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THE SERRE DUALITY THEOREM FOR RIEMANN SURFACES

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ABSTRACT. Given a Riemann surface S, there exists a finitely generated Fuchsian group G of the first kind acting on the upper half plane U, such that $S \cong U/G$. This isomorphism makes it possible to use Fuchsian group methods to prove theorems about Riemann surfaces. In this note we give a proof of the Serre duality theorem by Fuchsian group methods which is technically simpler than proofs depending on sheaf theoretic methods.

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1. INTRODUCTION.

For a compact Riemann surface S of genus $g > 1$, the uniformization theorem states that there exists a universal covering map $\pi: U \to S$, where U is the upper half plane. The covering group G is a finitely generated Fuchsian group of the first kind without elliptic or parabolic elements. As a consequence of this result, it is possible to prove theorems about Riemann surfaces in two different ways. For example, Riemann and his followers constructed meromorphic functions on compact Riemann surfaces by first constructing meromorphic differentials on the surface; the ratio of two such differentials would then be a meromorphic function. On the other hand Poincaré explicitly constructed certain series on U, which were automorphic forms for a given Fuchsian group G, and their ratios were automorphic functions for G. The isomorphism $S \cong U/G$ then gives the existence of meromorphic functions of S. The fact that U is simply connected sometimes makes it easier to work there. In this note we show that the Serre duality theorem for Riemann surfaces can be proven by Fuchsian group methods. Though the structure of the proof is the same as in, say, Gunning [1], certain technical simplifications are introduced by working in U. The main idea of the proof is already contained in Kra [2], where a particular case of the duality theorem is proved. To obtain the general case it is necessary to introduce appropriate definitions.

The proof of the duality theorem in Gunning's book uses the sheaf of germs of distributions on Riemann surfaces. Though we do not require distributions, we need the concept of a generalized or weak derivative. We therefore begin by stating the necessary definitions and results on weak derivatives.

Let D be an arbitrary open subset of the complex plane $\mathbb{C}$, and f a measurable function which is Lebesgue integrable over every compact subset of D. We say that g, a Lebesgue integrable function on D, is a generalized - $\bar{z}$ - derivative of f if

$$\iint_D \phi g \, dz\wedge d\bar{z} = -\iint_D \phi_{\bar{z}} f \, dz\wedge d\bar{z} \tag{1.1}$$

holds for all infinitely differentiable functions ϕ with compact support in D, and where $\phi_{\bar{z}} = \frac{1}{2}\left(\frac{\partial\phi}{\partial x} + i\frac{\partial\phi}{\partial y}\right)$. We write this relation between f and g as $f_{\bar{z}} = g$. In a similar way we can use

$$\iint_D \phi g \, dz\wedge d\bar{z} = -\iint_D \phi_z f \, dz\wedge d\bar{z} \tag{1.2}$$

where $\phi_z = \frac{1}{2}\left(\frac{\partial\phi}{\partial x} - i\frac{\partial\phi}{\partial y}\right)$, to define the generalized - z - derivative of f.

We now state two lemmas which we require in the sequel. The first lemma essentially says that if some function satisfies the Cauchy-Riemann relations in the weak sense then it is holomorphic. The second lemma, which is technical, constructs a partition of unity for a Fuchsian group G on U. Proofs of both lemmas are contained in Kra [3].

LEMMA 1. Suppose f is a measurable function, which is Lebesgue integrable over every compact subset of D, and $f_{\bar{z}} = 0$. Then there is a holomorphic function g on D such that g = f almost everywhere.

LEMMA 2. For a Fuchsian group G acting on U, there exists a function $\eta \in C^\infty(U)$ such that

(i) $0 \leq \eta \leq 1$

(ii) for each $z \in U$, there is a neighborhood V of z and a finite set $J \leq G$, such that $\eta|_{g(V)} = 0$ for each $g \notin J$, and

(iii) $\sum_{g\in G} \eta(gz) = 1$, $z \in U$.

