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ON A METHOD FOR INVERSE THEOREMS FOR (C,1) AND GAP (C,1) SUMMABILITY

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ABSTRACT. A simple method for proving several theorems of Tauberian and Mercerian type for (C,1) and gap (C,1) summability is presented.

KEY WORDS AND PHRASES. *(C,1), gap (C,1) summability, Tauberian, numerical Tauberian, Mercerian theorems.*

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1. INTRODUCTION AND RESULTS.

Let a_ν be reals and

$$s_n = \sum_{\nu=o}^{n} a_\nu, \qquad \sigma_n = (n+1)^{-1} \sum_{\nu=o}^{n} s_\nu.$$

We present here an elementary procedure for proving, in a simple and unified way, the following five results relating the asymptotics of σ_n to that of s_n:

1.1. G. Hardy's classical Tauberian theorem with E. Landau's convergence condition, [1. §6.1].

THEOREM 1. If

$$\sigma_n \to s, \quad n \to \infty \tag{1.1}$$

and

$$a_n > -M/n, \quad M > 0, \tag{1.2}$$

then $s_n \to s$, $n \to \infty$.

The theorem still holds if the condition (1.2) is replaced by the weaker one of R. Schmidt, [1. §6.2]:

$$\liminf_{n\to\infty}(s_q - s_p) \geqslant 0, \quad q > p, \quad q/p \to 1. \tag{1.2'}$$

We prove in addition the following more general

THEOREM 2. From (1.1) and

$$\left.\begin{aligned} &\liminf_{p\to\infty} \frac{1}{p}\sum_{\nu=1}^{p} (s_{n+\nu}-s_n) \geq 0, \\ &\liminf_{p\to\infty} \frac{1}{p}\sum_{\nu=1}^{p-1} (s_n-s_{n-\nu}) \geq 0, \end{aligned}\right\} \quad p=o(n) \tag{1.2''}$$

there follows $s_n \to s$, $n \to \infty$.

Although Theorem 1 is implied by Theorem 2 the proof of the former is also given as being simpler than that of the implication.

1.2. Gap (C,1) summability. In all above results only the convergence condition has been gradually relaxed from the original Hardy's $a_n = O(1/n)$ to (1.2") whereas (1.1) - being considered exclusively as a summability method - is kept unaltered. Contrary to that, we fix here the original Hardy's convergence condition but treat (1.1) as an assumption that may be relaxed by changing the notion of the (C,1) summability itself in such a way, that this, together with the convergence condition, still leads to $s_n \to s$. To that effect we introduce the notion of the "gap (C,1) summability" by the following:

DEFINITION 1. A series Σa_ν is said to be gap (C,1) summable to s if for a sequence $\{n_k\}$ of positive integers there follows $\sigma_{n_k} \to s$, $k \to \infty$.

We prove the following

THEOREM 3. Let $\{n_k\}$ be a sequence of positive integers such that

$$n_{k+1} - n_k = o(n_k), \qquad k \to \infty. \tag{1.3}$$

If

$$\sigma_{n_k} \to s, \quad k \to \infty \tag{1.4}$$

and

$$a_n = O(1/n), \tag{1.5}$$

then

$$s_n \to s, \qquad n \to \infty.$$

Notice that o-and O-symbols can be interchanged in (1.3) and (1.5) without affecting the statement of the theorem.

1.3. A Tauberian theorem with remainder. The above mentioned convergence conditions, and in particular, the R. Schmidt's one - (1.2´), (characterizing slowly decreasing sequences) are related to the class $R(\gamma,\Gamma)$ of O-regularly varying functions as introduced in [2] and [3].

We exploit the class here for proving a Tauberian theorem with a remainder term ("Numerical" Tauberian theorem) (cf. [4. §1.8]).

DEFINITION 2. A function f(x) is almost increasing if there exists a constant $A > 1$ such that $x_1 < x_2$ implies $f(x_1) < A\, f(x_2)$; almost decreasing functions are defined likewise.

DEFINITION 3. A positive continuous function $g(x)$ is of class $R(\gamma,\Gamma)$ if there exist $\gamma > 0$, $\Gamma > 0$ such that $x^{\gamma}g(x)$ is almost increasing and $x^{-\Gamma}g(x)$ is almost decreasing for sufficiently large x.

We use here the definition from [3] but the class of functions in question has been introduced and studied already in [2] in a more general form.

