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QUASICONFORMAL EXTENSIONS FOR SOME GEOMETRIC SUBCLASSES OF UNIVALENT FUNCTIONS

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ABSTRACT. Let S denote the set of all functions f which are analytic and univalent in the unit disk D normalized so that $f(z) = z + a_2z^2 + \cdots$. Let S^* and C be those functions f in S for which $f(D)$ is starlike and convex, respectively. For $0 \leq k < 1$, let S_k denote the subclass of functions in S which admit $(1+k)/(1-k)$-quasiconformal extensions to the extended complex plane. Sufficient conditions are given so that a function f belongs to $S_k \cap S^*$ or $S_k \cap C$. Functions whose derivatives lie in a half-plane are also considered and a Noshiro-Warschawski-Wolff type sufficiency condition is given to determine which of these functions belong to S_k. From the main results several other sufficient conditions are deduced which include a generalization of a recent result of Fait, Krzyz and Zygmunt.

KEY WORDS AND PHRASES: Quasiconformal mapping, starlike and convex functions, univalent functions, subordination.
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1. INTRODUCTION

If F is a sense-preserving homeomorphism of a domain $\Omega \subset \mathbb{C}$ which is absolutely continuous on almost all lines parallel to the coordinate axes and whose complex dilitation $\mu_F(z) = F_{\bar{z}}/F_z$ is bounded by k, $0 \leq k < 1$, almost everywhere, then F is said to be a $\frac{1+k}{1-k}$-quasiconformal mapping of Ω. Let S denote the class of functions f which are analytic and univalent in the unit disk D normalized so that $f(0) = 0$ and $f'(0) = 1$. A function f belongs to the class S_k if it belongs to S and has a $\frac{1+k}{1-k}$-quasiconformal extension F defined on the extended complex plane $\hat{\mathbb{C}}$.

If $f(z) = z + a_2z^2 + \cdots$ is analytic in D there are many sufficient conditions implying univalence (see Nehari [1,2], Duren et al [3] and Becker [4]). For instance, if f satisfies either of the conditions

$$(1 - |z|^2)\left|\frac{zf''(z)}{f'(z)}\right| \le 1 \quad \text{or} \quad \frac{1}{2}(1 - |z|^2)|\{f,z\}| \le 1,$$

$z \in D$, where $\{f,z\} = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$ is the Schwarzian derivative of f, then $f \in S$. If we replace both right-hand sides of the above inequalities by k we obtain two sufficient conditions implying $f \in S_k$ (Becker [4], Ahlfors and Weill [5]). It is well-known that if $f(z) = z + a_2z^2 +\cdots$ is analytic in D then each of the following implies $f \in S$:

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z} \tag{1.1}$$

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+z}{1-z} \tag{1.2}$$

$$f'(z) \prec \frac{1+cz}{1-z}, \qquad |c| \le 1, \tag{1.3}$$

where $\prec$ denotes subordination. Conditions (1.1) and (1.2) say that $f(D)$ is a starlike and convex region respectively. Condition (1.3) is due to Noshiro [6], Warschawski [7] and Wolff [8] and says that f' lies in a half-plane. Thus (1.1)-(1.3) are geometric conditions implying univalence. The purpose of this note is to show that by replacing the right-hand sides of (1.1)-(1.3) by appropriate functions we can have f still retain the above geometric properties and also belong to S_k. From these results we obtain further sufficient conditions for quasiconformal extensions. We should point out that some of our results may be obtained by showing that certain functions are initial links of a Loewner chain and then use a result of Becker [4]. However, there is no indication as to how the geometry comes into play. Our approach is more direct and shows precisely how the geometry gives rise to the quasiconformal extensions.

2. STARLIKE AND CONVEX CASE

Let S^* and C denote the subclass of starlike and convex functions in S.

