Internat. J. Math. & Math. Sci.
Vol. 7 No. 1 (1984) 97-102

REDUCING INCREASING MONOTONICITY OF KERNELS

RINA LING

Department of Mathematics
California State University
Los Angeles, California 90032

(Received October 14, 1982 and in revised form May 13, 1983)

ABSTRACT. Integral equations with positive increasing kernels are transformed into ones with positive decreasing kernels, and using a similar technique, more positive increasing kernels are reduced to ones with less increasing monotonicity.

KEY WORDS AND PHRASES. Integral equations, monotone increasing kernals, monotone decreasing kernels.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODE. 45H05, 44A15.

1. INTRODUCTION.

The behavior of and the approximations to solutions of integral equations have been studied; see, for example, Refs. 1-11. Some of these papers are on equations with positive decreasing kernels and some study positive increasing kernels, satisfying a convexity condition, namely, that the logarithm of the kernel be convex in the first case and that the logarithm of the derivative of the kernel be convex in the second case.

In this paper, equations of the form

$$f(t) = 1 - \int_0^t K(t - \tau) f(\tau) d\tau, \tag{1.1}$$

where the kernel K(t) is positive increasing, are transformed into the form

$$f(t) = \phi(t) - \int_0^t L(t - \tau) f(\tau) d\tau, \tag{1.2}$$

where the new kernel L(t) is positive decreasing or is positive increasing with reduced increasing monotonicity. In the process, the logarithmic convexity condition is preserved.

2. REDUCING INCREASING MONOTONICITY OF KERNELS

By Theorem 1.1.1 in Ref. 8, equation (1.1) is equivalent to

$$f(t) = e^{-\gamma t} - \int_0^t L(t - \tau) f(\tau) d\tau, \tag{2.1}$$

where γ is any constant,

$$L(t) = (a - \gamma) e^{-\gamma t} + \int_0^t K'(t - \tau) e^{-\gamma t} \, d\tau, \tag{2.2}$$

and $a = K(0)$.

THEOREM 2.1. If (1) $K(t) > 0$ with $K'(t) > 0$ and $K''(t) < 0$, (2) $\ell n\, K'(t)$ is convex (i.e. $\frac{K''}{K'}$ is nondecreasing) and (3) $a + \frac{c}{b} \geq 2b^{\frac{1}{2}}$, where $a = K(0)$, $b = K'(0)$ and $c = K''(0)$, then, by an appropriate choice of γ, (1) $L(t) > 0$, $L'(t) < 0$ and (2) $\ell n\, L(t)$ is convex (i.e. $\frac{L}{L'}$ is nondecreasing).

PROOF. Note first that $L(t) > 0$ for $\gamma < a$. Differentiation of (2.2) leads to

$$L'(t) = (\gamma^2 - a\gamma + b)e^{-\gamma t} + K'' * e^{-\gamma t}. \tag{2.3}$$

On the other hand,

$$L'(t) = K'(t) - \gamma L(t) \tag{2.4}$$

and

$$L''(t) = K''(t) - \gamma K'(t) + \gamma^2 L(t);$$

therefore,

$$LL'' - L'^2 = K''L + \gamma K'L - K'^2. \tag{2.5}$$

Let $P(t) = L(t)L''(t) - L'^2(t)$; we want to show that $P(t) \geq 0$, for a certain value of γ. Substitution of (2.2) into (2.5) leads to

$$\begin{aligned} e^{\gamma t}P(t) &= \gamma(a-\gamma)K'(t) + \gamma K'(t)\int_0^t e^{\gamma\tau} K'(\tau)d\tau + (a-\gamma)K''(t) \\ &\quad + K''(t)\int_0^t e^{\gamma\tau}K'(\tau)d\tau - e^{\gamma t}K'^2(t). \end{aligned} \tag{2.6}$$

Since

$$\int_0^t e^{\gamma\tau}K'(\tau)d\tau = \frac{1}{\gamma}\, e^{\gamma t}K'(t) - \frac{1}{\gamma}\, b - \frac{1}{\gamma}\int_0^t e^{\gamma\tau}K''(\tau)d\tau,$$

(2.6) becomes

$$\begin{aligned} e^{\gamma t}P(t) &= \gamma(a-\gamma)K'(t) + (a-\gamma)K''(t) - bK'(t) \\ &\quad + \int_0^t e^{\gamma\tau}[K''(t)K'(\tau) - K'(t)K''(\tau)]d\tau. \end{aligned}$$

