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THE MEHLER-FOCK TRANSFORM OF GENERAL ORDER AND ARBITRARY INDEX AND ITS INVERSION

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ABSTRACT. An integral transform involving the associated Legendre function of zero order, $P_{-\frac{1}{2}+i\tau}(x)$, $x \in [1,\infty)$, as the kernel (considered as a function of τ), is called Mehler-Fock transform. Some generalizations, involving the function $P^{\mu}_{-\frac{1}{2}+i\tau}(x)$, where the order μ is an arbitrary complex number, including the case when $\mu = 0,1,2,\ldots$, have been known for some time. In this present note, we define a general Mehler-Fock transform involving, as the kernel, the Legendre function $P^{\mu}_{-\frac{1}{2}+t}(x)$, of general order μ and an arbitrary index $-\frac{1}{2}+t$, $t = \sigma + i\tau$, $-\infty < \tau < \infty$. Then we develop a symmetric inversion formulae for these transforms. Many well-known results are derived as special cases of this general form. These transforms are widely used for solving many axisymmetric potential problems.

KEY WORDS AND PHRASES. *Mehler-Fock transform, Kontorovich-Lebedev transforms, Legendre functions of first and second kinds, Macdonald function, Bessel functions.*
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1. INTRODUCTION.

To solve many axisymmetric potential problems, integral transforms involving associated Legendre functions as kernels, are used widely. When the associated Legendre function of the first kind, $P_{-\frac{1}{2}+i\tau}(x)$, $x \in [1,\infty)$, is used as the kernel, the transform is called Mehler-Fock transform of order zero, [1, p.175(8,9)], and index $-\frac{1}{2} + i\tau$, $-\infty < \tau < \infty$. Some generalizations of Mehler-Fock transformation involve more general associated Legendre function, namely, $P^{m}_{-\frac{1}{2}+i\tau}(x)$, m, a non-negative integer [2, p.390] and $P^{\mu}_{-\frac{1}{2}+i\tau}(x)$, where $\mu \neq 0$ [3], have been known for some time, [cf 4]. In all these generalizations, the emphasis has been to generalize the order μ of the Legendre function $P^{\mu}_{\nu}(x)$, but no attempt has been made to extend the results for an arbitrary index ν. In this note we define transforms, involving a more general form of the kernel function, the associated Legendre function of the first kind $P^{\mu}_{-\frac{1}{2}+t}(x)$, with index $t = \sigma + i\tau$, $-\infty < \tau < \beta$ and μ an arbitrary complex number. An inversion formula is developed, also, involving the Legendre function $P^{\mu}_{-\frac{1}{2}+t}(x)$ as kernel, thus establishing a symmetric inversion theory for general Mehler-Fock transformation. Also, a Parseval type relation for these general transforms is produced. All the well-known results are derived as special cases.

2. THE PRELIMINARY RESULTS.

We note below, for future reference, some of the properties of the associated Legendre function of the first kind, $P^{\mu}_{\nu}(z)$ of order μ and index ν for unrestricted ν,μ and z [1,III]. The function is one-valued and regular in the complex plane supposed cut along the real axis from 1 to $-\infty$.

$$P^{\mu}_{\nu}(z) = P^{\mu}_{-\nu-1}(z) \tag{2.1}$$

$$P^{m}_{\nu}(z) = \frac{\Gamma(\nu+m+1)}{\Gamma(\nu-m+1)} P^{-m}_{\nu}(z), \quad m = 0,1,2,\ldots \tag{2.2}$$

An important case is when $\mu = 0$ and ν is an non-negative integer. Then

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n,$$

the Legendre polynomial.

Now, consider the representation, [1, p.128 (28)],

$$P^{\mu}_{\nu}(z) = 2^{\mu}(z^2-1)^{-\frac{1}{2}\mu} \frac{(z+\sqrt{z^2-1})^{\nu+\mu}}{\Gamma(1-\mu)} {}_2F_1\left(-\nu-\mu, \frac{1}{2} - \mu; 1-2\mu; \frac{2\sqrt{z^2-1}}{z+\sqrt{z^2-1}}\right)$$

Re $z > 1$ and Re $\mu < 1$.