THEOREM 1. If $f(z) = z + a_2z^2 +\cdots$ is analytic in D and satisfies

$$\frac{zf'(z)}{f(z)} \prec \frac{1+kz}{1-kz}, \quad z \in D, \tag{2.1}$$

then $f \in S^* \cap S_k$. An explicit extension is given by

$$F(z) = \begin{cases} f(z), & |z| \le 1 \\ |z|f(\frac{z}{|z|}), & |z| \ge 1. \end{cases}$$

PROOF. We first point out that the definition of F makes sense. Indeed, from (2.1) we see that

$$\ell og\left|\frac{f(z)}{z}\right| = \int_0^{|z|} \mathrm{Re}\left\{\frac{1+kte^{i\theta}}{1-kte^{i\theta}} - 1\right\}\frac{dt}{t}$$

$$\leq \int_0^{|z|}\left\{\frac{1+kt}{1-kt} - 1\right\}\frac{dt}{t} = \ell og(1-k|z|)^{-2}.$$

Hence $|f(z)| \leq |z|/(1-k|z|)^2$ and observe that (2.1) then gives

$$|f'(z)| \leq \left(\frac{1+k|z|}{1-k|z|}\right)\left|\frac{f(z)}{z}\right| \leq \frac{1+k|z|}{(1-k|z|)^3}.$$

Thus f has a continuous extension to $|z| \leq 1$ and also $f(e^{i\theta})$ is absolutely continuous. This follows from a theorem of Riesz [9] for example. Observe that (2.1) implies that $f \in S^*$. Now the value of F for $z = re^{i\theta}$, $r > 1$, is found by radially extending the value of $f(e^{i\theta})$. Hence starlikeness is essential.

Suppose f satisfies (2.1). Choose a sequence $\{t_n\}$ such that $0 < t_n < 1$ and t_n increases to 1. Consider the dilated function F_n defined by

$$F_n(z) = \begin{cases} f(t_n z)/t_n & , \ |z| \leq 1 \\ |z| f(t_n z/|z|)/t_n, & |z| \geq 1. \end{cases}$$

Now since $f \in S^*$ we see that $f(t_n z)/t_n$ maps $|z| \leq 1$ onto a starlike region. Thus each F_n maps $\hat{\mathbb{C}}$ continuously one-to-one and onto $\hat{\mathbb{C}}$ with $F_n(\infty) = \infty$. Consequently, since $\hat{\mathbb{C}}$ is compact, each F_n is a homeomorphism from $\hat{\mathbb{C}}$ onto $\hat{\mathbb{C}}$ (see for example Whyburn [10; p. 21]). Clearly each F_n is a C^1 mapping. To compute the complex dilitation we consider $|z| = r \geq 1$ and observe that $F_n(re^{i\theta}) = rf(t_n e^{i\theta})/t_n$. It is then easy to see that

$$|\mu_{F_n}(z)| = \left|\frac{1 - \left\{\dfrac{t_n e^{i\theta} f'(t_n e^{i\theta})}{f(t_n e^{i\theta})}\right\}}{1 + \left\{\dfrac{t_n e^{i\theta} f'(t_n e^{i\theta})}{f(t_n e^{i\theta})}\right\}}\right| \leq k.$$

The last equality is equivalent to

$$\frac{t_n e^{i\theta} f'(t_n e^{i\theta})}{f(t_n e^{i\theta})} = \frac{zf'(z)}{f(z)} \prec \frac{1+kz}{1-kz},$$

which follows by (2.1). Since $|\mu_{F_n}| \leq k < 1$, this implies that the Jacobian $J_{F_n} = |(F_n)_z|^2 - |(F_n)_{\bar{z}}|^2 > 0$. Hence each F_n is sense-preserving. We can now conclude that the limit function F is a $(1+k)/(1-k)$ -quasiconformal mapping of $\hat{\mathbb{C}}$ onto $\hat{\mathbb{C}}$ (Lehto and Virtanen [11; p.74]). Thus $f \in S_k$.

Some sufficiency conditions in terms of the coefficients of f can now be given.

