

**DYNAMICAL PROPERTIES OF MAPS DERIVED FROM MAPS
WITH STRONG NEGATIVE SCHWARZIAN DERIVATIVE**

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ABSTRACT. A strong Schwarzian derivative is defined, and it is shown that the convolution of a function with a map from an interval into itself having negative strong Schwarzian derivative is a function with negative Schwarzian derivative. Such convolutions have 0 as a stable periodic point and at most one other stable periodic orbit in the interior of the domain.

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1. INTRODUCTION.

Let $f: [0,1] \rightarrow [0,1]$ be a C^3 map, i.e., it has 3 continuous derivatives. The Schwarzian derivative at a point x is given by

$$(f, x) = \frac{f'''(x)}{f'(x)} - \frac{3(f''(x))^2}{2(f'(x))^3}. \quad (1.1)$$

This derivative was first formulated by H.A. Schwarz and has been used in the theory of differential equations [1]. Recently, it has found important application in the study of bifurcation of periodic orbits [2]. In [3,4], the Schwarzian derivative is used to study the limiting behaviour of dynamical systems.

The main result of [2] is

THEOREM 1. Let $f: [0,1] \rightarrow [0,1]$ be a C^3 map and let it satisfy

(i) $f(0) = f(1) = 0$

(ii) f has a unique critical point c in $(0,1)$

and (iii) $(f, x) < 0$ for all $x \in [0,1] - c$.

Then f has at most one stable orbit in $(0,1)$. If it exists it is the ω -limit set of c .

Note that the wording of Theorem 1 allows for the possibility that 0 is also

a stable fixed point of f . Indeed, Theorem 1 holds even if the slope at 0 (and at 1) is equal to 0. Furthermore, the requirement that $f'''(x)$ is continuous at c can be relaxed. In [4, p. 100] it is shown that Theorem 1 is true even if only $f'(x)$ is continuous at c . We collect these observations in a version of Theorem 1 which we shall need in the sequel.

THEOREM 2. Let $f: [0,1] \rightarrow [0,1]$ be a C^3 map everywhere in $(0,1)$ except possibly at c where it is C^1 . Assume conditions (i), (ii) and (iii) of Theorem 1. Then f has at most one stable periodic orbit in $(0,1)$. If it exists it is the ω -limit set of c .

In this note we define a strong Schwarzian derivative and show that the convolution of a function with a map having negative Schwarzian derivative is a function with negative Schwarzian derivative.

In practice one is concerned with the structural stability of the map f , i.e. with what happens to the dynamical properties of f once it is perturbed. Since $\{f, x\} < 0$ is an open condition, it is clear that for maps which are C^3 and close to f , this condition will be retained. Thus, for small, smooth perturbations of f , the negative Schwarzian property is not destroyed. However, for any given perturbation, the Schwarzian derivative must be computed to verify that $\{f, x\} < 0$. In this note we consider a class of large perturbations of f derived by convoluting f with a known function, g . We define a new derivative of f called the strong Schwarzian derivative, denoted by Sf . The main result of this note shows that if $Sf < 0$, then the map $F = f * g$ has negative Schwarzian derivative. Thus, we can draw dynamical conclusions about maps which are large perturbations of f .

2. A SIMPLE LEMMA.

Let $(Sf)(x) = f'''(x)f'(x) - \frac{3}{2}(f''(x))^2$. We define the strong Schwarzian derivative $\bar{S}f$ at points a and b in $[0,1]$, $a < b$, by

$$(\bar{S}f)(a,b) = f'(a)f'''(b) - \frac{3}{2}f''(a)f''(b). \tag{2.1}$$

Clearly, if $(\bar{S}f)(a,b) < 0$ for all $a < b$, then $(Sf)(x) < 0$ for all x in $[0,1]$.

LEMMA 1. Let $f: [0,1] \rightarrow [0,1]$ be a unimodal map with 3 continuous derivatives such that $f'(x) > 0$ on $[0,c)$ and $f'(x) < 0$ on $(c,1]$. Furthermore, assume:

- 1) $(Sf)(x) < 0$
- 2) $f''(x) < 0$, $f'''(x) \geq 0$, $f^{IV}(x) \leq 0$ for all $x \in (0,1)$.

