

## RINGS ALL OF WHOSE ADDITIVE ENDOMORPHISMS ARE LEFT MULTIPLICATIONS

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ABSTRACT. Motivated by Cauchy's functional equation  $f(x+y) = f(x) + f(y)$ , we study in §1 special rings, namely, rings for which every endomorphism  $f$  of their additive group is of the form  $f(x) \equiv ax$ . In §2 we generalize to  $R$  algebras ( $R$  a fixed commutative ring) and give a classification theorem when  $R$  is a complete discrete valuation ring. This result has an interesting consequence, Proposition 12, for the theory of special rings.

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### 1. INTRODUCTION.

Our starting point is Cauchy's functional equation

$$(*) \quad f(x+y) = f(x) + f(y)$$

where  $f$  is a real function with domain  $(-\infty, \infty)$  and the equality holds for every real  $x, y$ . If  $f$  is assumed to be continuous throughout  $(-\infty, \infty)$ , or even merely at one point, or merely bounded above or below on some set of positive measure, then necessarily  $f(x) \equiv f(1)x$ . On the other hand, using a Hamel basis, one can construct noncontinuous solutions of (\*). For more details and references the reader is referred to [1], pp. 31-36.

By using (\*) merely for integral (or rational)  $x$  and  $y$ , one easily concludes [1, pp. 31-32] that  $f(x) = f(1)x$  for every integral (respectively, rational)  $x$ . This raises the following general problem: what are the rings  $A$  for which every endomorphism  $f$  of their additive group is of the form  $f(x) \equiv ax$ ? We call such rings special.

In [2, Problem 45, p. 232] L. Fuchs raised the related question of characterizing the rings  $A$  which are isomorphic to the endomorphism ring  $\text{End}(A^+)$  of  $A^+$ , the additive group of  $A$ . This question was studied by P. Schultz [5] who found a number of interesting results.

In §1 we study special rings. Some of this material was considered in a different form by P. Schultz in [5]. In §2 we generalize our problem to  $R$ -algebras ( $R$  a fixed commutative ring) and give a classification theorem when  $R$  is a complete discrete valuation ring. This result has an interesting consequence, Proposition 12, for the theory of special rings. It would be nice to give a complete classification of special rings themselves, but this, so far, has eluded us.

## 2. SPECIAL RINGS.

PROPOSITION 1: If  $A$  is a special ring, then  $A$  is commutative and has an identity.

PROOF: The identity endomorphism is given by left multiplication by an element  $e$ . Similarly, the endomorphism  $x \mapsto xe$  is given by left multiplication by an element  $e'$ . Thus  $e = e^2 = e'e$ . For every  $x \in A$  we find  $xe = e'x = e'ex = e^2x = ex = x$ . This shows that  $e$  is an identity. We set  $e = 1$  from now on.

If  $a \in A$ , consider the map  $x \mapsto ax - xa$ . This is in  $\text{End}(A^+)$  so  $ax - xa = \lambda x$  for some  $\lambda$  and all  $x$ . Taking  $x = 1$  shows  $\lambda = 0$ , and so  $ax = xa$  for all  $x$ .

PROPOSITION 2: If  $A$  is special and  $A = M + N$  as abelian groups, then  $M$  and  $N$  are ideals.

PROOF: Let  $\pi$  be the projection on  $M$ . Since  $\pi \in \text{End}(A^+)$ , we have  $\pi x = ex$  for some  $e$  and all  $x$ . It follows that  $M = Ae$ . ■

PROPOSITION 3: Suppose  $A$  is special and  $A = M + N + P$ . Then  $\text{Hom}(M, N) = (0)$ .

PROOF: Let  $f \in \text{Hom}(M, N)$ . Define  $f^*: A \rightarrow A$  by  $f^*(m+n+p) = f(m)$  where  $m \in M$ ,  $n \in N$  and  $p \in P$ . Then  $f^* \in \text{End}(A^+)$  and so  $f^*(x) = rx$  for some  $r$  and all  $x$ . Since  $M, N$  and  $P$  are ideals by Proposition 2, we see that  $f(m) = rm$  for all  $m \in M$  and  $n \in N$ . Take now  $n = 0$ . ■

PROPOSITION 4: Suppose  $A = M_1 + M_2 + \dots + M_t$  as groups. If  $A$  is a special ring, then so is each  $M_i$  and, moreover,  $\text{Hom}(M_i, M_j) = (0)$  whenever  $i \neq j$ . Conversely, if each  $M_i$  is special and  $\text{Hom}(M_i, M_j) = (0)$  whenever  $i \neq j$ , then  $A$  is special.