Therefore

$$\frac{e^{\gamma t}P(t)}{K'(t)} = \gamma(a-\gamma) + (a-\gamma)\,\frac{K''(t)}{K'(t)} - b + \int_0^t e^{\gamma\tau}\left[\frac{K''(t)K'(\tau) - K'(t)K''(\tau)}{K'(t)}\right]d\tau. \tag{2.7}$$

Since $\frac{K''}{K'}$ is nondecreasing, we have, for $\gamma < a$,

$$\begin{aligned} &\gamma(a-\gamma) + (a-\gamma)\,\frac{K''(t)}{K'(t)} - b \\ &\geq \gamma(a-\gamma) + (a-\gamma)\frac{c}{b} - b \\ &= \gamma(a-\gamma) - \gamma(a-\gamma) + b - b, \text{ if } (a-\gamma)\frac{c}{b} = -\gamma(a-\gamma) + b, \\ &= 0. \end{aligned} \tag{2.8}$$

One root of the above quadratic equation is

$$\gamma_0 = \frac{1}{2}\,[a - \frac{c}{b} - \sqrt{(a + \frac{c}{b})^2 - 4b}],$$

where $\gamma_0 < a$ if and only if

$$-\sqrt{(a + \frac{c}{b})^2 - 4b} < a + \frac{c}{b}\ ,$$

which is true since $a + \frac{c}{b} > 0$. So $L(t) > 0$. The second part of the integrand in (2.7) is equal to

$$\left[\frac{K''(t)}{K'(t)}\right] K'(\tau) - K''(\tau)$$

$$\geq \left[\frac{K''(\tau)}{K'(\tau)}\right] K'(\tau) - K''(\tau)$$

$$= 0,$$

so the integral is positive. It follows from (2.7) that $P(t) \geq 0$ and $\ln L(t)$ is convex, for $\gamma = \gamma_0$.

Equation (2.8) leads to the fact that $\gamma_0^2 - a\gamma_0 + b < 0$ and, from (2.3), $L'(t) < 0$. The proof is now complete.

Under certain conditions, the next class of positive increasing kernels, with increasing rate of growth, can be reduced to the class of positive increasing kernels with decreasing rate of growth.

THEOREM 2.2. If (1) $K(t) > 0$ with $K'(t) > 0$, $K''(t) > 0$, and $K'''(t) < 0$, (2) $\ln K''(t)$ is convex, and (3) $ac - d \leq \frac{3c}{2}(a - \sqrt{a^2 - 4b})$ with $ac - d \geq \sqrt{3c(bc - ad)}$ and $ac - d \geq 3\sqrt[3]{c^2(c^2 - bd)}$, where $a = K(0)$, $b = K'(0)$, $c = K''(0)$, and $d = K'''(0)$, with at least one inequality being a strict inequality, then (1) $L(t) > 0$, $L'(t) > 0$, $L''(t) < 0$ and (2) $\ln L'(t)$ is convex.

PROOF. As before, note first that $L(t) \geq 0$ for $\gamma \leq a$, and $L'(t) > 0$ for $\gamma^2 - a\gamma + b > 0$. Differentiation of (2.3) leads to

$$L''(t) = [-\gamma(\gamma^2 - a\gamma + b) + c]e^{-\gamma t} + K'''*e^{-\gamma t}. \tag{2.9}$$

Differentiation of (2.4) leads to

$$L''(t) = K''(t) - \gamma L'(t)$$

and

$$L'''(t) = K'''(t) - \gamma K''(t) + \gamma^2 L'(t);$$

therefore,

$$L'L''' - L''^2 = K'''L' + \gamma K''L' - K''^2. \tag{2.10}$$

Let $P(t) = L'(t)L'''(t) - L''^2(t)$. Substitution of (2.3) into (2.10) leads to

$$e^{\gamma t}P(t) = \gamma(\gamma^2 - a\gamma + b)K''(t) + \gamma K''(t)\int_0^t e^{\gamma t}K''(\tau)d\tau$$
$$+ (\gamma^2 - a\gamma + b)K'''(t) + K'''(t)\int_0^t e^{\gamma\tau}K''(\tau)d\tau - e^{\gamma t}K''^2(t). \tag{2.11}$$

Since

$$\int_0^t e^{\gamma\tau} K''(\tau)d\tau = \frac{1}{\gamma} e^{\gamma t}K''(t) - \frac{1}{\gamma}c - \frac{1}{\gamma}\int_0^t e^{\gamma\tau}K'''(\tau)d\tau,$$

