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A CONSTRUCTION OF THE GENERAL RELATIVISTIC BOLTZMANN EQUATION

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ABSTRACT. Our work depends essentially on the notion of a one-particle seven-dimensional state-space. In constructing a general relativistic theory we assume that all measurable quantities arise from invariant differential forms. In this paper, we study only the case when instantaneous, binary, elastic collisions occur between the particles of the gas. With a simple model for colliding particles and their collisions, we derive the kinetic equation, which gives the change of the distribution function along flows in state-space.

KEY WORDS AND PHRASES. *General Relativistic Kinetic Theory, phase space, Boltzmann distribution function.*
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NOTATION: The Latin a , b , c , d take the values 0 , 1 , 2 , 3 and the Greek indices κ , λ , μ the values 1 , 2 , 3 . The symbols ε_{abcd} or $\varepsilon_{\kappa\lambda\mu}$ are the totally antisymmetric ones. The symbols Γ^{κ}_{ab} are the Christoffel symbols of the space-time metric.

1. THE STATE-SPACE OF A PARTICLE

The *state* of a particle is represented by the pair (x,p) , where $x = (x^0,x^1,x^2,x^3)$ is the position and $p = (p^0,p^1,p^2,p^3)$ is the momentum of the particle. The oriented space-time manifold is denoted by X and its metric tensor $g_{ab} = g_{ab}(x)$ has signature + --- . The momentum-space at a space-time point x , denoted by $P(x)$, is the local Minkowskian tangent plane to X at x [1,2,3,4,5].

We define the real non-negative map m on $P(x)$ by

$$m(p) = \sqrt{g_{ab}p^a p^b} = \sqrt{g_{00}(p^0)^2+2g_{0\kappa}p^0p^\kappa+g_{\kappa\lambda}p^\kappa p^\lambda} \quad . \tag{1.1}$$

A particle having proper mass m_j is called a j-particle. The physical momentum-space of a j-particle at x is denoted by $P_j(x)$ and is defined by

$$P_j(x) = \{p\varepsilon P(x):m(p)=m_j,\ p_0=g_{0a}p^a=g_{00}p^0+g_{0\kappa}p^\kappa > 0\} \quad . \tag{1.2}$$

Now, from (1.1) and (1.2), we obtain the relations [4]

$$p_0 = \sqrt{g_{00}m_j^2+(g_{0\kappa}g_{0\lambda}-g_{00}g_{\kappa\lambda})p^\kappa p^\lambda} \tag{1.3}$$

$$p^0 = \frac{-g_{0\kappa}p^\kappa \pm \sqrt{g_{00}m_j^2+(g_{0\kappa}g_{0\lambda}-g_{00}g_{\kappa\lambda})p^\kappa p^\lambda}}{g_{00}} \quad . \tag{1.4}$$

The element of the volume in X and in $P(x)$ is given respectively by η and ρ, where

$$\eta = \frac{\sqrt{-g(x)}}{4!} e_{abcd} dx^a{}_\Lambda dx^b{}_\Lambda dx^c{}_\Lambda dx^d = \sqrt{-g(x)}\, dx^0{}_\Lambda dx^1{}_\Lambda dx^2{}_\Lambda dx^3 \qquad (1.5)$$

$$\rho = \frac{\sqrt{-g(x)}}{4!} e_{abcd} dp^a{}_\Lambda dp^b{}_\Lambda dp^c{}_\Lambda dp^d = \sqrt{-g(x)}\, dp^0{}_\Lambda dp^1{}_\Lambda dp^2{}_\Lambda dp^3 \quad . \qquad (1.6)$$

The element of the volume in $P_j(x)$ must be an invariant differential three-form, independent of dm and non-vanishing when $m = 0$. Differentiating (1.1) we obtain $dp^0 = (mdm - p_\kappa dp^\kappa)/p_0$, and substituting in (1.6) we have $\rho = mdm_\Lambda \rho_j$, where ρ_j is obviously the required element of the volume [3] and it is given by

$$\rho_j = \frac{\sqrt{-g(x)}}{p_0} dp^1{}_\Lambda dp^2{}_\Lambda dp^3 = \frac{\sqrt{-g(x)}}{3!p_0} e_{\kappa\lambda\mu} dp^\kappa{}_\Lambda dp^\lambda{}_\Lambda dp^\mu \quad . \qquad (1.7)$$