PROOF: Straightforward, using the last two propositions. ■

COROLLARY 1: If  $A$  is special and a  $Q$ -algebra, then  $A \cong Q$ . ( $Q$  is the field of rational numbers.)

COROLLARY 2: If  $A$  is special and  $A^+$  is finitely generated, then either  $A \cong Z$  or  $A \cong Z/nZ$  for some  $n \in Z$ . ( $Z$  is the ring of ordinary integers.)

COROLLARY 3: If  $A$  is special and  $A^+$  is torsion-free and completely decomposable, then  $A \cong M_1 + M_2 + \dots + M_t$ , where the  $M_i$  are subrings of  $Q$  of incomparable types as abelian groups. The converse is also true.

PROOF: The decomposition of  $A$  into rank one groups has only finitely many

many summands because these summands must be ideals and  $A$  has an identity. The rest follows from Proposition 4 and Proposition 85.4 of [3].

**PROPOSITION 5:** Let  $A$  be a commutative ring with identity and  $I = \{I\}$  a collection of ideals, closed under finite intersection. Suppose: i)  $f(I) \subseteq I$  for all  $f \in \text{End}(A^+)$  and  $I \in I$ , ii)  $A/I$  is special for all  $I \in I$ , and iii)  $A \cong \varprojlim A/I$ . Then  $A$  is special.

**PROOF:** If  $f \in \text{End}(A^+)$ , then, by i),  $f$  induces  $f_I \in \text{End}(A/I)$ . Thus, by ii), there is an  $r_I \in A/I$  such that  $f_I(\bar{x}) = r_I \bar{x}$  for all  $\bar{x} \in A/I$ . The collection  $\{r_I\}$  defines an element in the inverse limit and so, via iii), an element  $r \in A$ . It is easily seen that  $f(x) = rx$  for all  $x \in A$ . ■

**COROLLARY 1:**  $\hat{Z} = \varprojlim Z/nZ$  and  $Z_p = \varprojlim Z/p^m Z$  are both special rings. Here  $p$  is a prime and  $Z_p$  is the ring of  $p$ -adic integers.

**COROLLARY 2:** Suppose  $\{A_j\}$  is a collection of special rings and, for each  $j_0$ ,  $\text{Hom}(\prod_{j \neq j_0} A_j, A_{j_0}) = (0)$ . Then  $\prod A_j$  is a special ring.

**COROLLARY 3:** For each prime  $p$ , let  $C_p$  be either cyclic of  $p$ -power order, or isomorphic to  $Z_p$ . Then  $\prod C_p$  is special.

**PROOF:** Almost immediate, from Corollary 2. ■

**PROPOSITION 6:** If  $A$  is special and  $A = B + C$ , with  $B \cong Q$ , then  $C$  is torsion.

**PROOF:** By Proposition 3,  $\text{Hom}(C, B) \cong \text{Hom}(C, Q) = (0)$ . It is standard, however, that  $\text{Hom}(C, Q) = (0)$  if and only if  $C$  is torsion. ■

**PROPOSITION 7:**  $A$  is special and  $A^+$  is torsion if and only if  $A$  is cyclic.

**PROOF:** Let  $1$  be the identity of  $A$ . If  $A^+$  is torsion,  $n1 = 0$  for some positive integer  $n$ . But then  $na = 0$  for all  $a \in A$ . Thus  $A^+$  is a bounded torsion group and so, a direct sum of cyclic groups (Theorem 17.2 of [3]). Since  $A$  has an identity, and the summands are ideals, there are only finitely many of them. The result now follows rapidly from Proposition 3. ■

**PROPOSITION 8 (P. Schultz):** If  $A$  is special, and  $A^+$  is not reduced, then  $A \cong Q + Z/nZ$  and conversely.