(2.11) becomes

$$e^{\gamma t}P(t) = \gamma(\gamma^2 - a\gamma + b)K''(t) + (\gamma^2 - a\gamma + b)K'''(t) - cK''(t)$$
$$+ \int_0^t e^{\gamma\tau}[K'''(t)K''(\tau) - K''(t)K'''(\tau)]d\tau.$$

Therefore,

$$\frac{e^{\gamma t}P(t)}{K''(t)} = \gamma(\gamma^2 - a\gamma + b) + (\gamma^2 - a\gamma + b)\frac{K'''(t)}{K''(t)} - c$$
$$+ \int_0^t e^{\gamma\tau}\left[\frac{K'''(t)K''(\tau) - K''(t)K'''(\tau)}{K''(t)}\right] d\tau. \tag{2.12}$$

Since $\frac{K'''}{K''}$ is nondecreasing, for $\gamma^2 - a\gamma + b > 0$,

$$\gamma(\gamma^2 - a\gamma + b) + (\gamma^2 - a\gamma + b)\frac{K'''(t)}{K''(t)} - c$$

$$\geq \gamma(\gamma^2 - a\gamma + b) + (\gamma^2 - a\gamma + b)\frac{d}{c} - c$$

$$= \gamma(\gamma^2 - a\gamma + b) - \gamma(\gamma^2 - a\gamma + b) + c - c$$

$$= 0,$$

if γ also satisfies

$$(\gamma^2 - a\gamma + b)\frac{d}{c} = -\gamma(\gamma^2 - a\gamma + b) + c \tag{2.13}$$

or

$$\gamma^3 + (\frac{d}{c} - a)\gamma^2 + (b - \frac{ad}{c})\gamma + (\frac{bd}{c} - c) = 0. \tag{2.14}$$

Let $q(\gamma)$ be the polynomial in (2.14). Note first that $q(0) < 0$. The roots of $q'(\gamma) = 0$ are

$$\gamma = \frac{(ac-d) \pm \sqrt{(ac-d)^2 - 3c(bc-ad)}}{3c}$$

and the inflection of $q(\gamma)$ occurs at

$$\gamma = \frac{1}{3}\left(\frac{ac - d}{c}\right) .$$

By the second and third inequalities in assumption (3) of this theorem,

$$q\left[\frac{1}{3}\left(\frac{ac-d}{c}\right)\right] = \frac{1}{27}\left(\frac{ac-d}{c}\right)^3 + \left(\frac{bd-c^2}{c}\right) + \left(\frac{d-ac}{c}\right)\frac{1}{9}\left(\frac{ac-d}{c}\right)^2$$

$$+ \left(\frac{bc-ad}{c}\right)\frac{1}{3}\left(\frac{ac-d}{c}\right)$$

$$\geq \frac{1}{c}\left[\frac{(ac-d)^3}{27c^2} + (bd - c^2)\right] + \left(\frac{d-ac}{9c^3}\right)3c(bc-ad) + \frac{(bc-ad)(ac-d)}{3c^2}$$

$$= \frac{1}{c}\left[\frac{(ac-d)^3 + 27c^2(bd-c^2)}{27c^2}\right] + 0$$

$$\geq 0.$$

The fact that one of the three inequalities is strict implies that there is a zero, say γ_0, of $q(\gamma)$, satisfying

$$0 < \gamma_0 \leq \frac{1}{3}\left(\frac{ac-d}{c}\right)$$

and

$$\gamma_0 < \frac{1}{2}(a - \sqrt{a^2 - 4b}). \tag{2.15}$$

From (2.15), it follows that $\gamma_0^2 - a\gamma_0 + b > 0$. So $L'(t) > 0$. Thus, γ_0 has the two desired properties stated earlier. Since $\frac{K'''}{K''}$ is nondecreasing, the integral in (2.12) is positive. Equation (2.12) now leads to $P(t) \geq 0$ and so $\ell n\, L'(t)$ is convex, for $\gamma = \gamma_0$.

From (2.15), $\gamma_0 < a$, and so $L(t) > 0$. And from (2.13), we see that, for $\gamma = \gamma_0$, $L''(t) < 0$.

The case of $a^2 < 4b$ is examined in the following theorem.

THEOREM 2.3. If (1) $K(t) > 0$ with $K'(t) > 0$, $K''(t) > 0$, and $K'''(t) < 0$, (2) $\ln K''(t)$ is convex and (3) $a^2 < 4b$ and $abc + bd - c^2 > 0$, then (1) $L(t) > 0$, $L'(t) > 0$, $L''(t) < 0$ and (2) $\ln L'(t)$ is convex.