The state-space of the j-particle, denoted by M_j, is a fibre bundle over X, i.e. $M_j = P_j(X) = \underset{x \in X}{\cup} P_j(x)$. So, its element of the volume is given by

$$W_j = \eta_\Lambda \rho_j \quad . \qquad (1.8)$$

Before studying a gas, we consider a single j-particle, which moves in a given gravitational field. Its world-line, $x(s_j)$, must be a geodesic in X [2,3,4,5]. So it moves according to the laws

$$\frac{dx^a}{ds_j} = p^a \quad , \quad \frac{dp^\kappa}{ds_j} = - \Gamma^\kappa_{ab} p^a p^b \qquad (1.9)$$

The affine parameter s_j is the proper-time of the j-particle. The equations (1.9) define the vector field $\underset{\sim}{\frac{d}{ds_j}}$ in M_j and the differential operator $\underset{\sim}{d}$ completely

$$\underset{\sim}{\frac{d}{ds_j}} = p^a \frac{\partial}{\partial x^a} - \Gamma^\kappa_{ab} p^a p^b \frac{\partial}{\partial p^\kappa} \quad , \quad \underset{\sim}{d} = p^a ds_j \frac{\partial}{\partial x^a} - \Gamma^\kappa_{ab} p^a p^b ds_j \frac{\partial}{\partial p^\kappa} \quad . \qquad (1.10)$$

The integral curves $(x^a(s_j), p^\kappa(s_j))$ in M_j form the flow of the states in M_j generated by the vector field $\underset{\sim}{d/ds_j}$. Physically the flow of the states represents the set of all possible trajectories of the j-particle in M_j [3,4].

The element of the volume in an hypersurface in M_j, S, non-tangential to $\underset{\sim}{d/ds_j}$ must be an invariant differential six-form independent of ds_j. Using (1.8) and (1.9) and putting

$$\sigma_a = \frac{\sqrt{-g(x)}}{3!} e_{abcd} dx^b{}_\Lambda dx^c{}_\Lambda dx^d \quad , \quad t_\kappa = \frac{\sqrt{-g(x)}}{2!} e_{\kappa\lambda\mu} dp^\lambda{}_\Lambda dp^\mu \qquad (1.11)$$

we obtain

$$W_j = w_{j\Lambda} ds_j \quad , \qquad (1.12)$$

where $w_j = \frac{1}{4} p^a \sigma_a \Lambda \rho_j - \frac{1}{3} \Gamma^\kappa_{ab} p^a p^b t_\kappa{}_\Lambda \eta$ is obviously the required element of the volume.

Inserting (1.5) and (1.7) into (1.12) and applying the differential operator $\underset{\sim}{d}$, we have $\underset{\sim}{d}W_j = 0$. But, from (1.12) we have $\underset{\sim}{d}W_j = dw_{j\Lambda} ds_j$, and since $ds_j \neq 0$ we can immediately see that $\underset{\sim}{d}w_j = 0$.

Applying the differential operator $\underset{\sim}{d}$ on σ_a and using the known formula $\frac{1}{\sqrt{-g(x)}} \cdot \frac{\partial(\sqrt{-g(x)})}{\partial x^a} = \Gamma^b_{ba}$, we obtain the relation $\underset{\sim}{d\sigma}_a = \Gamma^b_{ba}\eta$.

2. THE DESCRIPTION OF THE RELATIVISTIC GAS

A gas in this paper consists of a large number of particles interacting through gravitational forces. We can obtain a relativistic description of such a system by the use of the state-space of one particle. Our description depends essentially on only two phenomena: (a) the collision of two particles, and (b) the motion of one particle in the intervals between collisions in the smoothed-out mean gravitational field, generated by all particles together.

A *simple* j-*gas* is the gas consisting of only j-particles. Here we are to study a gas which is the mixture of $N + 1$ simple gases and in which only instantaneous, binary, elastic collisions occur between the particles. A binary elastic jk-collision is a collision by which a j-particle collides with a k-particle and the result is also a j-particle and a k-particle. The only collisions we can observe in M_j are all the jk-collisions for $k = 0,1,\ldots,N$. Due to elastic jk-collisions, j-particles are removed from their states in M_j to other states in M_j . We say that the j-particle is annihilated at the state from which it is removed by the collision and a new j-particle created at the state to which it is removed by the collision. The newly created particle moves in M_j on a trajectory defined by (1.9). This new moving particle may also be annihilated. The parts of a trajectory between creation and annihilation are called *excited* parts.

