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SEPARATION METRICS FOR REAL-VALUED RANDOM VARIABLES

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ABSTRACT. If W is a fixed, real-valued random variable, then there are simple and easily satisfied conditions under which the function d_W, where $d_W(X,Y) =$ the probability that W "separates" the real-valued random variables X and Y, turns out to be a metric. The observation was suggested by work done in [1].

KEY WORDS AND PHRASES. Random variables, probability spaces, distribution functions metrics, metrics on random variables

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THEOREM 1. Let Ω be a probability space with probability measure P, and let W be a fixed, real-valued random variable on Ω. Then the function d_W defined by

$$d_W(X,Y) = P[X \le W < Y \text{ or } Y \le W < X]$$

is a pseudo metric on the space of real-valued random variables defined on Ω.

PROOF. We need check only the triangle inequality. Let X, Y, and Z be real-valued random variables on Ω and set

$$A = \{\omega \in \Omega : X(\omega) \le W(\omega) < Y(\omega) \text{ or } Y(\omega) \le W(\omega) < X(\omega)\},$$

$$B = \{\omega \in \Omega : Y(\omega) \le W(\omega) < Z(\omega) \text{ or } Z(\omega) \le W(\omega) < Y(\omega)\}, \text{ and}$$

$$C = \{\omega \in \Omega : X(\omega) \le W(\omega) < Z(\omega) \text{ or } Z(\omega) \le W(\omega) < X(\omega)\}$$

Let $\omega \in C$. If $X(\omega) \le W(\omega) < Z(\omega)$, then $Y(\omega) \le W(\omega)$ implies $W(\omega)$ "separates" $Y(\omega)$ and $Z(\omega)$ in such a fashion that $\omega \in B$, and $W(\omega) < Y(\omega)$ implies $W(\omega)$ "separates" $X(\omega)$ and $Y(\omega)$ in such a fashion that $\omega \in A$. A similar conclusion holds in the case $Z(\omega) \le W(\omega) < X(\omega)$. Thus $C \subset A \cup B$ and $P(C) \le P(A) + P(B)$ which is the triangle inequality.

REMARK. Random variables which differ only on a set of probability measure 0 will be considered to be identical.

THEOREM 2. Let Ω, P, and W be as in Theorem 1. Let R be some given collection of real-valued random variables on Ω, and suppose that (1) W is independent of every pair of members of R in the sense that if X, Y $\in$ R and A, B, and C are intervals in $\mathbb{R}$, then

$$P\,[X \in A,\ Y \in B,\ W \in C] = P\,[X \in A,\ Y \in B] \cdot P\,[W \in C] \quad \text{and}$$

(2) for every open interval J in $\mathbb{R}$ we have

$$P\,[W \in J] > 0\ .$$

Then d_W, as defined in Theorem 1, is a metric on R.

PROOF. Let X, Y $\in$ R such that X $\neq$ Y. We have only to show $d_W(X,Y) > 0$. We may, without loss of generality, suppose that the set

$$A = \{\omega \in \Omega : X(\omega) < Y(\omega)\}$$

has positive P-measure. Then there must be rational numbers p and q such that

$$B = \{\omega \in \Omega : X(\omega) < p < q < Y(\omega)\}$$

has positive P-measure. It follows that

$$\begin{aligned} d_W(X,Y) &\geq P\,[X < W < Y] \\ &\geq P\,[X < p < W < q < Y] \\ &= p(B) \cdot P\,[p < W < q] > 0. \end{aligned}$$

REMARK 1. In connection with this last theorem, it can be shown that if F_W, the cumulative distribution function of W, is continuous, then

$$d_W(X,Y) = E(|F_W(X) - F_W(Y)|)$$

where E means expected value.

REMARK 2. Again in connection with the last theorem, it might be objected that for some R no W exists with the desired properties; this would be the case, for example, if R was the set of all real-valued random variables on Ω. However, one can always "embed" R in a larger space of real-valued random variables containing something suitable for use as W. Simply take another probability space Ω', let W be a real-valued random variable on Ω' taking on values in every open interval of $\mathbb{R}$ with positive probability, let Ω^* be the product space $\Omega \times \Omega'$, let each X in R be replaced X^* where $X^*(\omega,\omega') = X(\omega)$, and let W be replaced by W^* where $W^*(\omega,\omega') = W(\omega')$. It follows that $X_1^*, \ldots, X_n^*$ must have the same joint distribution function as $X_1, \ldots, X_n$ when $X_1, \ldots, X_n$ are members of R and that W^* is independent of all X^* such that X $\in$ R in the desired fashion. So one may construct the metric d on R defined by

$$d(X,Y) = d_{W^*}(X^*,Y^*).$$

REFERENCE

1. TAYLOR, Michael D., New metrics for weak convergence of distribution functions. To appear.