f \in L_1(G)$. $\square$

It is shown by Brown [11] that D with the weak operator topology of $B[L_2(G)]$ is a compact, convex Hausdorff semigroup, and that on the set D the weak operator topologies of $B[L_p(G)]$, $1 \leq p < \infty$, coincide. An elementary argument shows that M_r is a compact, convex subsemigroup of D .

THEOREM 1. The mapping $\mu \rightarrow T_{\mu}$ of Proposition 7 is a topological isomorphism between the compact convex semigroups $P(G)$ and M_r .

PROOF. It is straightforward to show that $T_{\mu}T_{\nu} = T_{\mu * \nu}$, $T_{\mu}^* = T_{\mu}^*$, and $T_{t\mu + (1-t)\nu} = tT_{\mu} + (1-t)T_{\nu}$ for $\mu, \nu \in P(G)$ and $t \in [0, 1]$. By Proposition 7 the mapping $\mu \rightarrow T_{\mu}$ is an isomorphism of $P(G)$ onto M_r . Note that the mapping is a regular representation of $P(G)$ (see Hewitt and Ross [12], 22.11).

To prove that the mapping $\mu \rightarrow T_{\mu}$ is a homeomorphism it is enough to show that the mapping is continuous, or equivalently, whenever a net (μ_{α}) converges to μ in $P(G)$, the net $(T_{\mu_{\alpha}})$ converges to T_{μ} in M_r . Let f and g be real continuous functions on G such that $|f(x)| \leq 1$, $|g(x)| \leq 1$ for $x \in G$. Since $f \circ \xi(x, y) = f(y^{-1}x)$ is right uniformly continuous on $G \times G$, there exists, for each $\varepsilon > 0$, a neighbourhood U of the identity e of G such that for all $y \in G$,

$$|f(y^{-1}x) - f(y^{-1}x')| < \varepsilon/8 \text{ for } x'x^{-1} \in U.$$

Since G is compact, an open covering $\{U_x: x \in G\}$ of G has a finite subcovering $\{U_j: j = 1, 2, \dots, n\}$, where $U_j = Ux_j$. Then we have that for all $y \in G$,

$$\sup_{x, x' \in U_j} |f(y^{-1}x) - f(y^{-1}x')| < \varepsilon/4, \quad j = 1, 2, \dots, n,$$

and that

$$\sup_{x, x' \in U_j} |P_\nu f(x) - P_\nu f(x')| < \varepsilon/4, \quad j = 1, 2, \dots, n,$$

for all $\nu \in P(G)$, where $P_\nu f(x) = \int_G f(y^{-1}x) d\nu(y)$. Let $h_\alpha(x) = P_{\mu_\alpha} f(x) - P_\mu f(x)$. It follows that for all α ,

$$\sup_{x, x' \in U_j} |h_\alpha(x) - h_\alpha(x')| < \varepsilon/2, \quad j = 1, 2, \dots, n.$$

Define a finite partition $\{E_j: j = 1, 2, \dots, n\}$ of G by $E_1 = U_1$ and $E_j = U_j - \bigcup_{i=1}^{j-1} U_i$ for $j \geq 2$. Choose a point a_j in E_j , $j = 1, 2, \dots, n$. Then we obtain that for all α ,

$$\sup_{x \in E_j} |h_\alpha(x) - h_\alpha(a_j)| < \varepsilon/2, \quad j = 1, 2, \dots, n.$$

A simple calculation yields

$$\begin{aligned} & \left| \int_G (x) (P_{\mu_\alpha} f(x) - P_\mu f(x)) dm(x) \right| \leq \int_G |h_\alpha(x)| dm(x) = \sum_{j=1}^n \int_{E_j} |h_\alpha(x)| dm(x) \\ & \leq \sum_{j=1}^n \int_{E_j} |h_\alpha(x) - h_\alpha(a_j)| dm(x) + \sum_{j=1}^n |h_\alpha(a_j)| \\ & < \varepsilon/2 + \sum_{j=1}^n |P_{\mu_\alpha} f(a_j) - P_\mu f(a_j)| \end{aligned}$$

for all α . Since $\mu_\alpha \rightarrow \mu$, there exists α_0 such that for all $\alpha \geq \alpha_0$,

$$|P_{\mu_\alpha} f(a_j) - P_\mu f(a_j)| < \varepsilon/2n, \quad j = 1, 2, \dots, n,$$

so that

$$\left| \int_G g(x) (P_{\mu_\alpha} f(x) - P_\mu f(x)) dm(x) \right| < \varepsilon \quad \text{for } \alpha \geq \alpha_0.$$