THEOREM 2. If $f(z) = z + a_2z^2 + \cdots$ is analytic in $|z| < 1$ and satisfies

$$\sum_{n=2}^{\infty} \left(n + \frac{k-1}{k+1}\right)|a_n| \le \frac{2k}{1+k}, \tag{2.2}$$

then $f \in S^* \cap S_k$.

PROOF. Since $0 \le k < 1$, it is clear that (2.2) implies that $\sum_{n=2}^{\infty} |a_n| < 1$. In fact, inequality (2.2) is equivalent to

$$\sum_{n=2}^{\infty} (n-1)|a_n| \le \frac{2k}{1+k}\left(1 - \sum_{n=2}^{\infty} |a_n|\right). \tag{2.3}$$

Next, observe that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| = \left|\frac{\sum_{n=2}^{\infty} (n-1)a_nz^n}{z + \sum_{n=2}^{\infty} a_nz^n}\right| \le \frac{\sum_{n=2}^{\infty} (n-1)|a_n|}{1 - \sum_{n=2}^{\infty} |a_n|}.$$

If we use (2.3), we see that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le \frac{2k}{1+k}. \tag{2.4}$$

A calculation now shows that $\frac{zf'}{f} \prec \frac{1+kz}{1-kz}$ holds if and only if

$$\left|\frac{zf'(z)}{f(z)} - \frac{1+k^2}{1-k^2}\right| \le \frac{2k}{1-k^2}. \tag{2.5}$$

It is geometrically evident that if (2.4) holds then (2.5) holds. Thus (2.1) holds and $f \in S^* \cap S_k$ by Theorem 1.

COROLLARY 1. (Fait, Krzyz and Zygmunt [12]). If $f(z) = z + a_2z^2 + \cdots$ is analytic in $|z| < 1$ and $\sum_{n=2}^{\infty} n|a_n| \le k$, then $f \in S_k$.

PROOF. $\sum_{n=2}^{\infty} \left(n + \frac{k-1}{k+1}\right)|a_n| \le \sum_{n=2}^{\infty} n|a_n| \le k \le \frac{2k}{1+k}$, since $0 \le k < 1$.

We should point out that (2.1) implies that f is starlike of order $\frac{1-k}{1+k}$, $S^*\left(\frac{1-k}{1+k}\right)$, since $\mathrm{Re}\left\{\frac{zf'}{f}\right\} > \frac{1-k}{1+k}$. Another sufficiency condition for functions with negative coefficients can be obtained.

COROLLARY 2. If $f(z) = z - \sum_{n=1}^{\infty} |a_n|z^n \in S^*\left(\frac{1-k}{1+k}\right)$ then $f \in S^*\left(\frac{1-k}{1+k}\right) \cap S_k$.

PROOF. Since $f \in S^*\left(\frac{1-k}{1+k}\right)$ we have

$$\operatorname{Re}\left\{\frac{rf'(r)}{f(r)}\right\} = \frac{r - \sum_{n=2}^{\infty} n|a_n|r^n}{r - \sum_{n-2}^{\infty} |a_n|r^n} > \frac{1-k}{1+k} .$$

Let r approach 1 to obtain

$$\frac{1 - \sum_{n=2}^{\infty} n|a_n|}{1 - \sum_{n=2}^{\infty} |a_n|} \geq \frac{1-k}{1+k} .$$

This implies that $\sum_{n=2}^{\infty} (n + \frac{k-1}{k+1})|a_n| \leq \frac{2k}{1+k}$. Use Corollary 1.

It should be pointed out that the condition (2.1) is in a sense "best possible" If we consider the starlike function $f(z) = z/(1-kz)^2$ then it is easy to see that $\frac{zf'}{f} = \frac{1+kz}{1-kz}$ and that the extension F, given in Theorem 1, satisfies $|\mu_F(z)| = k$ on $|z| = 1$.

Theorem 1 can be generalized as follows:

THEOREM 3. If $f(z) = z + a_2z^2 + \cdots$ is analytic in $|z| < 1$ and if there exists a fixed ζ, $|\zeta| < 1$, such that $f'(\zeta) \neq 0$ and

$$\frac{1-\bar{\zeta}z}{1-|\zeta|^2}\left(f'(z)\,\frac{z-\zeta}{f(z)-f(\zeta)}\right) \prec \frac{1+kz}{1-kz} . \tag{2.6}$$

then $f \in S^* \cap S_k$.

