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AN APPLICATION OF HYPERGEOMETRIC FUNCTIONS TO A PROBLEM IN FUNCTION THEORY

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ABSTRACT. In some recent work in univalent function theory, Aharonov, Friedland, and Brannan studied the series, $(1 + xt)^{\alpha}(1 - t)^{\beta} = \sum_{n=0}^{\infty} A_n^{(\alpha,\beta)}(x)t^n$. Brannan posed the problem of determining $S = \{(\alpha,\beta): |A_n^{(\alpha,\beta)}(e^{i\theta})| < |A_n^{(\alpha,\beta)}(1)|$, $0 < \theta < 2\pi$, $\alpha > 0, \beta > 0$, $n = 1,2,3,\ldots\}$. Brannan showed that if $\beta \geq \alpha \geq 0$, and $\alpha + \beta \geq 2$, then $(\alpha,\beta) \in S$. He also proved that $(\alpha,1) \in S$ for $\alpha \geq 1$. Brannan showed that for $0 < \alpha < 1$ and $\beta = 1$, there exists a θ such that $|A_{2k}^{(\alpha,1)}(e^{i\theta})| > |A_{2k}^{(\alpha,1)}(1)|$ for k any integer. In this paper, we show that $(\alpha,\beta) \in S$ for $\alpha \geq 1$ and $\beta \geq 1$.

KEY WORDS AND PHRASES. Hypergeometric Functions, Jacobi Polynomials, Maximum property, and positive maximum property.

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1. INTRODUCTION.

Let D be a disk $\{z: |z-a| \leq r\}$ where the center a is real. Let f be a function analytic in an open neighborhood of the disk D. It is well known that the maximum modulus of F on D is attained on the boundary $\{z: |z - a| = r\}$. If the maximum modulus is attained at a + r and only at a + r then we say that f has the maximum property on D. If in addition f (a + r) > 0, then f has the positive maximum property. If the disk D is not specified then it is assumed that D is the unit disk.

Let $(1 + zt)^{\alpha}(1 - t)^{\beta} = \sum_{k=0}^{\infty} A_k^{(\alpha,\beta)}(z)t^k$ and let $MP = \{(\alpha,\beta): \alpha \geq 0, \beta \geq 0$ and $A_n^{(\alpha,\beta)}(z)$ satisfies the positive maximum property for $n = 1,2,3,\ldots\}$. The main problem in this paper is to characterize the sets MP and PMP. An application to extreme point theory is given in [2] .

2. SOME FUNDAMENTAL RECURRENCE RELATIONS

Starting with

$$(1 + at)^{\alpha}(1 - t)^{-\beta} = \sum_{n=0}^{\infty} A_n^{(\alpha,\beta)}(z)t^n, \tag{2.1}$$

one can derive a number of recurrence relations. For example

$$(1 + zt)^{\alpha+\gamma}(1 - t)^{-\beta} = \sum_{n=0}^{\infty} A_n^{(\alpha+\gamma,\beta)}(z)t^n.$$

Indeed, since $(1 + zt)^{\gamma} = \sum_{n=0}^{\infty} \frac{(-\gamma)_n}{n!}(-zt)^n$, by taking the Cauchy product of this last series and the series in (2.1) we obtain

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{(-\gamma)_k(-z)^k}{k!} A_{n-k}^{(\alpha,\beta)}(z)\right)t^n = \sum_{n=0}^{\infty} A_n^{(\alpha+\gamma,\beta)}(z)t^n.$$

Hence

$$A_n^{(\alpha+\gamma,\beta)}(z) = \sum_{k=0}^{n} \frac{(-\gamma)_k(-z)^k}{k!} A_{n-k}^{(\alpha,\beta)}(z). \tag{2.2}$$

Similarly

$$A_n^{(\alpha,\beta+\gamma)}(z) = \sum_{k=0}^{n} \frac{(\gamma)_k}{k!} A_{n-k}^{(\alpha,\beta)}(z). \tag{2.3}$$

If we let $\gamma = 1$ in (2), we obtain

$$A_n^{(\alpha,\beta)}(z) + zA_{n-1}^{(\alpha,\beta)}(z) = A_{n+1}^{(\alpha+1,\beta)}(z). \tag{2.4}$$

Relations (2.3) and (2.4) are significant because if $(\alpha,\beta) \in$ PMP, then $(\alpha,\beta') \in$ PMP for all $\beta' > \beta$. Also, $(\alpha,\beta) \in$ PMP implies that $(\alpha + n,\beta) \in$ PMP, $n = 1,2,3,\cdots$.