2. STATEMENT OF THE DUALITY THEOREM.

We now give the necessary definitions to state the duality theorem. Let G be a Fuchsian group acting on the upper half plane U, such that, U/G is a compact Riemann surface. The group G may have elliptic elements. Let K^∞, 0 and 0* denote the sets of C^∞, holomorphic and nowhere vanishing holomorphic functions, respectively, on U. A factor of automorphy for G is a function

$$\phi: G \times U \to \mathbf{C}^* = \mathbf{C} - \{0\}, \tag{2.1}$$

such that, for a fixed $g \in G$, $\phi(g,z) \in 0^*$, and for all g_1, $g_2 \in G$ and all $z \in U$,

$$\phi(g_1 \circ g_2, z) = \phi(g_1, g_2(z))\phi(g_2, z). \tag{2.2}$$

The product of two factors of automorphy ϕ_1 and ϕ_2 is defined by

$$(\phi_1\phi_2)(g,z) = \phi_1(g,z)\phi_2(g,z), \text{ for } g \in G \text{ and } z \in U. \quad (2.3)$$

With this product the set of factors of automophy forms a group. Of special importance is the canonical factor of automorphy defined as

$$\kappa(g,z) = g'(z), \text{ for } g \in G \text{ and } z \in U. \quad (2.4)$$

It is clear that a factor of automorphy for G projects to a line bundle on U/G (see Gunning [1]).

Now let X denote either K^{∞} or 0. We define the action of a group G on X with respect to a given factor of automorphy ϕ as follows:

$$(f \cdot g)(z) = f(g(z))\phi(g,z) \text{ where } f \in X,\ g \in G \text{ and } z \in U. \quad (2.5)$$

The zeroth cohomology group with respect to ϕ and with coefficients in X is

$$H^{o}(G,X)_{\phi} = \{f \in X \mid f \cdot g = f \text{ for all } g \in G\}. \quad (2.6)$$

To any factor of automorphy ϕ there corresponds an integer $C(\phi)$, the Chern class of ϕ. This may be defined by choosing an $f \in H^{o}(G,0)_{\phi}$. Let deg f = the number of zeros of f in a fundamental domain of G, counting multiplicity. Clearly deg f is independent of the fundamental domain. Moreover, if f_1 is another function in $H^{o}(G,0)_{\phi}$, then it is obvious by using (2.5) and (2.6) that

$$\frac{f(z)}{f_1(z)} = \frac{f(g(z))}{f_1(g(z))} \text{ for all } g \in G. \quad (2.7)$$

Thus $\frac{f}{f_1}$ is an automorphic function for G and deg f = deg f_1. We define $C(\phi)$ = deg f for any $f \in H^{o}(G,0)_{\phi}$.

To state the duality theorem we also require the first cohomology group with respect to ϕ and with coefficients in X,

$H^{1}(G,X)$ = space of 1 - cocycles with coefficients in X/space of 1
1 - coboundaries. (2.8)

A 1-cocycle with coefficients in X is a map P taking $g \in G \to P_g \in X$, such that,

$$P_{g_1 \circ g_2} = P_{g_1} \cdot g_2 + P_{g_2}. \quad (2.9)$$

Moreover, for each $f \in X$ the map $g \in G \to f \cdot g - f \in X$, which is obviously a 1-cocycle, is called a 1-coboundary.

To enable us to define certain Banach spaces we would like to have a metric corresponding to any factor of automorphy ϕ. For example, the Poincaré metric,

$$\lambda_U(z) = \frac{1}{\text{Im}z} \text{ where } z \in U, \quad (2.10)$$

which satisfies the property

$$\lambda_U(gz)\,|g'(z)| = \lambda_U(z) \text{ for all } g \in G \quad (2.11)$$

will be viewed as a metric corresponding to the canonical factor of automorphy. For any other factor of automorphy ϕ and with η as in lemma 2, we define the metric

$$\lambda_\phi(z) = \sum_{h\in G} \eta(hz)|\phi(h,z)|, \text{ which satisfies} \tag{2.12}$$

$$\lambda_\phi(g,z)|\phi(g,z)| = \lambda_\phi(z) \text{ for all } g \in G. \tag{2.13}$$

We now define two Banach spaces corresponding to any factor of automophy ϕ. The first is $L^1(G,\phi\bar{K})$ = the set of all measurable functions μ on U which satisfy

$$\mu(gz)\phi(g,z)\overline{g'(z)} = \mu(z) \text{ for all } g \in G, \text{ and} \tag{2.14}$$

$$||\mu||_1 = \iint_{U/G}\lambda_U(z)\lambda_\phi^{-1}(z)|\mu(z)dz\wedge d\bar{z}| < \infty. \tag{2.15}$$

The other Banach space is $L^\infty(G,\phi^{-1}K)$ = the set of all measurable functions ν on U which satisfy

$$\nu(gz)\phi^{-1}(g,z)g'(z) = \nu(z) \text{ for all } g \in G, \text{ and} \tag{2.16}$$

$$||\nu||_\infty = \sup_{z\in U} \lambda_U^{-1}(z)\lambda_\phi(z)|\nu(z)| < \infty \tag{2.17}$$

For $\mu\in L^{-1}(G,\phi\bar{K})$ and $\nu\in L^\infty(G,\phi^{-1}K)$ we have the inner product

$$(\mu,\nu) = \iint_{U/G}\mu(z)\nu(z)\, dz\wedge d\bar{z}. \tag{2.18}$$

It is a consequence of the Riesz representation theorem that this inner product defines $L^\infty(G,\phi^{-1}K)$ as the dual of $L^1(G,\phi\bar{K})$. We note that $H^\circ(G,0)_{\phi^{-1}K}$ is a subspace of $L^\infty(G,\phi^{-1}K)$ and $H^\circ(G,K^\infty)_{\phi\bar{K}}$ is a subspace of $L^\infty(G,\phi\bar{K})$, because any C^∞ function on U is bounded on any fundamental domain of G.