PROOF. Since $f'(\zeta) \neq 0$, we consider the function

$$g(w) = \frac{f\left(\frac{w+\zeta}{1+\bar{\zeta}w}\right) - f(\zeta)}{f'(\zeta)(1-|\zeta|^2)} .$$

Then $g \in S^* \cap S_k$ if and only if $f \in S^* \cap S_k$. Put $z = (w+\zeta)/(1+w\bar{\zeta})$ and observe that

$$\frac{wg'(w)}{g(w)} = \frac{1-\bar{\zeta}z}{1-|\zeta|^2}\left(f'(z)\,\frac{z-\zeta}{f(z)-f(\zeta)}\right).$$

Now $w = w(z) = (z-\zeta)/(1-\bar{\zeta}z)$ and hence (2.6) says that the region of values of $wg'(w)/g(w)$ is contained in that of $(1+kz)/(1-kz)$. Since this function is univalent we conclude that $wg'(w)/g(w) \prec (1+kz)/(1-kz)$. Thus by Theorem 1, $g \in S^* \cap S_k$ and so $f \in S^* \cap S_k$.

For the convex case, we need the following result which generalizes results of several authors. For example see Libera [13] and MacGregor [14].

LEMMA. Suppose g is analytic and maps $|z| < 1$ onto a many-sheeted region starlike with respect to the origin. If H is a univalent convex mapping and f an analytic function in $|z| < 1$ with $\frac{f'(z)}{g'(z)} \prec H(z)$, then $\frac{f(z)}{g(z)} \prec H(z)$.

PROOF. Fix z_0, $|z_0| < 1$, and let $w_0 = g(z_0)$. Choose a branch of g^{-1} defined on one of the starlike sheets Ω so that $F(w) = f \circ g^{-1}(w)$ is analytic in Ω. Since Ω is starlike, the ray from 0 to w_0 lies in Ω. Let $s(t)$, $0 \le t \le 1$, be the preimage of this ray under g^{-1}. Then we can conclude

$$\frac{F(w_0)}{w_0} = \frac{1}{w_0}\int_0^{w_0} F'(\zeta)d\zeta = \int_0^1 F'(tw_0)dt$$

$$= \int_0^1 \frac{f'(s(t))}{g'(s(t))}\,dt \subseteq \text{convex hull of } \left\{\frac{f'(z)}{g'(z)} : |z| < 1\right\}$$

$$\subseteq \{H(z) : |z| < 1\}.$$

Hence $\frac{f(z_0)}{g(z_0)} \subseteq \{H(z) : |z| < 1\}$. The result now follows by the univalence of H.

THEOREM 4. If $f(z) = z + a_2z^2 + \cdots$ is analytic in $|z| < 1$ and

$$1 + \frac{zf''}{f'} \prec \frac{1 + kz}{1 - kz} \tag{2.7}$$

then $f \in C \cap S_k$.

PROOF. By (2.7) we have $f \in C$ and

$$\frac{(zf')'}{f'} \prec \frac{1 + kz}{1 - kz}\,.$$

We can now apply the lemma to conclude that $\frac{zf'}{f} \prec \frac{1 + kz}{1 - kz}$. The result now follows from Theorem 1.

Using Theorem 4, we can obtain a coefficient condition similar to that given in Theorem 2. Specifically, if $\sum_{n=2}^{\infty} n(n + \frac{k - 1}{k + 1})|a_n| \le \frac{2k}{1 + k}$, then $f \in C \cap S_k$.

3. BOUNDED DERIVATIVE CASE

As noted earlier, the Noshiro-Warschawski-Wolff Theorem asserts the univalence of a function f in $|z| < 1$ (more generally a convex region) if f' lies in a half-plane. We obtain sufficient conditions for $f \in S_k$ related to this theorem.