**PROOF:** Suppose  $A$  is special and let  $D$  be its maximal divisible subgroup.  $D$  is a direct summand of  $A$  and so, must be a special ring. If  $D \neq (0)$ , then  $D$  cannot be torsion, by the last proposition. Thus  $D \neq (0)$  implies that  $D$  contains a copy of  $Q$ . Since  $Q$  is divisible,  $A = B + C$  where  $B \cong Q$  and  $C$  is torsion, by Proposition 6. By Proposition 7,  $C$  must be cyclic. The converse follows from Proposition 4. ■

**PROPOSITION 9:** Suppose  $A$  is special and the torsion subgroup  $T$  of  $A^+$  is a proper direct summand. Then  $A \cong Z/nZ + C$  where  $C$  is a special ring which is torsion-free and divisible by  $n$ . The converse is also true.

**PROOF:** Immediate from Propositions 4 and 7. ■

PROPOSITION 10: Suppose  $A$  is special and satisfies the descending chain condition on ideals. Then  $A$  is isomorphic to either  $Z/nZ$ ,  $Q$  or  $Z/nZ + Q$ . The converse is also true.

PROOF: Using Theorem 122.4 of [3], we see that  $A^+$  is the direct sum of a torsion group and a  $Q$ -vector space. By previous results, the torsion group must be cyclic and the vector space must have dimension  $\leq 1$ . The converse is clear. ■

3. R-ALGEBRAS.

Let  $R$  be a fixed commutative ring. We consider the class of  $R$ -algebras  $A$ . The notion of a special ring can be generalized as follows. An  $R$ -algebra  $A$  is called special if every element of  $\text{End}_R(A)$  is given by a left multiplication by an element of  $A$ . We may now ask for a classification of special  $R$ -algebras. Using a theorem in [4], we will give such a classification when  $R$  is a complete discrete valuation ring, for example, when  $R$  is some  $Z_p$ .

We note that, suitably modified, Proposition 3 remains valid in the context of  $R$ -modules and  $R$ -algebras.

PROPOSITION 11: Let  $R$  be a complete discrete valuation ring, and  $K$  its quotient field. Let  $C$  represent a cyclic torsion module. An  $R$ -algebra  $A$  is special if and only if  $A$  is isomorphic to either  $C$ ,  $R$ ,  $K$  or  $K + C$ .

PROOF: If  $A$  is special as an  $R$ -algebra and not reduced as an  $R$ -module, then the proof of Proposition 8 is easily adapted to show that  $A$  is isomorphic to either  $K$  or  $K + C$ . Thus we can assume  $A$  is reduced. By Corollary 1 to Theorem 23 in [4], every reduced  $R$ -module has a cyclic direct summand. If  $A = B + D$  where  $B \cong R$ , then  $\text{Hom}_R(R, D) = (0)$  implies  $D = (0)$ , and so,  $A \cong R$ . If  $A = C + D$ , then  $\text{Hom}_R(C, D) = (0)$  implies  $D$  is torsion-free. If  $D$  were not trivial, it would have a direct summand isomorphic to  $R$ , but this cannot occur since  $\text{Hom}_R(R, C) \neq (0)$ . Thus, in this case,  $A = C$ .

Conversely,  $C$ ,  $R$ ,  $K$  and  $K + C$  are easily seen to be special. ■

We can derive a consequence of this proposition for the theory of special rings.

Let  $p$  be a prime. Let us call a ring  $A$   $p$ -complete if  $A = \varprojlim A/p^m A$ .

Such a ring is clearly a  $Z_p$  algebra. Moreover, every  $Z$ -endomorphism automatically commutes with the  $Z_p$  action. Since  $Z_p$  is a complete discrete valuation ring, we have

PROPOSITION 12: A ring  $A$  is special and  $p$ -complete if and only if  $A$  is isomorphic to either  $Z_p$  or  $Z_p/p^m Z_p$  for some  $m$ .

PROOF: This follows from Proposition 11.  $Q_p$ , the quotient field of  $Z_p$ , cannot occur since it is infinite dimensional over  $Q$ . ■

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