Thus $T_{\mu_\alpha} \rightarrow T_\mu$. $\square$

REMARK 1. We may show readily that there exists a homeomorphism $\mu \rightarrow T_\mu$ between the compact convex semigroups $P(G)$ and M_ℓ such that $T_\mu f = f * \mu$ for $f \in L_2(G)$, and that this mapping is not an isomorphism. (See Hewitt and Ross [12], 22.21)

COROLLARY 1. The mapping $\mu \rightarrow T_\mu$ of Theorem 1 carries $P^Z(G)$ onto M and is a topological isomorphism between the compact convex Abelian semigroups $P^Z(G)$ and M .

PROOF. In view of Proposition 1 and Theorem 1, it is enough to show that the mapping $\mu \rightarrow T_\mu$ is a bijection between $P^Z(G)$ and M . Note that $L_s = T_{\varepsilon_s}$ for $s \in G$. Then we have from Theorem 1, together with Theorem 2.5.1 of Stromberg [7], that $\mu \in P^Z(G)$ iff $\mu * \varepsilon_s = \varepsilon_s * \mu$ for $s \in G$ iff $T_\mu L_s = T_{\mu * \varepsilon_s} = T_{\varepsilon_s * \mu} = L_s T_\mu$ for $s \in G$ iff $T_\mu \in M$. $\square$

We also have from Theorem 1, the following corollary.

COROLLARY 2. M is the center of the semigroup M_ℓ .

As an immediate consequence of Proposition 3 and Corollary 1 we have

COROLLARY 3. The mapping $\mu \rightarrow T_\mu$ of Theorem 1 induces a topological isomorphism between the compact semigroups $P^1(G) \cap P^2(G)$ and $\{T \in M: T^2 = T\}$.

It is easy to see that the mapping $\mu \rightarrow T_\mu$ of Theorem 1 induces a homeomorphism between the compact sets $P^1(G)$ and $\{T \in M_\mu: T^2 = T\}$.

PROPOSITION 8. Let $\mu \in P(G)$ and $T \in M_\mu$ be the associated elements as in Theorem 1. The following assertions are equivalent:

- (i) $\mu * \mu = \mu$;
- (ii) $T^2 = T$;
- (iii) $T(fTg) = TfTg$ for $f, g \in L_\infty(G)$.

PROOF. The equivalence of (i) and (ii) is obvious. If we put $f = 1$ in (iii), then $T^2g = Tg$ for all $g \in L_\infty(G)$, so that $T^2 = T$. Therefore (iii) implies (ii).

Suppose that (i) holds. Then $\mu = m_H$, where H is a compact subgroup of G . It follows that for $f, g \in L_\infty$ and $x \in G$,

$$\begin{aligned} T(fTg)(x) &= \int_H f(y^{-1}x) Tg(y^{-1}x) d\mu(y) = \int_H f(yx) Tg(yx) d\mu(y) \\ &= \int_H f(yx) \left(\int_H g(z^{-1}yx) d\mu(z) \right) d\mu(y) = \int_H f(yx) \left(\int_H g(zyx) d\mu(z) \right) d\mu(y) \\ &= \int_H f(yx) \left(\int_H g(zx) d\mu(z) \right) d\mu(y) = Tf(x) Tg(x). \end{aligned}$$

Thus (i) implies (iii). $\square$

4. LEFT TRANSLATION OPERATORS.

Let $\underline{G}$ be the set of left translation operators, $\underline{G} = \{L_s: s \in G\}$. Then it is plain that $\underline{G}$ is a subgroup of the compact semigroup M_μ .

THEOREM 2. The mapping $s \rightarrow L_s$ is a continuous injection of G into M_μ and is a topological isomorphism of the compact groups G and $\underline{G}$.