Also, [2,p.78(1)],

$${}_2F_1(a,b;c;z) = \frac{\Gamma^2(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)} \int_0^1 \tau^{a-1}(1-\tau)^{c-a-1}\, d\tau \times \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-t\tau z)^c}\, dt,$$

Re $c >$ Re $b > 0$.

Using the inequality

$$\frac{1}{(1-t\tau z)^c} \le \frac{1}{(1-z)^c}, \qquad c > 0,$$

it is clear that

$${}_2F_1(a,b;c;z) \le \frac{1}{(1-z)^c}.$$

Thus

$$P^{\mu}_{\nu}(z) \le \frac{2^{\mu}(z+\sqrt{z^2-1})^{\nu-\mu+1}}{\Gamma(1-\mu)(z^2-1)^{\frac{1}{2}\mu}(z-\sqrt{z^2-1})^{1-2\mu}}, \qquad \text{Re } \mu < \frac{1}{2},$$

whence,

$$P^{\mu}_{\nu}(x) \sim x^{\nu} \text{ as } \quad x \to \infty \tag{2.3}$$

and

$$P^{\mu}_{\nu}(x) \sim (x^2-1)^{-\frac{1}{2}\mu} \qquad \text{as } x \to 1+ . \tag{2.4}$$

Also, that

$${}_2F_1(\sigma+i\tau,b;c;z) = 0(1), \qquad \text{as } |\tau| \to \infty,$$

whence

$$P^{\mu}_{\sigma+i\tau}(z) = 0(1) \qquad \text{as } |\tau| \to \infty, \tag{2.5}$$

provided Re $\mu < \frac{1}{2}$.

We shall also need the following result, [5, p.75(76)].

Lemma 1.

Let $f(\nu) = O(e^{-\nu})$, as $\nu \to \infty$

$= O(\nu^{\frac{1}{2}})$, as $\nu \to 0$.

If

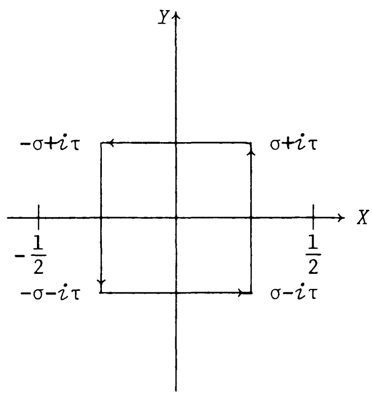
$$f(x) = \frac{1}{\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} tK_t(x)g(t)dt, \qquad \alpha > -\frac{1}{2}, \tag{2.6}$$

then $$g(t) = \int_0^\infty \frac{f(\nu)}{\nu} I_t(\nu)d\nu , \tag{2.7}$$

where $K_t(\nu)$ and $I_t(\nu)$ are the Macdonald function and the Bessel function respectively of order t, a complex number. Next let us consider the contour integral

$$I = \int_C z\, K_z(u)g(z)dz ,$$

where C is the closed contour in the z-plane as shown. Now $K_z(u)$ is an entire function of z and it is clear that $g(z)$ is regular inside and on C from (2.7) above. Thus, by Cauchy's theorem of residues,



$I = 0$,

or

$$\begin{aligned} 0 = i &\int_{-\tau}^{\tau} (\sigma+iy)\, K_{\sigma+iy}(u)g(\sigma+iy)dy \\ &+ \int_{\sigma}^{-\sigma} (x+i\tau)\, K_{x+i\tau}(u)g(x+i\tau)dx \\ &+ i\int_{\tau}^{-\tau} (-\delta+iy)\, K_{-\sigma+iy}(u)g(-\sigma+iy)dy \\ &+ \int_{-\sigma}^{\sigma} (x-i\tau)\, K_{x-i\tau}(u)g(x-i\tau)dx \\ &= I_1 + I_2 + I_3 + I_4, \text{ say,} \end{aligned}$$

Now the existance of the integral in (2.6), implies that

$$|(x+i\tau)\, K_{x+i\tau}(u)g(x+i\tau)| \to 0 \qquad \text{as } |\tau| \to \infty,$$

thus

$|I_2|$ and $|I_4|$ both vanish as $|\tau| \to \infty$.