PROOF. Since $a^2 < 4b$, $\gamma^2 - a\gamma + b > 0$ for all γ and so $L'(t) > 0$. Let $q(\gamma)$ be the same polynomial as before; then

$$\begin{aligned} q(a) &= ab + \frac{bd}{c} - c \\ &> 0. \end{aligned}$$

So $q(\gamma)$ has a zero, say γ_0, satisfying $0 < \gamma_0 < a$. The rest of the proof for the convexity of $\ln L'(t)$ is similar to that in Theorem 2.2.

Since $\gamma_0 < a$, we have $L(t) > 0$; at $\gamma = \gamma_0$, (2.13) shows that $L''(t) < 0$.

Some examples of kernels satisfying the conditions in Theorem 2.2 and Theorem 2.3 are, respectively, $K(t) = \frac{3}{2}t + e^{-\frac{1}{2}t} + 1$ and $K(t) = \frac{5}{2}t + e^{-\frac{1}{2}t} + 1$.

The next class of positive increasing kernels can be reduced accordingly.

THEOREM 2.4. If (1) $K(t) > 0$ with $K'(t) > 0$, $K''(t) > 0$, $K'''(t) > 0$, and $K^{(4)}(t) < 0$, (2) $\ln K'''(t)$ is convex and (3) $a^2 \le 3b$, $-ab + c > 0$, and $(ab - c)(a + \frac{e}{d}) + d < 0$, where $a = K(0)$, $b = K'(0)$, $c = K''(0)$, $d = K'''(0)$, and $e = K^{(4)}(0)$, then (1) $L(t) > 0$, $L'(t) > 0$, $L''(t) > 0$, $L'''(t) < 0$ and (2) $\ln L''(t)$ is convex.

PROOF. As before, $L(t) \ge 0$ for $\gamma \le a$ and $L'(t) \ge 0$ for $\gamma^2 - a\gamma + b \ge 0$. $L''(t) > 0$ for $-\gamma^3 + a\gamma^2 - b\gamma + c > 0$. Differentiation of (2.9) leads to

$$L'''(t) = [-\gamma(-\gamma^3 + a\gamma^2 - b\gamma + c) + d]e^{-\gamma t} + K^{(4)} * e^{-\gamma t}. \qquad (2.16)$$

Proceeding as before, we can show that $L''(t) > 0$ and $\ln L''(t)$ is convex. The equations corresponding to (2.10) and (2.12) are

$$L''L^{(4)} - L'''^2 = K^{(4)}L'' + \gamma K'''L'' - K'''^2$$

and

$$\begin{aligned} \frac{e^{\gamma t}P(t)}{K'''(t)} &= \gamma(-\gamma^3 + a\gamma^2 - b\gamma + c) + (-\gamma^3 + a\gamma^2 - b\gamma + c)\,\frac{K^{(4)}(t)}{K'''(t)} - d \\ &+ \int_0^t e^{\gamma\tau}\left[\frac{K^{(4)}(t)K'''(\tau) - K'''(t)K^{(4)}(\tau)}{K'''(t)}\right] d\tau. \end{aligned} \qquad (2.17)$$

The sum of the first three terms in (2.17) is nonnegative if

$$-\gamma^3 + a\gamma^2 - b\gamma + c > 0$$

and

$$(-\gamma^3 + a\gamma^2 - b\gamma + c)\frac{e}{d} = -\gamma(-\gamma^3 + a\gamma^2 - b\gamma + c) + d \qquad (2.18)$$

or

$$\gamma^4 + (\frac{e}{d} - a)\gamma^3 + (b - \frac{ae}{d})\gamma^2 + (\frac{be}{d} - c)\gamma + (d - \frac{ce}{d}) = 0. \qquad (2.19)$$

Let $C(\gamma) = -\gamma^3 + a\gamma^2 - b\gamma + c$ and $q(\gamma)$ be the polynomial in (2.19). Since $a^2 - 3b \le 0$, $C(\gamma)$ is nonincreasing. Now the fact that $C(0) > 0$ and $C(a) = -ab + c > 0$ leads to $C(\gamma) > 0$ for $0 < \gamma < a$. As for q, $q(0) > 0$ and

$$\begin{aligned} q(a) &= (ab - c)(a + \frac{e}{d}) + d \\ &< 0. \end{aligned}$$

Therefore $q(\gamma)$ has a zero at γ_0, satisfying $0 < \gamma_0 < a$. So, for $\gamma = \gamma_0$, $L''(t) > 0$.