Then $(\bar{S}f)(a,b) < 0$ for all $a, b \in (0,1)$.

PROOF. Let $a < b$. Since $(Sf)(a) < 0$ and $f'''(a) > 0$

$$f'(a) < \frac{3}{2} \frac{[f''(a)]^2}{f'''(a)}.$$

Then

$$f'(a)f'''(b) < \frac{3}{2} f''(a)f''(b) \frac{f''(a)}{f''(b)} \frac{f'''(b)}{f'''(a)}. \tag{2.2}$$

Since $f'''(a) \geq 0$, $f''(a) \leq f''(b)$, and since $f^{IV}(x) \leq 0$, $f'''(a) \leq f'''(b)$.

Hence

$$f'(a)f'''(b) < \frac{3}{2} f''(a)f''(b), \tag{2.3}$$

i.e., $(\bar{S}f)(a,b)$ for all $a < b$, $a \in (0,1)$.

To complete the proof, we must show that

$$f'''(a)f'(b) < \frac{3}{2} f''(a)f''(b), \tag{2.4}$$

where $a < b$. There are two cases.

CASE 1. $f'(b) > 0$. Since $a < b$, $f'(a) > 0$. Then

$$f'''(a) < \frac{3}{2} \frac{[f''(a)]^2}{f'(a)}$$

and

$$f'(b)f'''(a) < \frac{3}{2} [f''(a)f''(b)] \frac{f''(a)}{f''(b)} \frac{f'(b)}{f'(a)}. \tag{2.5}$$

Since $f'''(x) \geq 0$, $\frac{f''(a)}{f''(b)} \leq 1$, and since $f''(x) < 0$, $\frac{f'(b)}{f'(a)} < 1$. Hence the inequality (2.4) holds.

CASE 2. $f'(b) < 0$. Since $f'''(a) > 0$,

$$f'(b)f'''(a) - \frac{3}{2} f''(a)f''(b) < 0. \tag{2.6}$$

This completes the proof.

Q.E.D.

EXAMPLE 1. $f(x) = rx(1-x)$, $0 \leq r \leq 4$. Since $f_r'''(x) = 0$, and $f_r''(x) < 0$, $\bar{S}f_r(a,b) < 0$ for all (a,b) in $(0,1)$.

EXAMPLE 2. $f_\alpha(x) = xe^{-2x}$, $\alpha > 0$, defined on $[0, \frac{2}{\alpha}]$, where the critical point $c = \frac{1}{\alpha}$ and the point of inflection is at $x = 2/\alpha$. It is easy to verify that $f_\alpha(x)$ satisfies all the conditions of Lemma 1.

3. MAPS DEFINED BY CONVOLUTION.

Let $f: [0,1] \rightarrow [0,1]$ satisfy $f(0) = f(1) = 0$ and let $g: [0,1] \rightarrow [0, \infty)$ be a map such that $g(x) \geq 0$ and

$$\int_0^1 g(x) dx \leq 2. \tag{2.7}$$

We extend both f and g to $(-\infty, \infty)$ by letting $f(x) = g(x) = 0$ outside of $[0,1]$ and use the same symbols to denote these extended functions on $(-\infty, \infty)$. The convolution of f and g is given by

$$F(x) = \int_{-\infty}^{\infty} g(x-t)f(t)dt = g * f(x). \tag{2.8}$$

The support of F is $[0,2]$ and (2.7) guarantees that the range of F is contained in $[0,2]$. Hence $F: [0,2] \rightarrow [0,2]$ is a well-defined map.

LEMMA 2. Let $f: [0,1] \rightarrow [0,1]$, $f(0) = f(1) = 0$, be in C^3 and assume $(\bar{S}f)(a,b) < 0$ for all points a,b in $[0,1]$. Let $g: [0,1] \rightarrow [0, \infty)$ be in C^3 , let (2.7) be satisfied, and assume $g(0^+)$ and $g(1^-)$ exist. Then $F(x)$ has a continuous third derivative everywhere except possibly at $x = 1$, where at least $F'(x)$ is continuous, and $F(x)$ has negative Schwarzian derivative for all $x \in (0,2)$.