Let S be an hypersurface in M_j , independent of ds_j . We say that a trajectory is an *occupied* trajectory with respect to S if S is crossed by an excited part of this trajectory once and only once.

The *distribution function* [4] of the j-gas component of this mixture is denoted by $f_j(x,p)$, where $f_j(x,p)$ is a scalar function on the state-space of the j-particle M_j . If we denote by $N_j(S)$ the number of the occupied trajectories with respect to S , this number is obviously equal to the number of the j-particles crossing S once and only once. So, we can define $N_j(S)$ by

$$N_j(S) = \int_S f_j(x,p)w_j \quad . \tag{2.1}$$

Now, consider the volume D in M_j . We denote the boundary of D by ∂D and obviously ∂D is an hypersurface in M_j . So, by (2.1), we have

$$N_j(\partial D) = \int_{\partial D} f_j(x,p)w_j \quad . \tag{2.2}$$

In the case when $\partial D = \partial_1 I \cup \partial_2 D$, with $\partial_1 D \cap \partial_2 D = \emptyset$, and with $\partial_2 D$ lying in the future of $\partial_1 D$ (we say that $\partial_2 D$ lies in the future of $\partial_1 D$ if the vector field $\underset{\sim}{d/ds_j}$ is directed from $\partial_1 D$ to $\partial_2 D$) we have

$$N_j(\partial D) = N_j(\partial_1 D) - N_j(\partial_2 D) \quad . \tag{2.3}$$

Let us define:

$a_{jk}(D)$ = the number of j-particles annihilated in D due to jk-collisions,

$c_{jk}(D)$ = the number of j-particles created in D due to jk-collisions,

$a_j(D) = \sum_{k=0}^{N} a_{jk}(D)$, $c_j(D) = \sum_{k=0}^{N} c_{jk}(D)$. We obviously have

$$a_j(D) - c_j(D) = \sum_{k=0}^{N} (a_{jk}(D) - c_{jk}(D)) \quad . \tag{2.4}$$

We will try to shed some light on the foundations upon which the whole theory is constructed by establishing the following theorem [5].

THEOREM 1. $N_j(\partial D) = N_j(\partial_1 D) - N_j(\partial_2 D) = a_j(D) - c_j(D)$. (2.5)

PROOF. $N_j(\partial_1 D)$ = (the number of j-particles crossing $\partial_1 D$) = (the number of j-particles crossing $\partial_1 D$ and not crossing $\partial_2 D$) + (the number of j-particles crossing both $\partial_1 D$ and $\partial_2 D$) .

$N_j(\partial_2 D)$ = (the number of j-particles crossing $\partial_2 D$) = (the number of j-particles crossing $\partial_2 D$ without crossing $\partial_1 D$) + (the number of j-particles crossing both $\partial_1 D$ and $\partial_2 D$) .

So: $N_j(\partial_1 D) - N_j(\partial_2 D)$ = (the number of j-particles crossing $\partial_1 D$ and not crossing $\partial_2 D$) - (the number of j-particles crossing $\partial_2 D$ without crossing $\partial_1 D$) = (the number of j-particles crossing $\partial_1 D$ and annihilated in D) - (the number of j-particles created in D and crossing $\partial_2 D$) = {(the number of j-particles crossing $\partial_1 D$ and annihilated in D) + (the number of j-particles which are both created and annihilated in D)} - {(the number of j-particles created in D and crossing $\partial_2 D$) + (the number of j-particles which are both created and annihilated in D)} = $a_j(D) - c_j(D)$.

3. THE CONSTRUCTION OF THE RELATIVISTIC KINETIC EQUATION

Now, we are in the position to construct the relativistic kinetic equation. To achieve this we apply the theorem of Gauss to the relation (2.2), and analyse the mechanism of instantaneous, binary elastic collisions in the relation (2.4). The *kinetic equation* is the equation which gives the change of $f_j(x,p)$ along the flow of the states in M_j . This change is just the action of the vector field $\underset{\sim}{d/ds_j}$ on $f_j(x,p)$.