PROOF. We shall show that the mapping $s \rightarrow L_s$ is a continuous map from G into M_μ . For $s \in G$, let

$$U(L_s: f, g, \epsilon) = \{T: T \in M_\mu, |\langle g, (L_s - T)f \rangle| < \epsilon\},$$

where $f, g \in C(G)$ and $\epsilon > 0$. Since $h(x, s) = g(x) \bar{f}(s^{-1}x)$ is right uniformly continuous on $G \times G$, there exists a neighborhood V of e such that $|h(x, s) - h(x, t)| < \epsilon$ for all t in Vs and for all x in G , so that $L_t \in U(L_s: f, g, \epsilon)$ for all t in Vs . This proves that the mapping is continuous.

We verify easily that the mapping is an algebraic isomorphism of the groups G and $\underline{G}$. It follows that, since G is compact, $\underline{G}$ is a compact subgroup of M_μ , and so the assertion follows. $\square$

As an immediate consequence of Theorem 2 we obtain

COROLLARY 4. The mapping $s \rightarrow L_s$ of Theorem 2 carries the center G^Z of G onto the center $\underline{G}^Z$ of $\underline{G}$ and is a topological isomorphism between the compact Abelian subgroups G^Z and $\underline{G}^Z$.

We prove the following characterization of a left translation on G .

LEMMA 4. Let ϕ be a mapping from G into itself. The following assertions are equivalent:

- (i) $\phi(xy) = \phi(x)y$ for all $x, y \in G$;
 (ii) there exists an element s in G such that $\phi(x) = sx$ for all $x \in G$.

PROOF. If (i) holds, we put $s = \phi(e)$, so that $\phi(x) = \phi(ex) = \phi(e)x = sx$ for all $x \in G$. Clearly (ii) implies (i). $\square$

Similarly we prove that a mapping ϕ from G into G is a right translation iff $\phi(xy) = x\phi(y)$ for all $x, y \in G$.

LEMMA 5. $\phi_1 \cap M_R = \phi \cap M_R = \underline{G}$.

PROOF. Since $\underline{G} \subset \phi_1 \cap M_R \subset \phi \cap M_R$, it suffices to show that $\phi \cap M_R \subset \underline{G}$. If $T_\phi \in \phi \cap M_R$, then $R_{y^{-1}}T_\phi = T_\phi R_{y^{-1}}$ for each $y \in G$, so that $\phi(xy) = \phi(x)y$ for $x, y \in G$. By Lemma 4 there exists $s \in G$ such that $\phi(x) = sx$ for all $x \in G$, and so $T_\phi = L_{s^{-1}} \in \underline{G}$. $\square$

THEOREM 3. Let T be in M_R . The following assertions are equivalent:

- (i) T is an isometry of $L_p(G)$ into itself for all $p \in [1, \infty)$;
 (ii) T is an isometry of $L_p(G)$ into itself for some $p \in [1, \infty)$;
 (iii) T is in $\underline{G}$.

PROOF. Clearly (i) implies (ii). Suppose that (ii) holds. Let $T \in M_R$ and $\mu \in P(G)$ be the associated elements as in Proposition 7. Let P be the Markov operator on $C(G)$ defined by $Pf(x) = \int_G P(x, dy)f(y)$, where $P(x, A) = \mu(xA^{-1})$. It follows from (ii), together with Jensen's inequality, that, for each nonnegative f in $C(G)$, $|Pf|^p(x) = P|f|^p(x)$ m -a.e. If the equality holds at a point $x \in G$, then the measure $P(x, \cdot)$ is a unit mass at some point $\sigma(x) \in G$, that is, $P(x, \cdot) = \epsilon_{\sigma(x)}(\cdot)$. Thus we have $P(x, \cdot) = \epsilon_{\sigma(x)}(\cdot)$ m -a.e. The mapping $\sigma(x)$ is defined on G m -a.e., but it can be defined everywhere on G in the usual manner. We show readily that $T = T_\sigma \in \phi \cap M_R$, and so by Lemma 5 T is in $\underline{G}$. Therefore (iii) holds. Clearly (iii) implies (i). $\square$

COROLLARY 5. Let $T \in M_R$. The following assertions are equivalent:

- (i) T is an isometric, algebra isomorphism of $L_1(G)$;
 (ii) T is an isometry on $L_1(G)$;
 (iii) T is in $\underline{G}$.