And we now have,

$I_1 = -I_3$ or $|\tau| \to \infty$

or,

$$i\int_{-\infty}^{\infty} (\sigma+iy)\, K_{\sigma+iy}(u)g(\sigma+iy)dy = -i\int_{\infty}^{-\infty} (-\sigma+iy)K_{-\sigma+iy}(u)\, g(-\sigma+iy)dy$$

on simplifying, we obtain

$$\int_{\sigma-i\infty}^{\sigma+i\infty} zK_z(u)g(z)dt = -\int_{\sigma-i\infty}^{\sigma+i\infty} zK_z(u)g(-z)dz$$

$$= if(u),$$

using (2.6) and the fact that $K_{-z}(u) = K_z(u)$.

Thus,

$$f(u) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} zK_z(u)[g(z)-g(-z)]dz \tag{2.8}$$

From the definition of the function g given by (2.7),

$$g(z) - g(-z) = \int_0^\infty \frac{f(\nu)}{\nu}[I_z(\nu) - I_{-z}(\nu)]d\nu ,$$

or

$$-\frac{\pi}{2}\frac{g(z)-g(-z)}{\sin(\pi z)} = \int_0^\infty \frac{f(\nu)}{\nu} K_z(\nu)d\nu , \tag{2.9}$$

If we let

$$-\frac{\pi}{2}\frac{g(z) - g(-z)}{\sin(\pi z)} = F(z), \text{ say,}$$

then (2.8) and (2.9) reduce to

$$f(u) = \frac{i}{\pi^2}\int_{\sigma-i\infty}^{\sigma+i\infty} z \sin(\pi z)K_z(u)F(z)dz$$

and

$$F(z) = \int_0^\infty \frac{f(v)}{v} K_z(v)dv ,$$

respectively.

Now we have a generalized Kontorovich-Lebedev transforms in a symmetrical form, (cf. 6]. Thus, we have shown:

Lemma 3. Let $f(v) = 0(e^{-v})$, as $v \to \infty$

$= 0(v^{\frac{1}{2}})$, as $v \to 0$.

If

$$f(v) = \frac{i}{\pi^2}\int_{\sigma-i\infty}^{\sigma+i\infty} z \sin(\pi z)K_z(v)F(z)dz , \tag{2.10}$$

then

$$F(z) = \int_0^\infty \frac{f(v)}{v} K_z(v)dv , \tag{2.11}$$

where $|\sigma| < \frac{1}{2}$.

If we set $\sigma = 0$, the above pair reduces to

$$f(v) = \frac{2}{\pi^2}\int_0^\infty \tau \sinh(\pi\tau)K_{i\tau}(v)F(i\tau)d\tau$$

where

$$F(i\tau) = \int_0^\infty \frac{f(v)}{v} K_{i\tau}(v)dv,$$

giving us the usual Kontorovich-Lebedev transformation [2, p.361].