Applying the theorem of Gauss in (2.2) we have

$$N_j(\partial D) = \int_{\partial D} f_j(x,p) w_j = \int_{\partial D} \underset{\sim}{d}(f_j(x,p) w_j) = \int_{\partial D} (\underset{\sim}{d} f_j(x,p))_\wedge w_j =$$

$$= \int_{\partial D} \frac{df_j(x,p)}{\underset{\sim}{ds_j}} ds_{j\wedge} w_j = \int_D \frac{d}{\underset{\sim}{ds_j}} (f_j(x,p)) W_j \quad . \tag{3.1}$$

Our aim now is to express $a_j(D) - c_j(D)$ as a volume-integral over D . We achieve this by the construction of a simple model for colliding particles. In doing this, we need to define the concepts of the j-*range* and the j-*volume* of the collision.

Before proceeding, we want to note that a binary, elastic collision does not occur at a space-time point, but it is an interaction between two particles located at different space-time points. When we say that such a collision is elastic and instantaneous we mean that (a) the interaction affects only the momenta of the colliding particles and not their space-time positions, and (b) the proper-duration of the interaction is zero. From these remarks, we can see that the collision occurs on an hypersurface U of X, defined by $\underset{\sim}{ds_j} = 0$, i.e. both particles must lie on an hypersurface upon which s_j = constant, while interacting.

The element of the volume on such an U is given by

$$\eta_j = \frac{1}{4} p^a \sigma_a \quad . \tag{3.2}$$

This is obvious because we have $\eta = \frac{\sqrt{-g(x)}}{4!} e_{abcd} dx^a{}_\Lambda dx^b{}_\Lambda dx^c{}_\Lambda dx^d = \frac{1}{4} dx^a{}_\Lambda \sigma_a =$ $= ds_{j\Lambda}(\frac{1}{4} p^a \sigma_a)$.

The particles constituting the gas, will be thought of as interacting through an inter-particle potential. Let us study the jk-collision by which the j-particle at x with momentum p emerges with momentum p', after colliding with the k-particle at y with momentum q. To examine the phenomenon we restrict our study to the approximation $y = x$. *This restriction will be seen not to violate the generality, in view of a very natural assumption*(*) *concerning momenta at the same space-time point to be introduced later.* The law of momentum-conservation is then valid, and is given by: $p - q = p' - q'$. So, $q' = p - q - p'$, and we can describe the collision by using only p, q, p'. The range of interaction of the j-particle for this collision, called j-range of the collision, must (a) be a neighbourhood of x in U, and (b) be independent of x and (c) depend on p, q, p' continuously. So, the j-range must be given by means of a continuous, positive function of p, q, p'. Let us denote this function by $\sigma_{jk}(p,q,p')$. Now, since we know the element of the volume of U, namely η we can put:

$$\text{j-range of the collision} = \sigma_{jk}(p,q,p')\eta_j \quad . \tag{3.3}$$

We say that the jk-collision occurs, when the k-particle lies inside the j-range of the collision and that the jk-collision occurs in propertime ds_j when the k-particle finds itself inside the j-range of the collision in proper-time ds_j.

Since we know p and q we can find the relative velocity between the two particles. The relative velocity is given by the formula [4]

$$v_{pq} = \sqrt{(p,q)^2 - m_j^2 m_k^2} \,/\, (p,q) \quad , \qquad \text{where} \quad (p,q) = g_{ab} p^a q^a \quad . \tag{3.4}$$

The relative velocity multiplied by ds_j gives the proper-distance between the two particles, and since ds_j is perpendicular to the j-range of the collision we have:

$$\text{proper-distance of k-particle from j-range} = v_{pq} ds_j \quad . \tag{3.5}$$

Multiplying (3.3) with (3.5), we obtain a volume which we call j-volume of the collision. In fact $v_{pq} ds_{j\Lambda}(\sigma_{jk}(p,q,p')\eta_j) = v_{pq}\sigma_{jk}(p,q,p')\eta$. Now, using the concept of the j-volume of the collision we can say that the jk-collision occurs in proper-time ds_j when the k-particle lies inside the j-volume of the collision.

For convenience we make the following assumption: For every y lying inside the j-volume of the collision, we have $f_k(y,q) = f_k(x,q)$. (*)

The collision-number assumption or Boltzmann's *"Stosszahlensatz"*, in our case can be expressed by means of the following propositions: (a) The gas is sufficiently *dilute* so that each j-volume contains at most one k-particle with momentum $q \varepsilon \rho_k$.