We are now in a position to state the duality theorem:

THEOREM 1. The vector spaces $H^\circ(G,0)_{\phi^{-1}K}$ and $H^1(G,0)_\phi$ are finite dimensional and canonically dual to each other.

3. FINITE DIMENSIONALITY OF $H^\circ(G,0)_\phi$.

As a first step toward the duality theorem we prove

THEOREM 2. $H^\circ(G,0)_\phi$ is a finite dimensional vector space over the field of complex numbers.

PROOF. Let $n = C(\phi) + 1$ and let $z_1, z_2, \ldots, z_n$ be n points in a fundamental domain of G. We claim that $\dim H^\circ(G,0)_\phi \le n$. For, if $\dim H^\circ(G,0)_\phi > n$, then the linear map

$$T:H^\circ(G,0)_\phi \to \mathbb{C}^n \text{ given by } T(f) = (f(z_1), \ldots, f(z_n)) \tag{3.1}$$

has a nontrivial kernel. Suppose $f \in \text{Ker } T$, $f \neq 0$. Then $\deg f \geq n > C(\phi)$ which contradicts the fact that $\deg f = C(\phi)$.

4. A DE RHAM TYPE THEOREM.

We need the following lemma to prove an analogue of de Rham's theorem. It is contained in Kra [2,3].

LEMMA 3. The cohomology group $H^1(G,K^\infty)_\phi = \{0\}$.

PROOF. Let η be the C^∞ function of lemma 2 and let $P \in H^1(G,K^\infty)_\phi$. Define the C^∞ function

$$\theta(z) = -\sum_{g\in G} \eta(gz)P_g(z). \tag{4.1}$$

If $A \in G$, then

$$(\theta\cdot A)(z) - \theta(z) = -\sum_{g\in G} [\eta(gAz)P_g(Az)\phi(A,z) - \eta(gz)P_g(z)] \tag{4.2}$$

$$= -\sum_{g\in G} [\eta(gAz)(P_{g\circ A}(z) - P_A(z)) - \eta(gz)P_g(z)] \tag{4.3}$$

$$= \sum_{g\in G} \eta(gAz)P_A(z) = P_A(z). \tag{4.4}$$

Thus P is a 1-coboundary and the lemma is proved.

THEOREM 3. $H^1(G,0)_\phi \simeq \dfrac{H^\circ(G,K^\infty)_{\phi\bar{K}}}{\bar{\partial}H^\circ(G,K^\infty)_\phi}$.

PROOF. A proof of this result using exact sequences can be given. However, we shall prove it directly. Consider a function $f \in H^\circ(G,K^\infty)_{\phi\bar{K}}$; thus

$$f(gz)\phi(g,z)g'(z) = f(z) \text{ for all } g \in G. \tag{4.5}$$

Suppose $\theta(z)$ is a C^∞ solution of the differential equation $\dfrac{\partial\theta}{\partial\bar{z}} = f(z)$ in U. Then equation (4.5) implies that

$$\frac{\partial}{\partial\bar{z}}(\theta(gz)\phi(g,z) - \theta(z)) = 0, \tag{4.6}$$

and we see that $\theta\cdot g - \theta = P_g$ is a holomorphic function. It is easy to check that $g \to P_g$ is a holomorphic 1-cocyle and that we have obtained a well defined map

$$T:H^\circ(G,K^\infty)_{\phi\bar{K}} \to H^1(G,0)_\phi. \tag{4.7}$$

This map is onto. For suppose $P \in H^1(G,0)_\phi$. By lemma 3 there is a C^∞ function θ such that

$$\theta(gz)\phi(g,z) - \theta(z) = P_g(z) \text{ for all } g \in G. \tag{4.8}$$

It is clear that $\dfrac{\partial\theta}{\partial\bar{z}} \in H^\circ(G,K^\infty)_{\phi\bar{K}}$ and that it will be mapped to P. It remains to be shown that the kernel of T is $\bar{\partial}H^\circ(G,K^\infty)_\phi$, but this is straightforward.