Lemma 4. If $\psi(t) \in L(\sigma-i\infty, \sigma+i\infty)$, $|\sigma| < \frac{1}{2}$, and Re $\mu < 1$,

then

$$\int_1^\infty \frac{e^{-xy}}{(y^2-1)^{\mu/2}} dy \int_{\sigma-i\infty}^{\sigma+i\infty} \psi(t)P^{\mu}_{-\frac{1}{2}+t}(y)dt = \sqrt{\frac{2}{\pi}}\, x^{\mu-\frac{1}{2}} \int_{\sigma-i\infty}^{\sigma+i\infty} \psi(t)K_t(x)dt .$$

Proof. Consider the double integral

$$\left|\int_1^\infty \frac{e^{-xy}}{(y^2-1)^{\mu/2}} dy \int_{\sigma-i\infty}^{\sigma+i\infty} \psi(t)P^{\mu}_{-\frac{1}{2}+t}(y)dt\right|$$

$$\leq \int_1^\infty \int_{\sigma-i\infty}^{\sigma+i\infty} \left| \frac{e^{-xy}\psi(t)}{(y^2-1)^{\mu/2}} P^{\mu}_{-\frac{1}{2}+t}(y) \right| dtdy$$

$$= \int_1^\infty \int_{-\infty}^{\infty} \frac{e^{-xy}}{(y^2-1)^{\mu/2}} \psi(\sigma+i\tau)P^{\mu}_{-\frac{1}{2}+\sigma+i\tau}(y) \Big| d\tau dy$$

$$= \int_1^\delta \int_{-\infty}^{\infty} | \cdot |d\tau dy + \int_\delta^\infty \int_{-\infty}^{\infty} | \cdot |d\tau dy , \quad \delta > 1.$$

$$\leq \int_1^\delta \int_{-\infty}^{\infty} \left| \frac{e^{-xy}}{(y^2-1)^{\mu/2}} \psi(\sigma+i\tau)(y^2-1)^{\mu/2} d\tau dy + \int_\delta^\infty \int_{-\infty}^{\infty} \right| \frac{e^{-xy}}{(y^2-1)^{\mu/2}} \psi(\sigma+i\tau) \Big| d\tau dy$$

$$= \int_1^\delta \frac{e^{-xy}}{(y^2-1)^{\mu}} dy \int_{-\infty}^{\infty} |\psi(\sigma+i\tau)|d\tau + \int_\delta^\infty \frac{e^{-xy}}{(y^2-1)^{\mu/2}} dy \times$$

$$\int_{-\infty}^{\infty} |\psi(\sigma+i\tau)|d\tau < \infty, \text{ due to the hypotheses.}$$

Hence

$$\int_1^\infty \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{e^{-xy}}{(y^2-1)^{\mu/2}} \psi(t)P^{\mu}_{-\frac{1}{2}+t}(y)dtdy$$

is absolutely convergent, and therefore the change of order of integration is possible.

And,

$$\int_1^\infty \frac{e^{-xy}}{(y^2-1)^{\mu/2}} dy \int_{\sigma-i\infty}^{\sigma+i\infty} \psi(t)P^{\mu}_{-\frac{1}{2}+t}(y)dt$$

$$= \int_{\sigma-i\infty}^{\sigma+i\infty} \psi(t)dt \int_1^\infty \frac{e^{-xy}}{(y^2-1)^{\mu/2}} P^{\mu}_{-\frac{1}{2}+t}(y)dy$$

$$= \sqrt{\frac{2}{\pi}}\, x^{\mu-\frac{1}{2}} \int_{\sigma-i\infty}^{\sigma+i\infty} \psi(t)K_t(x)dt, \qquad \text{Re } \mu < 1.$$

by evaluating the y-integral, [6,p.179(1)].

3. The main results and special cases.

Theorem 1.

Let $f(y) = 0(e^{-y})$, as $y \to \infty$

$= 0(1)$, as $y \to 1+$

If

$$f(y) = \frac{i}{2\pi} \int_{\sigma-i\infty}^{\sigma+i\infty} t \sin(\pi t)\Gamma(\tfrac{1}{2} - \mu - t)\Gamma(\tfrac{1}{2} - \mu + t)P^{\mu}_{-\frac{1}{2}+t}(y)F(t)dt \tag{3.1}$$

then

$$F(t) = \int_1^\infty f(y)P^{\mu}_{-\frac{1}{2}+t}(y)dy \tag{3.2}$$

where $t = \sigma + i\tau$, $-\infty < \tau < \infty$, $|\sigma| < \frac{1}{2} - \mu$, $\text{Re } \mu < 1$.