The existence of such k-particles leads to jk-collisions. (b) The number of k-particles, with momentum $q\varepsilon\rho_k$, lying inside the total volume of the j-volumes is assumed to be equal to the number obtained by multiplying the total volume of the j-volumes with the number of k-particles with $q\varepsilon\rho_k$, lying in ρ_k. Now

number of j-particles with $p\varepsilon\rho_k$, lying in the j-range of the collision $= f_j(x,p)\rho_{j\Lambda}(\sigma_{jk}(p,q,p')\eta_j)$ (3.6)

and

number of k-particles with $q\varepsilon\rho_k$, of which the proper-distance from the j-range is smaller than $v_{pq}ds_j = f_k(x,q)\rho_k\Lambda(v_{pq}ds_j)$. (3.7)

THEOREM 2. The number of the jk-collisions occuring in proper-time ds_j, with $p\varepsilon\rho_j$, $q\varepsilon\rho_k$ resulting in $p'\varepsilon\rho'_j$ is given by multiplying (3.6) with (3.7) and with ρ'_j, i.e. this number is given by

$$f_j(x,p)f_k(x,q)v_{pq}\sigma_{jk}(p,q,p')\rho_k\Lambda\rho'_j\Lambda W_j \quad . \tag{3.8}$$

(In arriving at (3.8) we have used (1.8) and (3.2)).

PROOF. The number of j-particles with $p\varepsilon\rho_j$ is $f_j(x,p)\rho_j$. Imagine that to each of these particles there is attached one j-volume. The number of such j-volumes is obviously $f_j(x,p)\rho_j$. So the total volume occupied by the j-volumes is $f_j(x,p)\rho_{j\Lambda}(v_{pq}\sigma_{jk}(p,q,q')\eta) = f_j(x,p)v_{pq}\sigma_{jk}(p,q,p')W_j$. Using this result, combining the assumptions (a) and (b) of the "Stosszahlansatz" and multiplying with ρ'_j we obtain the required number which is the one given by (3.8).

Integrating (3.8) over D, we obtain the number of collisions annihilating j-particles inside D. But this number is just the number of annihilations inside D due to jk-collisions, i.e. $a_{jk}(D)$. So,

$$a_{jk}(D) = \int_D\left(\int_{P'_j(x)}\left(\int_{P_k(x)} f_j(x,p)f_k(x,q)v_{pq}\sigma_{jk}(p,q,p')\rho_k\right)\rho_j\right)W_j \quad . \tag{3.9}$$

By the same method we find that the number of creations inside D due to jk-collisions, i.e. $c_{jk}(D)$, is given by the relation

$$c_{jk}(D) = \int_D\left(\int_{P'_j(x)}\left(\int_{P'_k(x)} f_j(x,p')f_k(x,q')v_{p'q'}\sigma_{jk}(p',q',p)\rho'_k\right)\rho_j\right)W_j \quad . \tag{3.10}$$

Now, from (2.4) we have

$$a_j(D) - c_j(D) = \sum_{k=0}^{N}(a_{jk}(D)-c_{jk}(D)) = \int_D I_j(x,p)W_j \quad , \tag{3.11}$$

where $I_j(x,p)$, the collision-integral is defined by the relation

$$I_j(x,p) = \sum_{k=0}^{N}\left(\int_{P'_j(x)} f_j(x,p)\left(\int_{P_k(x)} f_k(x,q)v_{pq}\sigma_{jk}(p,q,p')\rho_k\right)\rho_j - \right.$$
$$\left. - \int_{P'_j(x)} f_j(x,p') \int_{P'_k(x)} f_k(x,q')v_{p'q'}\sigma_{jk}(p',q',')\rho'_k\right)\rho_j\Bigg) \quad . \tag{3.12}$$

Now, from (2.5), (3.1) and (3.11) we have that $\int_D \frac{d}{d\underset{\sim}{s}_j} (f_j(x,p))W_j = \int_D I_j(x,p)W_j$ for every D in M_j. So

$$\frac{d}{d\underset{\sim}{s}_j} (f_j(x,p)) = I_j(x,p) \quad . \tag{3.13}$$

This is the kinetic equation [4].

When $a_j(D) = c_j(D) = 0$ (case of the free-equilibrium) and when $a_j(D) \neq 0$, $c_j(D) \neq 0$ with $a_j(D) - c_j(D) = 0$ (case of the balanced-equilibrium) we obviously have $I_j(x,p) = 0$ and so

$$\frac{d}{d\underset{\sim}{s}_j} (f_j(x,p)) = 0 \quad . \tag{3.14}$$

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