We prove

THEOREM 4. Let $\phi_i(x) \in R(\gamma_i,\Gamma_i)$, $\quad i = 1,2,$

$$0 < c_1 \leq \phi_1(x)\phi_2(x) \leq c_2x^2 \qquad \text{for some } c_i \tag{1.6}$$

and

$$\phi_2(x) \to \infty, \quad \phi_1(x) = o\{\phi_2(x)\}, \qquad x \to \infty.$$

If

$$\sigma_n = s+O(1/\phi_2(n)), \qquad n \to \infty \tag{1.7}$$

and

$$a_n > -\phi_1(n)/n, \tag{1.8}$$

then

$$s_n = s + O\{(\phi_1(n)/\phi_2(n))^{1/2}\}, \qquad n \to \infty. \tag{1.9}$$

1.4. A Mercerian theorem. We prove

THEOREM 5. If

$$s_n + c_n\sigma_n \to 0, \qquad n \to \infty \tag{1.10}$$

and

$$\liminf_{n\to\infty} c_n > -1, \tag{1.11}$$

then

$$s_n \to 0, \quad n \to \infty.$$

This contains as special cases the original Mercer's theorem where $c_n = c$ $(c > -1)$, [1. §5.9], and a result of Vijayaraghavan where besides (1.11) one assumes $c_n = O(1)$, [6].

2. PROOFS.

All mentioned results follow directly from the following simple identities used occasionally by the second author, and for the first time in [5] in the case of Fourier series.

For $m > n$ one has

$$m\sigma_{m-1} - n\sigma_{n-1} = (m-n)s_n + \sum_{\nu=1}^{m-n-1} (s_{n+\nu} - s_n) \tag{i}$$

or, with $p > 1$,

$$(n+p+1)\sigma_{n+p} - (n+1)\sigma_n = ps_n + \sum_{\nu=1}^{p} (s_{n+\nu} - s_n). \tag{i'}$$

For $m < n$ one has

$$m\sigma_{m-1} - n\sigma_{n-1} = -(n-m)s_n + \sum_{\nu=1}^{n-m} (s_n - s_{n-\nu}) \tag{ii}$$

or, with $p > 1$,

$$(n-p+1)\sigma_{n-p} - (n+1)\sigma_n = -ps_n + \sum_{\nu=1}^{p-1} (s_n - s_{n-\nu}) \tag{ii'}$$

REMARK. It is also useful to replace the sums in the right-hand sides of (i´) and (ii´) by $\Sigma_1^p(p-\nu+1)a_{n+\nu}$, and $\Sigma_1^{p-1}(p-\nu)a_{n-\nu+1}$ respectively.

The proof of these identities follows by a straightforward calculation and hence is omitted.

Throughout the proofs we tacitly take $s = 0$ for otherwise one could consider $s_n - s$ instead. Also, all occuring inequalities hold for large values of the variable in question.

PROOF OF THEOREM 1. Suppose contrary to the statement of the theorem that there exists a sequence $\{n_k\}$ tending to infinity with k and such that for $n = n_k$

$$s_n \geqslant A > 0. \tag{2.1}$$

Let m denote an (arbitrary large) positive integer independent of n_k and, having in mind the Remark, write the identity (i´) with $n = n_k$ as

$$\frac{mps_n}{n+1} = m\,\frac{n+p+1}{n+1}\,\sigma_{n+p} - m\sigma_n - \frac{m}{n+1}\sum_{\nu=1}^{p}(p-\nu+1)a_{n+\nu}. \tag{2.2}$$

Now choose $p = [n_k/m]$; then, because of (1.1), first two terms on the right-hand side of (2.2) tend to zero for $k \to \infty$. The third term is, due to (1.2), majorized by M_1/m, $M_1 > 0$ and hence can be made arbitrary small. The left-hand side of (2.2) is, however, due to (2.1), bounded away from zero. If $s_n \leqslant -A < 0$ for $n = n_k$, then the identity (ii´) would lead to a similar contradiction which completes the proof.

PROOF OF THEOREM 2. Suppose that there exists a sequence $\{n_k\}$ such that for $n = n_k$, $s_n \geqslant A > 0$. Then, write the identity (i´) for $n = n_k$ as

$$\frac{n+1}{p}\,\sigma_n - \frac{n+p+1}{p}\,\sigma_{n+p} + \frac{1}{p}\sum_{\nu=1}^{p}(s_{n+\nu} - s_\nu) = -s_n.$$

Let ε be an (arbitrary small) positive number independent of $n(=n_k)$ and put $p = [\varepsilon n]$ in the preceding identity. Then, the first two terms on its left-hand side are, due to (1.1), small for large n. Hence, because of the first condition (1.2"), the whole left-hand side is either small or nonnegative while the right-hand side one is negative and bounded away from zero. If for $n = n_k$, $s_n \leqslant A < 0$ then the identity (ii´) and the second condition (1.2") are used in the same way to obtain a contradiction.

PROOF OF THEOREM 3. It is sufficient to prove that $\sigma_\lambda \to 0$, $\lambda \to \infty$ and then apply Hardy's original Tauberian theorem for (C,1 method (i.e. Theorem 1. with $a_n = O(1/n)$ instead of (1.2)).