Proof. Using the estimates (2.3) and (2.4) of the function $P^{\mu}_{-\frac{1}{2}+t}(y)$, along with the conditions imposed on f, the integral in (3.2) defining the function g, exists. Note that in order to ensure the existence of the integral in (3.1), representing the function f, one must have $t^{1-2\mu}F(t) \in L(\sigma-i\infty,\sigma+i\infty)$ at least.

Now let,

$$\phi(t) \equiv \frac{\pi}{2}\, \Gamma(\tfrac{1}{2} - \mu - t)\Gamma(\tfrac{1}{2} - \mu + t)F(t) \tag{3.3}$$

then from (3.1)

$$f(y) = \frac{i}{\pi^2}\int_{\sigma-i\infty}^{\sigma+i\infty} t\,\sin(\pi t)\phi(t)P^{\mu}_{-\frac{1}{2}+t}(y)dt\ . \tag{3.4}$$

Next, we define an integral operator.

$$Lf \equiv L_{\mu}[f(y);\ y \to x] \equiv \int_1^{\infty} \frac{e^{-xy}}{(y^2-1)^{\mu/2}} f(y)dy, \qquad \text{Re}\ \mu < 1.$$

Note that this is a linear, self adjoint operator, a sort of generalized incomplete Laplace operator. Now applying this operator to both sides of the equation (3.4) above, we obtain

$$\int_1^{\infty} \frac{e^{-xy}}{(y^2-1)^{\mu/2}} f(y)dy = \frac{i}{\pi^2}\int_1^{\infty} \frac{e^{-xy}}{(y^2-1)^{\mu/2}}\,dy \int_{\sigma-i\infty}^{\sigma+i\infty} t\,\sin(\pi t)\phi(t)\cdot P^{\mu}_{-\frac{1}{2}+t}(y)dt$$
$$= \frac{i}{\pi^2}\sqrt{\frac{2}{\pi}}\,x^{\mu-\frac{1}{2}}\int_{\sigma-i\infty}^{\sigma+i\infty} t\,\sin(\pi t)\phi(t)K_t(x)dt,$$

due to lemma 4.

Or,

$$\psi(x) \equiv \sqrt{\frac{\pi}{2}}\,x^{\frac{1}{2}-\mu}\int_1^{\infty} \frac{e^{-xy}}{(y^2-1)^{\mu/2}} f(y)dy = \frac{i}{\pi^2}\int_{\sigma-i\infty}^{\sigma+i\infty} t\,\sin(\pi t)\phi(t)K_t(x)dt,$$

whence, according to lemma 3, we have the inversion,

$$\phi(t) = \int_0^{\infty} \frac{\psi(v)}{v} K_t(v)dv\ ,$$

provided the integral exists. Also,

$$\phi(t) = \sqrt{\frac{\pi}{2}}\int_0^{\infty} K_t(v)v^{-\frac{1}{2}-\mu}\,dv \int_1^{\infty} \frac{e^{-vy}}{(y^2-1)^{\mu/2}} f(y)dy$$
$$= \sqrt{\frac{\pi}{2}}\int_1^{\infty} \frac{f(y)}{(y^2-1)^{\mu/2}}\,dy \int_0^{\infty} e^{-vy}\,v^{-\frac{1}{2}-\mu}\,K_t(v)dv\ .$$

The change of order of integration can be justified due to absolute convergence, by making use of the estimates of the Macdonald's function $K_t(v)$ along with the conditions imposed on the function f. Now the v-integral can be evaluated, [6,p.198(27)], to give,

$$\phi(t) = \frac{\pi}{2}\,\Gamma(\frac{1}{2} - \mu - t)\Gamma(\frac{1}{2} - \mu + t)\int_1^{\infty} f(y)P^{\mu}_{-\frac{1}{2}+t}(y)dy\ .$$

And from (3.3), we have finally,

$$F(t) = \int_1^{\infty} f(y)P^{\mu}_{-\frac{1}{2}+t}(y)dy\ ,$$

as required.