COROLLARY 6. Let $T \in M_R$. The following assertions are equivalent:

- (i) T is a unitary operator on $L_2(G)$;
 (ii) T is an isometry on $L_2(G)$;
 (iii) T is in $\underline{G}$.

By a measure-preserving set isomorphism ψ on $(G, B(G), m)$ we shall mean a mapping ψ of the measure algebra $\langle B(G), m \rangle$ into itself such that $\psi(G) = G$, $\psi(G-A) = G - \psi(A)$, $\psi(\bigcup_{j=1}^{\infty} A_j) = \bigcup_{j=1}^{\infty} \psi(A_j)$, and $m(\psi(A)) = m(A)$. Let Ψ be the family of such set mappings ψ . Each $\psi \in \Psi$ defines a unique operator $T_\psi \in D$ such that $T_\psi \chi_A = \chi_{\psi(A)}$. We also write Ψ for $\{T_\psi : \psi \in \Psi\}$. In particular, if G is a compact metrizable group, then each set-mapping $\psi \in \Psi$ is induced by a point-mapping

$\phi \in \Phi$, so that $T_{\Psi} X_A = T_{\phi} X_A$, that is, $\Psi = \phi$. See Lamperti [13]; Royden [14] for details. We prove the following analogue of Theorem 3 for D .

PROPOSITION 9. Let T be in D . The following assertions are equivalent:

- (i) T is an isometry of $L_p(G)$ into itself for all $p \in [1, \infty)$;
- (ii) T is an isometry of $L_p(G)$ into itself for some $p \in [1, \infty)$;
- (iii) T is in Ψ .

PROOF. Clearly (i) implies (ii). We next show that (ii) implies (iii). If T is an isometry of $L_p(G)$ for some p : $1 \leq p < \infty$, $p \neq 2$, then by Theorem 3.1 of Lamperti [13], we have $T \in \Psi$. If T is an isometry of $L_2(G)$, then, by the argument of Brown [11], pages 22, 23, we obtain $T \in \Psi$. It is straightforward to show that (iii) implies (i). $\square$

The following corollary follows from Theorem 3 and Proposition 9.

COROLLARY 7. $\Psi \cap M_r = \underline{G}$.

REMARK 2. By the Kawada-Wendel theorem (Kawada [15]; Wendel [1,16]), we have that for each T in D , T is an isometric algebra isomorphism of $L_1(G)$ onto itself if and only if $T = T_{\phi} \in \Phi_1$, where ϕ is a homeomorphic automorphism of G .

For a convex subset K of a real or complex vector space, let $\text{ext } K$ be the set of extreme points of K . We write G and G^Z for the sets $\{\epsilon_x: x \in G\}$ and $\{\epsilon_x: x \in G^Z\}$. We verify easily that $\text{ext } P(G) = G$ and

$$G^Z = P^Z(G) \cap \text{ext } P(G) \subset \text{ext } P^Z(G).$$

Example 1 will show that $\text{ext } P^Z(G) \neq G^Z$ in general. It is known (Stromberg [7]) that for each $\mu \in P(G)$, the measure $\tilde{\mu}$ defined by $\tilde{\mu}(E) = \int_G \mu(xEx^{-1}) dm(x)$ is an element of $P^Z(G)$.

LEMMA 6. If μ is a measure in $P^Z(G) - G^Z$ such that $e \in S(\mu)$, then it is not an extreme point of $P^Z(G)$.