3. SOME EXPLICIT FORMULAS FOR $A_n^{(\alpha,\beta)}(z)$

Taking the Cauchy product of the series

$$(1 + zt)^{\alpha} = \sum_{n=0}^{\infty} \frac{(-\alpha)_n(-z)^n}{n!} t^n, \text{ and}$$

$$(1 - t)^{-\beta} = \sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} t^n, \text{ we have}$$

$$A_n^{(\alpha,\beta)}(z) = \sum_{k=0}^{n} \frac{(-\alpha)_k(\beta)_{n-k}}{k!(n - k)!}(-z)^k. \tag{3.1}$$

Using the fact that $(n - k)! = (1)_{n-k}$ and $(a)_{n-k} = \frac{(a)_n(-)^k}{(1-a-n)_k}$, we obtain

$$A_n^{(\alpha,\beta)}(z) = \frac{(\beta)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k(-\alpha)_k}{(1-\beta-n)_k k!}$$

Using ${}_2F_1\left({a,b \atop c};z\right) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} z^k$, we obtain

$$A_n^{(\alpha,\beta)}(z) = \frac{(\beta)_n}{n!}\ {}_2F_1\left({-n,-\alpha; \atop 1-\beta-n}-z\right) . \tag{3.2}$$

The Jacobi polynomials are defined as

$$P_n^{(\alpha,\beta)}(z) = \frac{(\alpha+1)_n}{n!}\ {}_2F_1\left({-n,n+\alpha+\beta+1 \atop \alpha+1}; \frac{1-z}{2}\right).$$

Hence

$$A_n^{(\alpha,\beta)}(z) = (-)^n P_n^{(-\beta-n,\beta-\alpha-1)}(2z+1). \tag{3.3}$$

Replacing k by n - k in (3.1), and following the same procedure we get

$$A_n^{(\alpha,\beta)}(z) = z^n P_n^{(\alpha-n,\beta-\alpha-1)}\left(1+\frac{2}{z}\right). \tag{3.4}$$

Using Pfaff's tranformation [1, p. 64]

$${}_2F_1\left({a,b \atop c};z\right) = (1-z)^{-a}\ {}_2F_1\left({a,c-b \atop c}; \frac{z}{z-1}\right),\ c \neq 0,-1,-2,\cdots$$

we can write $A_n^{(\alpha,\beta)}(z)$ as

$$A_n^{(\alpha,\beta)}(z) = \frac{(\beta)_n}{n!}(1+z)^{\alpha}\ {}_2F_1\left({-\alpha,1-\beta \atop 1-\beta-n}; \frac{z}{z+1}\right) . \tag{3.5}$$

Setting $\beta \to 1$, we get

$$A_n^{(\alpha,1)}(z) = (1+z)^{\alpha}\left(1 + \frac{(-)^n(-\alpha)_{n+1}}{(n+1)!}\left(\frac{z}{z+1}\right)^{n+1}\ {}_2F_1\left({n+1,n+1-\alpha \atop n+2}; \frac{z}{z+1}\right)\right) . \tag{3.6}$$

4. SOME MAXIMALITY PROPERTIES FOR $A_n^{(\alpha,\beta)}(z)$

It has been proven in 3 that $(\alpha,\beta) \in MP$ for $\beta = 1$ and $\alpha \geq 1$. We can now strengthen that result.

THEOREM 1. $(\alpha,\beta) \in PMP$ for $\alpha \geq 1$ and $\beta \geq 1$.

PROOF: It is evident from (3.2) that all coefficients of $A_n^{(\alpha,\beta)}(z)$ are positive for $\alpha \geq n$. So clearly $A_n^{(\alpha,1)}(z)$ will satisfy the positive maximum property for $\alpha \geq n$.

The theorem follows form (2.3) upon showing that $A_n^{(\alpha,1)}(1) > 0$ for $1 < \alpha < n$.

Assume that $1 < \alpha < n$. Then it follows from (3.6) that if

$$\left| \frac{(-\alpha)_{n+1}}{(n+1)!} \left(\frac{1}{2}\right)^{n+1} {}_2F_1 \left({n+1, n+1-\alpha \atop n+2} ; \frac{1}{2} \right) \right| < 1, \tag{3.7}$$

then $A_n^{(\alpha,1)}(1) > 0$.

Note that all terms of the ${}_2F_1$ in (3.7) are positive. Moreover

$${}_2F_1\left({n+1, n+1-\alpha \atop n+2} ; \frac{1}{2}\right) < {}_1F_0\left({n+1-\alpha \atop -} ; \frac{1}{2}\right) = 2^{n+1-\alpha},$$

by the binomial theorem.

Hence the left side of (3.7) is less than $|(-\alpha)_{n+1} 2^{-\alpha}/(n+1)!|$. Let m be an integer such that $m - 1 < \alpha \leq m$. Then

$$\left|\frac{(-\alpha)_{n+1}}{(n+1)!}\right| = \frac{|(-\alpha)(1-\alpha)\cdots(m-\alpha-1)(m-\alpha)\cdots(n-\alpha)|}{(n+1)!} =$$

$$\frac{\alpha(\alpha-1)\cdots(\alpha-m+1)(m-\alpha)\cdots(n-\alpha)}{(n+1)!} \leq$$

$$\frac{m(m-1)\cdots 2\cdot 1\cdot 1\cdot 2\cdots(n-m+1)}{(n+1)!} = \binom{n+1}{m}^{-1} < 1.$$

Consequently $|(-\alpha)_{n+1} 2^{-\alpha}(n+1)!| < 1$, and (3.7) is established. Brannan [3], showed that $(\alpha,1) \in MP$ for $\alpha \geq 1$. Hence $(\alpha,1) \in PMP$ for all $\alpha > 1$, and by (2.3), $(\alpha,\beta) \in PMP$ for all $\alpha > 1$ and $\beta \geq 1$.

The author feels that the properties of Jacobi polynomials as given in (3.3) and (3.4) will be useful in answering other questions of Brannan's regarding the series (2.1).

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