Thus we have a symmetric transformation, in that, if

$$f(y) = \frac{i}{2\pi}\int_{\sigma-i\infty}^{\sigma+i\infty} t\,\sin(\pi t)\Gamma(\frac{1}{2} - \mu - t)\Gamma(\frac{1}{2} - \mu + t)P^{\mu}_{-\frac{1}{2}+t}(y)F(t)dy,$$

then

$$F(t) = \int_1^{\infty} f(y)P^{\mu}_{-\frac{1}{2}+t}(y)dy,$$

defining a more general form of the Mehler-Fock transform of order μ and arbitrary index $-\frac{1}{2} + t$, where $t = \sigma + i\tau$, $-\infty < \tau < \infty$.

Corollary: If the function f, satisfies the conditions of the above theorem, then

$$f(y) = \frac{i}{2\pi}\int_{\sigma+i\infty}^{\sigma+i\infty}\int_1^\infty t\,\sin(\pi t)\Gamma(\tfrac{1}{2}-\mu-t)\Gamma(\tfrac{1}{2}-\mu+t)P^\mu_{-\frac{1}{2}+t}(y)P^\mu_{-\frac{1}{2}+t}(u)\,f(u)\,du\,dt. \tag{3.5}$$

Note that from the definition of g, in (3.3), we have

$$g(-t) = g(t) \tag{3.6}$$

Now we shall look at some of the special cases.

Let $\sigma = 0$. Then (3.1) reduces to

$$f(y) = \frac{i}{2\pi}\int_{-i\infty}^{i\infty} t\,\sin(\pi t)\Gamma(\tfrac{1}{2}-\mu-t)\Gamma(\tfrac{1}{2}-\mu+t)P^\mu_{-\frac{1}{2}+t}(y)F(t)\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \tau\,\sinh(\pi\tau)\Gamma(\tfrac{1}{2}-\mu-i\tau)\Gamma(\tfrac{1}{2}-\mu+i\tau)P^\mu_{-\frac{1}{2}+i\tau}(y)\,F(i\tau)\,d\tau\,.$$

Hence

$$f(y) = \frac{1}{\pi}\int_0^\infty \tau\,\sinh(\pi\tau)\Gamma(\tfrac{1}{2}-\mu-i\tau)\Gamma(\tfrac{1}{2}-\mu+i\tau)P^\mu_{-\frac{1}{2}+i\tau}(y)F(i\tau)\,d\tau,$$

using the properties (2.1) and (3.6),

where $F(i\tau) = \int_1^\infty f(y)P^\mu_{-\frac{1}{2}+i\tau}(y)\,dy,$

giving us a generalized Mehler-Fock transformation [3].

Further if we put $\mu = 0$, then the above pair is reduced to

$$f(y) = \int_0^\infty \tau\,\coth(\pi\tau)P_{-\frac{1}{2}+i\tau}(y)F(i\tau)\,d\tau$$

and

$$F(i\tau) = \int_1^\infty f(y)P_{-\frac{1}{2}+i\tau}(y)\,dy;$$

the usual Mehler-Fock transformation of zero order [2, p.389]. Next, we shall derive, formally, a Parseval type relation. Let us define the function $F(t)$ and $G(t)$ to be the generalized Mehler-Fock transforms of the function $f(y)$ and $g(y)$ respectively, so that