THEOREM 5. If $f(z) = z + a_2z^2 + \cdots$ is analytic in $|z| < 1$ and if there exists a complex constant λ such that

$$|z^2(\lambda f'(z) - 1)| \le k, \quad |z| < 1 \tag{3.1}$$

then $f \in S_k$.

It is simple consequence of the Maximum Principle and the fact that $|z| < 1$ that (3.1) is actually equivalent to

$$|\lambda f'(z) - 1| \le k, \quad |z| < 1. \tag{3.2}$$

PROOF. Suppose f satisfies (3.1) or (3.2) and suppose first that f is analytic in $|z| \le 1$. Since λ is nonzero, we consider the function

$$G(z) = \begin{cases} f(z), & |z| \le 1 \\ f\left(\frac{1}{\bar{z}}\right) + \frac{1}{\lambda}\left(z - \frac{1}{\bar{z}}\right), & |z| > 1. \end{cases}$$

Clearly G is a C^{∞}-mapping of $\hat{\mathbb{C}}$ into $\hat{\mathbb{C}}$. Its complex dilitation satisfies

$$\mu_G(z) = -\frac{1}{\bar{z}^2}\left[\lambda f'\left(\frac{1}{\bar{z}}\right) - 1\right] \quad \text{for } |z| > 1.$$

Thus by (3.1), we see that

$$|\mu_G(z)| \le \begin{cases} 0, & |z| \le 1 \\ k, & |z| > 1 \end{cases}. \tag{3.3}$$

Thus $J_G(z) > 0$ and so G is sense-preserving in $\mathbb{C}$. To show that G is globally one-to-one in $\mathbb{C}$, we resort to the idea used in the proof of the original Noshiro-Warschawski-Wolff Theorem. Observe first that $G_z \ne 0$ in $\hat{\mathbb{C}}$. In fact, we have

$$G_z(z) = \begin{cases} f'(z), & |z| \le 1 \\ 1/\lambda, & |z| > 1 \end{cases}. \tag{3.4}$$

The inequality (3.2) implies that $|\arg \lambda f'(z)| \le \sin^{-1}k$ for $|z| \le 1$. This together with (3.4) shows that

$$|\arg\{\lambda G_z(z)\}| \le \begin{cases} \sin^{-1}k, & |z| \le 1 \\ 0 & |z| > 1 \end{cases}. \tag{3.5}$$

Making use of (3.3) and (3.5) we see that

$$|\arg\{\lambda G_z(z)\}| < \frac{\pi}{2} - \sin^{-1}|\mu_G(z)|, \; z \in \mathbb{C}. \tag{3.6}$$

Now let $z_1, z_2 \in \mathbb{C}$ with $z_1 \ne z_2$. Then we have

$$\lambda[G(z_2) - G(z_1)] = \int_{z_1}^{z_2} \lambda G_z(z)dz + \lambda G_{\bar{z}}(z)d\bar{z},$$

where the path of integration is along the straight line from z_1 to z_2. If we let $z(t) = z_1 + t(z_2-z_1)$ and $e^{i\nu} = \overline{(z_2-z_1)}/(z_2-z_1)$, then

$$|\lambda[G(z_2) - G(z_1)]| \; |z_2 - z_1| \int_0^1 \mathrm{Re}\{\lambda[G_z(z(t)) + e^{i\nu}G_{\bar{z}}(z(t))]\}dt. \tag{3.7}$$

Note that $|\lambda[G_z + e^{i\nu}G_{\bar{z}}]| \geq |\lambda|[|G_z| - |G_{\bar{z}}|] = |\lambda G_z|(1-|\mu_G|) > 0$. Thus a simple geometric argument shows that

$$|\arg\{\lambda(G_z + e^{i\nu}G_{\bar{z}})\}| \leq |\arg\{\lambda G_z\}| + \sin^{-1}|\mu_G|.$$

Inequality (3.6) now gives $|\arg\{\lambda(G_z + e^{i\nu}G_{\bar{z}})\}| < \pi/2$, so that $\mathrm{Re}\{\lambda(G_z + e^{i\nu}G_{\bar{z}})\} > 0$. Hence by (3.7) we see that $G(z_2) \neq G(z_1)$ and hence G is one-to-one in $\hat{\mathbb{C}}$. Thus since $\hat{\mathbb{C}}$ is compact, G is a homeomorphism in $\hat{\mathbb{C}}$ (Whyburn [10; p.21]).