PROOF. Notice that in (2.8), $0 \leq x-t \leq 1$ and $0 \leq t \leq 1$. Hence for $0 \leq x < 1$, $0 \leq t \leq x$, and we have

$$F(x) = \int_0^x g(x-t)f(t)dt. \quad (2.9)$$

Similarly, for $1 < x \leq 2$,

$$F(x) = \int_{x-1}^x g(x-t)f(t)dt. \quad (2.10)$$

Using Leibnitz's Rule, we obtain

$$F'(x) = \int_0^x g'(x-t)f(t)dt + g(0^+)f(x), \quad 0 \leq x < 1 \quad (2.11)$$

and

$$F'(x) = \int_{x-1}^x g'(x-t)f(t)dt + g(0^+)f(x) - g(1^-)f(x-1), \quad (2.12)$$

where $1 < x \leq 2$. Now $F'(1^-) = F'(1^+)$ if

$$g(0^+)f(1^-) = g(0^+)f(1^+) - g(1^-)f(0^+), \quad (2.13)$$

which is so since $f(0) = f(1) = 0$. Hence $F'(x)$ is continuous on $(0,2)$. That $F'''(x)$ is continuous on $(0,2)$, except possibly at $x = 1$, follows from the fact that $f, g \in C^3$.

To prove $(SF)(x) < 0$, we differentiate (2.8) to get

$$F'(x) = \int_{-\infty}^{\infty} g'(x-t)f(t)dt. \quad (2.14)$$

Integrating by parts,

$$\begin{aligned} F'(x) &= -g(x-t)f(t) \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} g(x-t)f'(t)dt \\ &= \int_{-\infty}^{\infty} g(x-t)f'(t)dt. \end{aligned} \quad (2.15)$$

Similarly,

$$F''(x) = \int_{-\infty}^{\infty} g(x-t)f''(t)dt \quad (2.16)$$

and

$$F'''(x) = \int_{-\infty}^{\infty} g(x-t)f'''(t)dt. \quad (2.17)$$

Thus,

$$\begin{aligned} (SF)(x) &= \int_{-\infty}^{\infty} g(x-t)f'(t)dt \int_{-\infty}^{\infty} g(x-y)f'''(y)dy \\ &\quad - \frac{3}{2} \int_{-\infty}^{\infty} g(x-t)f''(t)dt \int_{-\infty}^{\infty} g(x-y)f''(y)dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x-t)g(x-y) \left[f'(t)f'''(y) - \frac{3}{2}f''(t)f''(y) \right] dt dy. \end{aligned} \quad (2.18)$$

Since $g(\xi) \geq 0$ and $(\bar{S}f)(a,b) < 0$ for all $a,b \in (0,1)$ by Lemma 1, we have $(SF)(x) < 0$ and hence $\{F,x\} < 0$ for all $x \in (0,2)$. Q.E.D.

Note that if f and g can be extended smoothly at the endpoints 0 and 1, then boundary conditions can be derived which will ensure that $F \in C^3$ everywhere. Under such conditions it would be possible to avoid $f(0) = 0, f(1) = 0$.

From (2.9) we see that $F(0) = 0$ and from (2.11) it follows that $F'(0^+) = g(0^+)f(0) < 1$. Hence 0 is a stable fixed point.

THEOREM 3. Let f and g be as in Lemma 2 and assume that f is a unimodal map, strictly increasing on $[0,c)$ and strictly decreasing on $(c,1]$. Then $F(x) = g * f(x)$ has 0 as a stable periodic point and at most one other stable periodic orbit in $(0,2)$.

PROOF. $F(x)$ is unimodal on $[0,2]$ with critical point $\bar{c} = 1$, since f is unimodal on $[0,1]$. By Lemma 2, $F'''(x)$ is continuous everywhere except possibly at 1, where $F'(x)$ is continuous, and $\{F,x\} < 0$. Clearly 0 is a stable periodic point. Applying Theorem 2 we obtain the desired result. Q.E.D.