5. SERRE'S DUALITY THEOREM.

If B is a Banach space then let B* denote the dual space of B; that is, members of B* are bounded linear functionals on B. With this notation we have:

THEOREM 4. $(H^1(G,0)_\phi)^* \simeq H^\circ(G,0)_{\phi^{-1}K}$.

PROOF. This proof is a minor modification of the proof of a particular case in Kra [3]. By theorem 3 it suffices to prove that

$$\left(\frac{H^\circ(G,K^\infty)_{\phi\bar{K}}}{\bar{\partial}H^\circ(G,K^\infty)_\phi}\right)^* \simeq H^\circ(G,0)_{\phi^{-1}K}. \tag{5.1}$$

If $f \in H^\circ(G,0)_{\phi^{-1}K}$, then f defines a linear functional ℓ on $H^\circ(G,K^\infty)_{\phi\bar{K}}$ given by

$$\ell(\psi) = \iint_{U/G} f(z)\psi(z)dz\wedge d\bar{z} \text{ for } \psi \in H^{\circ}(G,K^{\infty})_{\phi\bar{K}} . \tag{5.2}$$

Moreover, if $\psi = \bar{\partial}\tau$ where $\tau \in H^{\circ}(G,K^{\infty})_{\phi}$, then

$$\ell(\psi) = \iint_{w} f(z)\bar{\partial}\tau \, dz\wedge d\bar{z} = \iint_{w}\bar{\partial}(f\tau)\, dz\wedge d\bar{z} = -\int_{\partial w} f\tau dz = 0. \tag{5.3}$$

In (5.3) we have chosen U/G to be the standard fundamental domain w (also called the normal polygon) for G and applied Stokes' theorem. Thus ℓ is a linear functional on $H^{\circ}(G,K^{\infty})_{\phi\bar{K}}/\bar{\partial}H^{\circ}(G,K^{\infty})_{\phi}$. To show that different functions in $H^{\circ}(G,0)_{\phi^{-1}K}$ define distinct linear functionals, suppose that for some $f \in H^{\circ}(G,0)_{\phi^{-1}K}$,

$$\iint_{w} f\psi dz\wedge d\bar{z} = 0 \text{ for all } \psi \in H^{\circ}(G,K^{\infty})_{\phi\bar{K}}. \tag{5.4}$$

We consider only real valued functions ψ in (5.4) and prove that this equation implies Imf = 0. From this we shall conclude that f = 0. Equation (5.4) gives

$$\iint_{w} \text{Re}(-2if)\psi \frac{dz\wedge d\bar{z}}{-2i} = 0 \text{ for all real valued } \psi \in H^{\circ}(G,K^{\infty})_{\phi\bar{K}}. \tag{5.5}$$

If Re(-2if) is nonzero at some point z interior to w, then Re(-2if) has the same sign in a neighborhood V of z. We can construct a function $\psi \in H^{\circ}(G,K^{\infty})_{\phi\bar{K}}$ such that ψ has support in $V\cap w$, $\psi \geq 0$ in $V\cap w$ and $\psi(z) > 0$. Since such a ψ would contradict (5.5) we must have f = 0.

It remains to prove that every bounded linear functional on $H^{\circ}(G,K^{\infty})_{\phi\bar{K}}/\bar{\partial}H^{\circ}(G,K^{\infty})_{\phi}$ is a member of $H^{\circ}(G,0)_{\phi^{-1}K}$. Now ℓ can be viewed as a functional on $H^{\circ}(G,K^{\infty})_{\phi\bar{K}}$ which vanishes on $\bar{\partial}H^{\circ}(G,K^{\infty})_{\phi}$. By the Hahn-Banach theorem ℓ can be extended to a bounded linear functional on $L^{1}(G,\phi\bar{K})$ and by the Riesz representation theorem there is a $\nu \in L^{\infty}(G,\phi^{-1}K)$, such that,

$$\ell(\mu) = (\mu,\nu) \text{ for all } \mu \in L^{1}(G,\phi\bar{K}). \tag{5.6}$$

Since ℓ vanishes on $\bar{\partial}H^{\circ}(G,K^{\infty})_{\phi}$, therefore,

$$(\bar{\partial}\theta,\nu) = 0 \text{ for all } \theta \in H^{\circ}(G,K^{\infty})_{\phi}. \tag{5.7}$$

Lemma 1 now implies that ν is holomorphic. This proves theorem 4. The duality result follows by combining theorems 2 and 4.

The reader may consult Kra[3] for other examples of theorems about Riemann surfaces proven by Fuchsian group methods.

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