To prove the theorem for f analytic in $|z| < 1$, we let $g_t(z) = f(tz)/t$, $0 < t < 1$. Observe first that (3.2) holds for g_t and consequently so does (3.1). We now let t increase to 1 through a sequence to complete the proof of the theorem.

Several sufficient conditions are direct consequences of this theorem.

COROLLARY 1. If $f(z) = z + a_2z^2 + \cdots$ is analytic in $|z| < 1$ and $f'(z) \prec 1 + kz$, then $f \in S_k$.

PROOF. Let $\lambda = 1$ in the theorem. This result is also "best possible" in the sense that the function $f(z) = z + \frac{k}{2}z^2$ gives $f'(z) = 1 + kz$ and the extension G, given in the proof of the theorem, satisfies $|\mu_G(z)| = k$ on $|z| = 1$.

COROLLARY 2. If $f(z) = z + a_2z^2 + \cdots$ is analytic in $|z| < 1$ and $|\sum_{n=2}^{\infty} n\, a_n z^{n-1}| \leq k$, then $f \in S_k$.

PROOF. Let $\lambda = 1$ and use (3.2).

We can also obtain sufficient conditions for functions to belong to Σ_k: i.e., those functions $g(\xi) = \xi + b_0 + \frac{b_1}{\xi} + \frac{b_2}{\xi^2} + \cdots$ meromorphic and univalent in $|\zeta| > 1$ which has a $(1+k)/(1-k)$-quasiconformal extension to $\hat{\mathbb{C}}$. Although our results will only apply to nonvanishing functions in $|\zeta| > 1$, this can always be achieved by simply adjusting b_0 and this does not affect the dilitation. The next two results should be compared with those of Krzyz [15].

COROLLARY 3. Let $g(\zeta) = \zeta + b_0 + \frac{b_1}{\zeta} + \frac{b_2}{\zeta^2} + \cdots$ be meromorphic and nonzero in $|\zeta| > 1$. If there exists a nonzero complex constant λ such that all $|\zeta| > 1$

$$\left|\frac{\lambda g'(\zeta)}{g(\zeta)^2} - \frac{1}{\zeta^2}\right| \leq k, \quad |\zeta| > 1, \tag{3.8}$$

then $g \in \Sigma_k$.

PROOF. There exists an analytic function $f(z) = z + a_2z^2 + \cdots$ such that $g(z) = 1/f(1/z)$. Thus $g \in \Sigma_k$ if and only if $f \in S_k$. Now observe that (3.8) is equivalent to (3.1).

COROLLARY 4. If there exists a nonzero complex constant λ and a fixed $\zeta_0, |\zeta_0| > 1$, with $g'(\zeta_0) \neq 0$ and for all $|\zeta| > 1$

$$\left|\lambda \left(\frac{1-\bar{\zeta}_0}{1-|\zeta_0|^2}\right)^2 \frac{g'(\zeta)}{g'(\zeta_0)} - 1\right| \leq k, \ |\zeta| > 1, \tag{3.9}$$

then $g \in \Sigma_k$.

PROOF. Put $h(w) = \dfrac{(1-|\zeta_0|^2 g'(\zeta_0)}{g\left(\dfrac{1+\zeta_0 w}{w+\bar{\zeta}_0}\right)-g(\zeta_0)}$. Then it is clear that $h \in \Sigma_k$ if and only if $g \in S_k$. A calculation with $\zeta = (1+\zeta_0 w)/(w+\bar{\zeta}_0)$ then shows that (3.9) is equivalent to (3.2).

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