For any positive integer λ there is a k such that

$$n(k) \leq \lambda \leq n(k+1), \tag{2.3}$$

where we put $n_k = n(k)$ for convenience. With such a k write σ_λ as

$$\sigma_\lambda = (\lambda+1)^{-1}\{(n(k)+1)\sigma_{n(k)}+s_{n(k)+1}+\dots+s_\lambda\}. \tag{2.4}$$

We shall show that s_ν are bounded for any index ν between $n(k)+1$ and λ. To that end choose a subsequence $m(k)$ of $n(k)$ such that there exist two integers A, B, $B > A > 1$ such that

$$An(k) < m(k) \leq Bn(k) \tag{2.5}$$

which is possible due to (1.3). Inequalities (2.5) imply also

$$0 < m_2 \leq \frac{m(k)n(k)}{n(k)} \leq M_2. \tag{2.6}$$

Further, having in mind the Remark, rewrite the identity (i) with $p = m - n - 1$,take $m - 1 = m(k)$, $n = n(k)$ and put $\ell(k) = m(k) - n(k)$ to obtain

$$s_{n(k)}= \frac{m(k)+1}{\ell(k)}\ \sigma_{m(k)}- \frac{n(k)+1}{\ell(k)}\ \sigma_{n(k)}- \sum_{\nu=1}^{\ell(k)} (1- \frac{\nu-1}{\ell(k)})a_{n(k)+\nu}.$$

Using (1.4), (1.5), (2.5) and (2.6) to estimate the right-hand side of the previous identity one gets $s_{n(k)} = O(1)$.

Now, for any index in question, i.e. such that $n(k)+1 \leq \nu \leq \lambda$, one has

$$|s_\nu-s_{n(k)}| \leq |a_{n(k)+1}|+\dots+|a_\lambda|,$$

or, by (1.5), (2.3) and (1.3),

$$|s_\nu -s_{n(k)}|= o(1).$$

Hence s_ν is bounded since $s_{n(k)}$ is; consequently (2.4) and (2.3) lead to

$$|\sigma_\lambda| \leq |\sigma_{n(k)}|+ M_3 \frac{n(k+1)-n(k)}{n(k)}$$

and $\sigma_\lambda \to 0$, $\lambda \to \infty$, because of (1.3) and (1.4) qed.

PROOF OF THEOREM 4. Again, write the identity (ii) as

$$\frac{-ps_n}{n+1}\ \phi_2(n)= -\phi_2(n)\sigma_n+ \frac{n-p+1}{n+1}\ \frac{\phi_2(n)}{\phi_2(n-p)}\ \phi_2(n-p)\sigma_{n-p}- \frac{\phi_2(n)}{n+1}\ \sum_{\nu=1}^{p-1} (p-\nu)a_{n-\nu+1}, \tag{2.7}$$

and suppose first that $s_n < 0$ for a sequence $\{n_k\}$ (or for all n). Hence the left-hand side of (2.7) is non-negative for $n = n_k$. On the other hand, the first two terms on the right-hand side are bounded above. The former because of (1.7), and the later because of (1.7) and the fact that $n^{-\Gamma_2}\phi_2(n)$ is almost decreasing, implying

$$\frac{\phi_2(n)}{\phi_2(n-p)} \leq A(\frac{n}{n-p})^{\Gamma_2}$$

which is bounded for $p/n \leq k < 1$. The third term on the right hand side is majorized by $p^2\phi_1(n)\phi_2(n)n^{-2}$ by using (1.8) and by exploiting, similarly as above, the fact that $n^{\gamma_1}\phi_1(n)$ is almost increasing. Hence the identity (2.7) gives for $n = n_k$

$$0 \leq -s_n \leq M_4 \frac{n}{p\phi_2(n)} + M_5 \frac{p\phi_1(n)}{n} . \tag{2.8}$$

Now put $p = \left[M_6 n\{\phi_1(n)\phi_2(n)\}^{-1/2}\right]$, where $M_6 = (M_4/M_5)^{1/2}$, into the right-hand side of (2.8) which is thus minimized. Such a choice of p is legitimate since $p \geq 1$ and $p/n \leq k < 1$ due to (1.6). Whence for $n = n_k$, one obtains

$$0 \leq -s_n \leq M_7 \left(\frac{\phi_1(n)}{\phi_2(n)}\right)^{1/2} . \tag{2.9}$$

Next suppose $s_n > 0$ for a sequence $\{n_k\}$ (or for all n). By repeating the above argument using this time the identity (i′) one obtains the same conclusion as in (2.9) which completes the proof.

