

## A PROPERTY OF L-L INTEGRAL TRANSFORMATIONS

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ABSTRACT. The main result of this paper is the result that the collection of all integral transformations of the form  $F(x) = \int_0^\infty G(x,y)f(y)dy$  for all  $x \geq 0$ , where  $f(y)$  is defined on  $[0, \infty)$  and  $G(x,y)$  defined on  $D = \{(x,y): x \geq 0, y \geq 0\}$  has no identity transformation on  $L$ , where  $L$  is the space of functions that are Lebesgue integrable on  $[0, \infty)$  with norm  $\|f\| = \int_0^\infty |f(x)|dx$ . That is to say, there is no  $G(x,y)$  defined on  $D$  such that for every  $f \in L$ ,  $f(x) = \int_0^\infty G(x,y)f(y)dy$  for almost all  $x \geq 0$ . In addition, this paper gives a theorem that is an improvement of a theorem that is proved by J. B. Tatchell (1953) and Sunonchi and Tsuchikura (1952).

KEY WORDS AND PHRASES. *L-L Integral Transformation. Absolutely continuity of integrals. Lebesgue measurable. Lebesgue points.*

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### 1. INTRODUCTION.

The well known property of sequence to sequence transformations of the form  $(Ax)_n = \sum_{k \geq 1} a_{nk}x_k$  for which there is an identity mapping such that  $(Ax)_n = x_n$ , does not carry over to the function to function transformations of the form  $F(x) = \int_0^\infty G(x,y)f(y)dy$ ,  $x \geq 0$ . There is no identity mapping on  $L$  of the collection of all transformations of the form  $F(x) = \int_0^\infty G(x,y)f(y)dy$ . This is the main theorem of this paper. We list it as Theorem 2.

A result by Knopp and Lorentz [4] on sequence to sequence transformations of the form  $(Ax)_n = \sum_{k \geq 1} a_{nk}x_k$  gives necessary and sufficient condition for the sequence  $(Ax)_n$  to be defined and  $\sum_{k \geq 1} |(Ax)_k|$  convergent whenever  $\sum_{k \geq 1} |x_k|$  is convergent. Knopp and Lorentz proved that  $A$  is an  $\ell$ - $\ell$  matrix if and only if there is a number  $M$  such that for each  $K$ ,  $\sum_{n \geq 1} |a_{nk}| < M$ . Sunonchi and Tsuchikura [2] gives a similar result on function to function transformations of the form  $F(x) = \int_0^\infty G(x,y)f(y)dy$ , where  $x$  is a real variable and the kernel function  $G(x,y)$  is assumed to be a measurable function on the plane  $x \geq 0, y \geq 0$ . Then  $\int_0^\infty |F(x)|dx < \infty$  whenever  $\int_0^\infty |f(y)|dy < \infty$  if and only if  $L.u.b. \int_0^\infty |G(x,y)|dx < \infty$ . There are, however, nonmeasurable functions  $G(x,y)$  which define summable function  $F(x)$ . J. B. Tatchell has recently found the conditions for  $F(x)$  to be defined and  $F(x) \in L$  whenever

$f(y) \in L$ . We will first establish Theorem 1 and then use it in the proof of Theorem 2. Theorem 1 is an improvement of Tatchell's theorem.

## 2. NOTATION.

Although some of the symbols are the standard ones that are familiar to the reader, others are introduced here for the specific purpose of this paper.

The statement that  $f$  is integrable on  $[0, \infty)$  in some conditionally convergent sense means that for every  $u > 0$ ,  $f$  is integrable on  $[0, u]$  and that  $\int_0^u f(x) dx$  tends to a finite limit as  $u \rightarrow \infty$ .

$L$  - the space of functions that are Lebesgue integrable on  $[0, \infty)$  with norm  $\|f\| = \int_0^\infty |f(x)| dx$ .

$D$  - the first quadrant of the plane, i.e.,  $D = \{(x, y): x \geq 0, y \geq 0\}$ .

$G$  - an integral transformation,  $G: f \rightarrow F$ , of the form

$$(*) \quad F(x) = \int_0^\infty G(x, y) f(y) dy, \quad \text{for all } x \geq 0,$$

where  $f$  is defined on  $[0, \infty)$  and  $G(x, y)$  defined on  $D$ .

$L_G$  - the inverse image of  $L$  under the integral transformation  $G$  of the form  $(*)$ .

$G$  - the collection of all  $G$  of the form  $(*)$ .

$GL$  - the subcollection of  $G$  such that  $F \in L$  whenever  $f \in L$ , i.e.,  $GL = \{G \in G: L \subseteq L_G\}$ .