$$F(t) = \int_1^\infty f(y)P^\mu_{-\frac{1}{2}+t}(y)\,dy$$

and

$$G(t) = \int_1^\infty g(y)P^\mu_{-\frac{1}{2}+t}(y)\,dy,$$

where $t = \sigma + i\tau$, $|\sigma| < \frac{1}{2}$, Re $\mu < 1$ and $-\infty < \tau < \infty$. And of course, as proved above

$$f(y) = \frac{i}{2\pi}\int_{\sigma-i\infty}^{\sigma+i\infty} t\,\sin(\pi t)\Gamma(\tfrac{1}{2}-\mu-t)\Gamma(\tfrac{1}{2}-\mu+t)P^\mu_{-\frac{1}{2}+t}(y)F(t)\,dt,$$

and a similar formula for $g(y)$.

Consider the integral,

$$\frac{i}{2\pi}\int_{\sigma-i\infty}^{\sigma+i\infty} t\,\sin(\pi t)\Gamma(\tfrac{1}{2}-\mu-t)\Gamma(\tfrac{1}{2}-\mu+t)F(t)G(t)\,dt$$

$$\equiv \frac{i}{2\pi}\int_{\sigma-i\infty}^{\sigma+i\infty} \psi(t)F(t)G(t)\,dt, \text{ say}$$

$$= \frac{i}{2\pi}\int_{\sigma-i\infty}^{\sigma+i\infty}\psi(t)F(t)dt\int_1^\infty g(y)P^{\mu}_{-\frac{1}{2}+t}(y)dy$$

$$= \frac{i}{2\pi}\int_1^\infty g(y)dy\int_{\sigma-i\infty}^{\sigma+i\infty}\psi(t)F(t)P^{\mu}_{-\frac{1}{2}+t}(y)dy$$

$$= \int_1^\infty g(y)f(y)dy \ .$$

Thus we have,

Theorem 2.

Let F and G be the generalized Mehler-Fock transforms of f and g. Then,

$$\frac{i}{2\pi}\int_{\sigma-i\infty}^{\sigma+i\infty} t\,\sin(\pi t)\Gamma(\frac{1}{2}-\mu-t)\Gamma(\frac{1}{2}-\mu+t)F(t)G(t)dt = \int_1^\infty f(y)g(y)dy \tag{3.7}$$

This is a formal derivation, but the analysis can be justified by absolute convergence, using suitable conditions on the functions involved. If we set $\sigma = 0$ in (3.7), we can easily deduce that

$$\frac{1}{\pi}\int_0^\infty \tau\,\sinh(\pi\tau)\Gamma(\frac{1}{2}-\mu-i\tau)\Gamma(\frac{1}{2}-\mu+i\tau)F(i\tau)G(i\tau)d\tau = \int_1^\infty f(y)g(y)dy \ , \tag{3.8}$$

a known result [3]. If, further, we put $\mu = 0$, then (3.8) reduces to [2,p.394],

$$\frac{1}{\pi i}\int_0^\infty \tau\,\tanh(\pi\tau)F(i\tau)G(i\tau)d\tau = \int_1^\infty f(y)g(y)dy \ ,$$

where,

$$F(i\tau) = \int_1^\infty f(y)P^{\mu}_{-\frac{1}{2}+i}(y)dy$$

and a similar representation for $G(i\tau)$.

Although the functions F and f satisfying (3.1) and (3.2) are defined for restricted values of μ, one can formally extend the results as follows:

If

$$f(y) = \frac{i}{2\pi}\int_{\sigma-i\infty}^{\sigma+i\infty} t\,\sin(\pi t)\Gamma(\frac{1}{2}-m-t)\Gamma(\frac{1}{2}-m+t)P^{m}_{-\frac{1}{2}+t}(y)F(t)dt, \tag{3.9}$$

then

$$F(t) = \int_1^\infty f(y)P^{m}_{-\frac{1}{2}+t}(y)dy \tag{3.10}$$

where $t = \sigma + i\tau$, $|\sigma| < \frac{1}{2}$, $-\infty < \tau < \infty$ and $m = 0,1,2,\ldots$. And all the results derived above hold. For instance, if $\sigma = 0$, then we have from above,