PROOF OF THEOREM 5. a/ In addition to (1.11) assume first that

$$c_n < 0 \tag{2.10}$$

and observe the following two obvious facts:

1° s_n and σ_n are bounded. For, suppose that $\limsup s_n = A = +\infty$. Then,there exists a subsequence $n(k)$ such that $s_i < s_{n(k)}$ for all $i < n(k)$ and $s_{n(k)} \to \infty$. Hence

$$s_{n(k)} + c_{n(k)}\sigma_{n(k)} \geq \sigma_{n(k)}(1 + c_{n(k)}) \tag{2.11}$$

which is, due to (1.11), bounded away from zero, contradicting (1.10) ($\sigma_{n(k)}$ neither can be non-positive nor can tend to zero because of (1.11), (2.10) and $A = +\infty$).Thus $A \neq +\infty$ and similarly $\liminf s_n = B \neq -\infty$, so that s_n and σ_n are bounded.

2° This implies that if $c_{n(k)} \to 0$ (or trivially $c_{n(k)} = 0$), for any subsequence $n(k)$ of n, then also $s_{n(k)} \to 0$, due to (1.10).

Whence, throughout a/ we may consider sequences c_n not possessing zero as an accumulation point.

Now assume $A \geq |B|$ and substitute σ_m, σ_n from (1.10) into the identity (i) written as

$$(m+1)\sigma_m-(n+1)\sigma_n = s_{n+1}+\ldots+ s_m;$$

this gives

$$s_m \frac{m+1}{c_m} - s_n \frac{n+1}{c_n} + (s_{n+1}+\ldots+s_m) = o\left(\frac{m+1}{c_m}\right)+o\left(\frac{n+1}{c_n}\right). \tag{2.12}$$

Choose $n = n(k)$, $m = m(k)$ such that $s_{n(k)} \to A$, $s_{m(k)} \to A$, and $n(k)/m(k) \to 0$. Then, the order of the left-hand side of (2.12) is that of the first term, i.e. $s_m(m+1)/|c_m|$. For, the third term is majorized by $m(A + \varepsilon)$ which is of smaller order then the first term due to (1.11) and (2.10). The second term, however, is comparatively small due to the choice of m and n and since $c_{n(k)} \not\to 0$. But, for the same reason, the right-hand side of (2.12) is of order $o((m+1)/c_m)$, which is a contradiction unless $A = B = 0$. If

$|B| > A$ one repeats the procedure with s_n and s_m close to B instead of to A.

b/ Next suppose in addition to (1.11) that

$$c_n > 0. \tag{2.13}$$

Clearly, because of (1.10) and (2.13), A and B cannot be of the same sign, (e.g. negative).

So, assume $A > 0$ (or $A = +\infty$) and choose $m = m(k)$ such that $s_{m(k)} \to A$. To choose $n = n(k)$ consider $m(k)-i$, $i = 1,2,\dots$, and take $n(k) = m(k)-i_o$ where i_o is the first value of the index i for which $s_{m(k)-i}$ becomes negative. It must exist for otherwise one would have $B \geqslant 0$ which is already pointed out as impossible unless $A = B = 0$.

With such a choice of m and n one concludes, observing (2.13), that the order of the left-hand side of (2.12) is not less then $s_m(m+1)/c_m-s_n(n+1)/c_n$ whereas the one of right-hand side is $o((m+1)/c_m)+o((n+1)/c_n)$ which is impossible unless $A = 0$. Similarly $B = 0$.

To complete the proof we split the sequence c_n into two subsequences $c_{p(n)}$, $c_{q(n)}$ i.e. $\{c_{p(n)} \cup c_{q(n)}\}= c_n$ such that for both p(n) and q(n) (1.10) holds, and (1.11) and (2.10) hold for $c_{p(n)}$ (with the equality sign included), and for $c_{q(n)}$ holds (2.13). Then, from a/ and b/ there follows $s_n \to 0$, qed.

REFERENCES

1. HARDY, G.H. Divergent Series, Oxford, At the Clarendon Press 1949.
2. AVAKUMOVIĆ, V.G. Sur une extension de la condition de convergence des théoremes inverses de sommabilité. C.R. Sci. Paris 200(1935), 1515-1517.
3. BARI, N.K. , STEČKIN, S.B. Best approximations and differential properties of two conjugate functions (in Russian). Trudy Moskov. Mat. Obsc. 5(1965), 483-522.
4. CHANDRASEKHARAN, K., MINAKSHISUNDARAM, S. Typical Means. Oxford University Press 1952.
5. TOMIĆ, M. A note on lacunary Fourier series, PAMS 11(1960), 460-462.
6. VIJAYARAGHAVAN, T. A generalization of the theorem of Mercer, J. London Math. Soc. 3 (1928), 130-134.