$L^\infty$  - the space of functions which are measurable and essentially bounded on  $[0, \infty)$  with norm

$$\|f\| = \text{ess - sup}_{x \geq 0} |f(x)|.$$

$Q$  - the set of nonnegative rational numbers.

## 3. MAIN THEOREM.

The first theorem is an improvement of Theorem 2 of J. B. Tatchell [1] and Theorem 1 of Sunonchi and Tsuchikura [2]. We will refer to that Theorem (T.S.T.). Next a lemma is used to justify an inversion in an order of integration in the proof of Theorem (T.S.T.).

LEMMA. If  $G(x, y)$  is a function of  $y$  summable on every finite interval in  $[0, \infty)$  whenever  $x \geq 0$ , and if

$$g(x, t) = \int_0^t G(x, y) dy$$

is a function of  $x$  measurable on  $[0, \infty)$  whenever  $t \geq 0$ , then

$$G_*(x, t) = \lim_{h \rightarrow 0} \inf \frac{1}{h} \int_t^{t+h} G(x, y) dy$$

is measurable on  $D = \{(x, t): x \geq 0, t \geq 0\}$ .

PROOF.  $g(x, t)$  is a continuous function of  $t$  whenever  $x \geq 0$ , and, by hypotheses, for each  $t \geq 0$   $g(x, t)$  is a measurable function of  $x$  on  $[0, \infty)$ . It follows from a theorem by Ursell [3] that  $g(x, t)$  is measurable on  $D = \{(x, t): x \geq 0, t \geq 0\}$ , and this is sufficient to ensure that  $G_*(x, t)$  is measurable on  $D$ .

THEOREM 1. (T.S.T.). Necessary and sufficient conditions for  $F(x) = \int_0^\infty G(x, y) f(y) dy$  to be defined and summable on  $[0, \infty)$  whenever  $f(y)$  is summable on  $[0, \infty)$  are

- i) for each  $x \geq 0$ ,  $G(x,y)$  is a function of  $y$  measurable and essentially bounded on  $[0, \infty)$ ;  
 ii) for each  $t \geq 0$ ,  $g(x,t) = \int_0^t G(x,y)dy$  is a function of  $x$  measurable on  $[0, \infty)$ ;  
 iii) there is a real number  $H$  such that for almost all  $t \geq 0$ ,

$$\int_0^\infty |G_*(x,t)| dx \leq H \quad ,$$

where

$$G_*(x,t) = \liminf_{h \rightarrow 0} \frac{1}{h} \int_t^{t+h} G(x,y)dy \quad .$$

PROOF. It follows from Theorem 2 of J. B. Tatchell [1] that i) and ii) are true since they are the same as the i) and ii) of the Theorem 2 of J. B. Tatchell [1].

We now prove that condition iii) is a necessary and sufficient condition for  $F(x) = \int_0^\infty G(x,y)f(y)dy$  summable on  $[0, \infty)$  whenever  $f(y)$  is summable on  $[0, \infty)$ . The preceding lemma shows us that  $G_*(x,t)$  is measurable on  $D$ . It follows from Theorem 1 of G. I. Sunonchi and T. Tsuchikura [2] that the transformation  $\int_0^\infty G_*(x,t)f(t)dt$  is defined and summable on  $[0, \infty)$  whenever  $f(y)$  is summable on  $[0, \infty)$  if and only if there is a  $H > 0$  such that for almost all  $t \geq 0$

$$\int_0^\infty |G_*(x,t)| dx \leq H \quad .$$

If  $x \geq 0$ , then  $G_*(x,t) = G(x,t)$  for almost all  $t \geq 0$  and so

$$F(x) = \int_0^\infty G(x,y)f(y)dy = \int_0^\infty G_*(x,y)f(y)dy \quad .$$

Therefore, for any  $f(y) \in L$  the transformation  $F(x) = \int_0^\infty G(x,y)f(y)dy$  is defined and  $F(x) \in L$  if and only if there is a  $H > 0$ , such that for almost all  $t \geq 0$

$$\int_0^\infty |G_*(x,t)| dx \leq H \quad .$$

The proof is completed. Next is the main theorem.

THEOREM 2. The collection  $G$  of all transformations of the form (\*) has no identity transformation on  $L$ ; i.e., there is no transformation  $G$  in  $G$  such that for every  $f \in L$

$$f(x) = \int_0^\infty G(x,y)f(y)dy$$

for almost all  $x \geq 0$ .