$$f(y) = (-1)^m\int_0^\infty \tau\,\tanh(\pi\tau)\,\frac{\Gamma(\frac{1}{2}-m+i\tau)}{\Gamma(\frac{1}{2}+m+i\tau)}\,P^{m}_{-\frac{1}{2}+i\tau}(y)F(i\tau)d\tau$$

where

$$F(i\tau) = \int_1^\infty f(y)P^{m}_{-\frac{1}{2}+i\tau}(y)dy \ .$$

If we let

$$\mathcal{F}_m(\tau) \equiv \frac{\Gamma(\frac{1}{2} - m + i\tau)}{\Gamma(\frac{1}{2} + m + i\tau)} F(i\tau),$$

then

$$\mathcal{F}_m(\tau) = \int_1^\infty f(y) \frac{\Gamma(\frac{1}{2} - m + i\tau)}{\Gamma(\frac{1}{2} + m + i\tau)} P^m_{-\frac{1}{2}+i\tau}(y)dy,$$

or

$$\mathcal{F}_m(\tau) = \int_0^\infty f(y)\, P^{-m}_{-\frac{1}{2}+i\tau}\ (y)dy$$

using the property (2.2), where

$$f(y) = (-1)^m \int_0^\infty \tau \tanh(\pi\tau) P^m_{-\frac{1}{2}+i\tau}(y)\mathcal{F}_m(\tau)d\tau$$

giving us a pair of Mehler-Fock transform of order m, [2, p.416]. It may be pointed out here that one can prove Lemma 4, using the result of Theorem 1. In other words, assume (3.1) and (3.2) hold one can show that if

$$f(v) = \frac{i}{\pi^2} \int_{\sigma-i\infty}^{\sigma+i\infty} z \sin(\pi z) K_z(v) F(z) dz$$

then

$$F(z) = \int_0^\infty \frac{f(v)}{v} K_z(v)dv, \qquad |\sigma| < \frac{1}{2}$$

the so called generalized Kontrovich-Lebedev transform. Cosequently one can say that there is an equivalence between the generalized Mehler-Fock and generalized Kontrovich-Lebedev transform.

A slightly different form of the Mehler-Fock transformation and its inversion, can easily be established, by making use of the pair of Kontrovich-Lebedev transforms [3, p.75(76)],

$$f(x) = \frac{1}{\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} tK_t(x)g(t)dt \tag{3.11}$$

and

$$g(t) = \int_0^\infty \frac{f(v)}{v} I_t(v)dv \tag{3.12}$$

That is,

Theorem 3.

If

$$f(y) = \frac{1}{\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} zg(z)P^\mu_{-\frac{1}{2}+z}(y)dz, \qquad y \in [1,\infty) \tag{3.13}$$

then

$$g(z) = e^{i\mu\pi} \int_1^\infty f(v)\, Q^{-\mu}_{-\frac{1}{2}+z}\ (v)dv, \qquad \text{Re } \mu < 1, \qquad |\sigma| < \frac{1}{2} \tag{3.14}$$

where P^μ_ν and Q^μ_ν are the associated Legendre functions of first and second kind respectively.

Alternatively,

$$f(y) = \frac{e^{i\mu\pi}}{\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \int_1^\infty z\, P^\mu_{-\frac{1}{2}+z}\ (y) Q^{-\mu}_{-\frac{1}{2}+z}\ (v)dvdz$$

The proof of the above theorem is on the same lines as the proof of Theorem 1, the linear operator used in the proof will now be

$$L[f(y);\ y \to x] \equiv \int_0^\infty e^{-xy} y^{\mu-\frac{1}{2}} f(y)dy \ ,$$

which on applying to the equation (3.13), will reduce it to the form (3.11). Then using the inversion (3.10), will produce equation (3.14), as desired.

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