PROOF. Let $S(\mu) = H$. Then there exists a neighborhood U of e such that $0 < t = \mu(U) < 1$. Define the measures μ_1 and μ_2 in $P(G)$ by $\mu_1(E) = \mu(E \cap U)/t$ and $\mu_2(E) = \mu(E \cap (H-U))/s$, where $s = 1 - t$. It follows that $\mu_1 \neq \mu_2$, $\mu = t\mu_1 + s\mu_2$, and $\mu = \tilde{t}\tilde{\mu}_1 + \tilde{s}\tilde{\mu}_2$. Then there exists a neighbourhood V of e such that $xVx^{-1} \subset U$ for all $x \in G$. Observe that $\tilde{\mu}_2(V) = \frac{1}{s} \int_G \mu(xVx^{-1} \cap (H-U)) dm(x) = 0$. On the other hand, since $f(x) = \mu(xVx^{-1})$ is lower semicontinuous on G , and $f(e) = \mu(V) > 0$, there exists a neighborhood W of e such that $f(x) > f(e)/2$ for each $x \in W$. Thus $\tilde{\mu}_1(V) = \frac{1}{t} \int_G f(x) dm(x) \geq \frac{1}{t} \int_W f(x) dm(x) > f(e)m(W)/2t > 0$. Accordingly $\tilde{\mu}_1 \neq \tilde{\mu}_2$ so that μ is not an extreme point of $P^Z(G)$. $\square$

EXAMPLE 1. Let $G = S_3$, the symmetric group on three symbols. Let $G = \{a_1, a_2, \dots, a_6\}$, where $a_1 = e$, $a_2 = (1,2)$, $a_3 = (1,3)$, $a_4 = (2,3)$, $a_5 = (1,2,3)$, $a_6 = (1,3,2)$. For $a_j \in G$, let $[a_j]$ be the conjugacy class of G which contains a_j . Then $[a_1] = \{e\}$, $[a_j] = \{a_2, a_3, a_4\}$ for $j = 2, 3, 4$, and $[a_j] = \{a_5, a_6\}$ for $j = 5, 6$. Let $A = [a_2]$ and $B = [a_5]$. Note that $G^Z = \{e\}$.

Let μ be the uniform probability measure on A , that is, $\mu(a_j) = 1/3$ for $j = 2, 3, 4$. Then $\mu \in P^Z(G)$ and $e \notin A = s(\mu)$. Suppose that $\mu = t\mu_1 + (1-t)\mu_2$, where $\mu_1, \mu_2 \in P^Z(G)$ and $0 < t < 1$. We show readily that $S(\mu_j) = A$ for $j = 1, 2$, and that $\mu = \mu_1 = \mu_2$. Thus $\mu \in \text{ext } P^Z(G) - G^Z$.

Let ν be the uniform probability measure on B . Clearly $\mu \neq \nu$. Define $\lambda = (\mu + \nu)/2$. It follows that ν and λ are elements of $P^Z(G)$, $e \notin S(\lambda) = A \cup B$, and $\lambda \notin \text{ext } P^Z(G)$.

THEOREM 4. $\text{ext } M_r = \underline{G}$, and M_r is the closed convex hull of $\underline{G}$ in the strong operator topology of $B[L_2(G)]$.

PROOF. Since $\text{ext } P(G) = G$, we have at once from Theorem 1 that $\text{ext } M_r = \underline{G}$. It follows from the Krein-Milman theorem, together with Theorem 1, that M_r is the closed convex hull of $\underline{G}$ in the weak operator topology of $B[L_2(G)]$. Since the convex hull of $\underline{G}$ has the same closure in both the weak operator and the strong operator topologies of $B[L_2(G)]$ (Dunford and Schwartz [17]), the assertion follows. $\square$

Since $G^Z \subset \text{ext } P^Z(G)$, or equivalently, $\underline{G}^Z \subset \text{ext } M$, we have from the Krein-Milman theorem, together with Corollary 1, that M contains the closed convex hull of $\underline{G}^Z$ in the weak operator topology of $B[L_2(G)]$.

REMARK 3. Let $T \in M_r$ and $\mu \in P(G)$ be the associated elements as in Theorem 1. Since G and $\underline{G}$ are topologically isomorphic, we may view the probability measure μ as a probability measure on M_r supported by $\underline{G} = \text{ext } M_r$. It follows that for $f, g \in L_2(G)$,

$$\langle f, Tg \rangle = \int_G \int_G f(x) \bar{g}(y^{-1}x) d\mu(x) d\mu(y) = \int_G \langle f, L_y g \rangle d\mu(y) = \int_G \langle f, Lg \rangle d\mu(L),$$

so that μ is the only probability measure on M_r which represents T and which is supported by $\text{ext } M_r$. Thus a sharper form of the Choquet-Bishop-de Leeuw theorem (see Phelps [18], page 24) holds for M_r .

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