PROOF. Suppose that there is a  $\bar{G}(x,y)$  which defines an integral transformation  $G$  such that for every  $f \in L$

$$f(x) = \int_0^\infty \bar{G}(x,y)f(y)dy$$

for almost all  $x \geq 0$ . Then  $\bar{G} \in GL$ . It follows from Theorem (T.S.T.) that for each  $x \geq 0$ ,  $\bar{G}(x,y) \in L^\infty$ . Thus for any measurable set  $E$  with finite measure  $\int_E \bar{G}(x,y)dy < \infty$ . Hence for each  $x \in [0,1]$ ,  $\int_0^1 \bar{G}(x,y)dy < \infty$ . It follows from the absolute continuity of integrals that, given  $\epsilon = 1/2$ , there is a  $\delta_x > 0$ , such that for every measurable set  $e_x \subset [0,1]$  with  $m_{e_x} < \delta_x$

$$|\int_{e_x} \bar{G}(x,y)dy| < 1/2 \quad .$$

Now for each  $x \in [0,1]$ , we choose an interval  $[a,b] = e_x$ ,  $a \in Q \cap [0,1]$ ,  $b \in Q \cap [0,1]$ , containing  $y_0(=x)$  and  $0 < b - a < \delta_x$  so that

$$|\int_a^b \bar{G}(x,y)dy| < 1/2 \quad .$$

Let  $F = \{e_x : x \in [0,1]\}$ ,  
 $H_1 = \{[0,a] : [a,b] \in F\}$ ,  
 $H_2 = \{[b,1] : [a,b] \in F\}$ ,  
 $H = H_1 \cup H_2$   
 and  $\chi_\beta(y) = \begin{cases} 1, & \text{if } y \in \beta \\ 0, & \text{otherwise.} \end{cases}$

Hence for each  $\beta \in H$ ,  $\chi_\beta(y) \in L$  and

$$\begin{aligned} \chi_\beta(x) &= \int_0^\infty \bar{G}(x,y) \chi_\beta(y) dy \\ &= \int_\beta \bar{G}(x,y) dy \end{aligned}$$

for all  $x \geq 0$ , except a set  $E_\beta \subset [0,\infty)$  with  $mE_\beta = 0$ . Since  $H$  is a countable set so  $m \sum E_\beta = 0$ . Let  $K = [0,1] / \sum E_\beta$ , then  $mK = 1 - m \sum E_\beta = 1$ . Therefore for each  $x \in K$ ,  $\chi_\beta(x) = \int_0^\infty \bar{G}(x,y) \chi_\beta(y) dy$

$$= \int_\beta \bar{G}(x,y) dy$$

for all  $\beta \in H$ . It follows that for each  $x \in K \subset [0,1]$ , there is a measurable set  $e_x = [a,b] \in F$  with  $m_{e_x} < \delta_x$  so that

$$\left| \int_{e_x} \bar{G}(x,y) dy \right| < 1/2,$$

and there are  $[0,a] \in H$  and  $[b,1] \in H$  so that

$$\begin{aligned} \left| \int_0^1 \bar{G}(x,y) dy \right| &= \left| \int_0^a \bar{G}(x,y) dy + \int_a^b \bar{G}(x,y) dy + \int_b^1 \bar{G}(x,y) dy \right| \\ &\leq \left| \int_0^a \bar{G}(x,y) dy \right| + \left| \int_a^b \bar{G}(x,y) dy \right| + \left| \int_b^1 \bar{G}(x,y) dy \right| \\ &= |\chi_{[0,a]}(x)| + \left| \int_{e_x} \bar{G}(x,y) dy \right| + |\chi_{[b,1]}(x)| \\ &= 0 + \left| \int_{e_x} \bar{G}(x,y) dy \right| + 0 \\ &< 1/2. \end{aligned}$$

Thus for each  $x \in K \subset [0,1]$ ,  $mK = 1$  and

$$\begin{aligned} |\chi_{[0,1]}(x)| &= \left| \int_0^\infty \bar{G}(x,y) \chi_{[0,1]}(y) dy \right| \\ &= \left| \int_0^1 \bar{G}(x,y) dy \right| \\ &< 1/2. \end{aligned}$$

This is a contradiction which completes the proof.

COROLLARY. If  $f, g$  are measurable functions on  $[0,\infty)$  such that  $f(y)g(x-y) \in L$ ,  $x \in [0,\infty)$ , the convolution  $f * g$  of  $f$  and  $g$  at point  $x$  is defined by

$$(f * g)(x) = \int_0^\infty f(y)g(x-y) dy.$$

$(L,*)$  is a Banach algebra [5]. Then Banach algebra  $(L,*)$  has no unit element, i.e. there is no  $g \in L$  such that  $f * g = g * f = f$ , for all  $f \in L$ .

PROOF. It is clear this is a special case of preceding theorem where  $G(x,y) = g(x-y) \in L$ .

REMARK: This corollary is a well-known theorem, see [5], here is a new proof.

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