Since $F'(0^+) = 0$ there exists a point $0 < a < 1$, such that $F(a) = a$. Let $b = F^{-1}(a) \cap (1,2)$. Then clearly $[0,2) \cup (b,1]$ is in the domain of attraction of a .

COROLLARY. If $F(c) > b$, then F has 0 as a stable periodic point and no other stable periodic orbits.

PROOF. If F has a stable periodic orbit in $(0,2)$, Theorem 2 implies that it is the ω -limit set of $c = 1$. Here, clearly, the ω -limit set is equal to $\{0\}$. Q.E.D.

EXAMPLE 1. Let $f(x) = rx(1-x)$ and $g(x) = k$. Then, for $0 \leq x \leq 1$,

$$\begin{aligned}
 F(x) &= (g * f)(x) = k \int_0^x rt(1-t)dt \\
 &= kr x^2 \left(\frac{1}{2} - \frac{x}{3} \right). \tag{2.19}
 \end{aligned}$$

By symmetry, we have

$$F(x) = F(2-x), \quad 1 \leq x \leq 2. \tag{2.20}$$

Thus,

$$F(x) = \begin{cases} kr x^2 \left(\frac{1}{2} - \frac{x}{3} \right), & 0 \leq x \leq 1 \\ kr (x-2)^2 \left(\frac{x}{3} - \frac{1}{6} \right), & 1 \leq x \leq 2 \end{cases}. \tag{2.21}$$

For $F: [0,2] \rightarrow [0,2]$ to be well-defined we need

$$F(1) = \frac{kr}{6} \leq 2. \tag{2.22}$$

By Theorem 3, we know that $F(x)$ can have at most one stable periodic orbit in $(0,2)$.

For $kr = 6$, the point $c = 1$ is a stable fixed point.

Note that in this example $F'''(x)$ is not continuous at 1, but $F'(x)$ is.

EXAMPLE 2. Let $f(x) = rx(1-x)$ on $[0,1]$, $0 \leq r \leq 4$, and $g(x) = px(1-x)$

on $[0,1]$. Then $F(x) = g \circ f(x)$ is given by

$$F(x) = \begin{cases} \frac{rp}{6} (x^3 - x^4 + \frac{x^5}{5}) & \text{on } [0,1] \\ F(2-x) & \text{on } [1,2] \end{cases} \quad (2.23)$$

For F to be well-defined, we require

$$F(1) = \frac{rp}{30} \leq 2,$$

i.e., $rp \leq 60$. For $rp = 42$, $F(x) = x$ at $x_0 = .51$. Hence, by the symmetry of F , $[0, .51) \cup (1.49, 2.00]$ is in the domain of attraction of 0 . Since $F(1) = 1.4$, $c = 1$ is not in the domain of attraction of 0 , and hence it is possible for a stable periodic orbit to exist in $(.51, 1.49)$. However, for $rp = 48$, $[0, .46) \cup (1.56, 2.00]$ is in the domain of attraction of 0 . Since $F(1) = 1.6$, $c = 1$ is in the domain of attraction of 0 . Hence, by Theorem 3, 0 is the only stable periodic orbit. The same is true for $48 \leq rp \leq 60$.

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REFERENCES

- [1] HILLE, E. Lectures on Ordinary Differential Equations, Addison-Wesley, Reading, Mass., 1968.
- [2] SINGER, D. Stable Orbits and Bifurcations of Maps of the Interval, SIAM J. Appl. Math., Vol. 35, No. 2 (1978), 260-267.
- [3] GUCKHENHEIMER, J. Sensitive Dependence to Initial Conditions for One Dimensional Maps, Comm. Math. Phys., Vol. 70, No. 20 (1979), 133-160.
- [4] COLLET, P. and ECKMANN, J-P. Iterated Maps on the Interval as Dynamical Systems, Birkhauser, Boston, Mass., 1980.



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