Since $\frac{K^{(4)}}{K'''}$ is nondecreasing, it follows again that $P(t) \geq 0$ and $\ell n\ L''(t)$ is convex.

At $\gamma = \gamma_0$, $L(t) > 0$, and from (2.16) and (2.18), $L'''(t) < 0$. Finally, since $a^2 < 4b$, $\gamma^2 - a\gamma + b > 0$ for all γ and so $L'(t) > 0$.

An example kernel satisfying all the conditions in this theorem is $K(t) = \frac{7}{2} t^2 + t - e^{-t} + 3$.

3. DISCUSSION. It is clear that, in general, if (1) $K(t) > 0$, $K'(t) > 0,\ldots,K^{(n)}(t) > 0$, $K^{(n+1)}(t) < 0$, (2) $\ell n\ K^{(n)}(t)$ is convex and (3) there exists $\gamma = \gamma_0$ satisfying

$$P_1(\gamma) = a_0 - \gamma \geq 0,$$

$$P_2(\gamma) = -\gamma P_1(\gamma) + a_1 \geq 0,$$

$$P_3(\gamma) = -\gamma P_2(\gamma) + a_2 \geq 0,$$

$$\cdots\cdot$$

$$P_{n-1}(\gamma) = -\gamma P_{n-2}(\gamma) + a_{n-2} \geq 0,$$

$$P_n(\gamma) = -\gamma P_{n-1}(\gamma) + a_{n-1} > 0$$

and

$$P_n(\gamma)\left(\frac{a_{n+1}}{a_n}\right) = P_{n+1}(\gamma), \tag{3.1}$$

where $a_0 = K(0)$, $a_1 = K'(0),\ldots,a_{n+1} = K^{(n+1)}(0)$, then at $\gamma = \gamma_0$, (1) $L(t) > 0$, $L'(t) > 0,\ldots,L^{(n-1)}(t) > 0$, $L^{(n)}(t) < 0$ and (2) $\ell n\ L^{(n-1)}(t)$ is convex. Note that γ must be positive in order for (3.1) to be satisfied, and so requires that $a_i > 0$, $i = 0,1,2,..,n - 1$.

REFERENCES

1. FRIEDMAN, A. "On Integral Equations of Volterra Type," J. Analyse Math. 11 (1963) 381-413.
2. HANSION, F.B., KLIMAS, A., RAMANATHAN, G.V., and SANDRI, G. "Uniformly Valid Asymptotic Solution to a Volterra Equation on an Infinite Interval," J. Math. Phys. 14 (1973) 1592-1600.
3. HANSION, F.B., KLIMAS, A., RAMANATHAN, G.V., and SANDRI, G. "Analysis of a Model for Transport of Charged Particles in a Random Magnetic Field," J. Math. Anal. Appl. 44 (1973) 786-798.
4. LEVIN, J.J. "On a Nonlinear Volterra Equation," J. Math. Anal. Appl. 39 (1972) 458-476.
5. LING, R. "Uniformly Valid Solutions to Volterra Integral Equations," J. Math. Phys. 18 (1977) 2019-2025.
6. LING, R. "Asymptotic Behavior of Integral and Integrodifferential Equations," J. Math. Phys. 18 (1977) 1574-1576.
7. LING, R. "Uniform Approximations to Integral and Integrodifferential Equations," J. Math. Phys. 19 (1978) 1137-1140.
8. LING, R. "Integral Equations of Volterra Type," J. Math. Anal. Appl. 64 (1978) 381-397.
9. LONDEN, S.O. "On the Solutions of a Nonlinear Volterra Equation," J. Math. Anal. Appl. 39 (1972) 564-573.
10. MILLER, R.K. "On the Linearization of Volterra Integral Equations," J. Math. Anal. Appl. 23 (1968) 198-208.
11. WEIS, DENNIS G. "Asymptotic Behavior of Integral Equations Using Monotonicity," J. Math. Anal. Appl. 54 (1976) 49-58.


Advances in Operations Research
Advances in Decision Sciences
Journal of Applied Mathematics
Algebra
Journal of Probability and Statistics
The Scientific World Journal
International Journal of Differential Equations
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of Combinatorics
Advances in Mathematical Physics
Journal of Complex Analysis
Journal of Mathematics
Mathematical Problems in Engineering
Abstract and Applied Analysis
Discrete Dynamics in Nature and Society
International Journal of Mathematics and Mathematical Sciences
Journal of Discrete Mathematics
Journal of Function Spaces
International Journal of Stochastic Analysis
Journal of Optimization