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ON THE ASYMPTOTIC SERIES SOLUTION OF SOME HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS AT TURNING POINTS

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ABSTRACT. The Lagrange manifold (WKB) formalism enables the determination of the asymptotic series solution of second-order "wave type" differential equations at turning points. The formalism also applies to higher order linear differential equations, as we make explicit here illustrating with some 4th order equations of physical significance.

KEYWORDS AND PHRASES: *Higher order linear differential equations, Lagrange manifold, turning points, WKB.*
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1. INTRODUCTION.

While the classical WKB technique was developed for second order differential equations, it has long been applied to higher, especially 4th, order differential equations as well [1]. As might be expected, problems analogous to those encountered at turning points of second order differential equations occur in the higher order equations. The Lagrange manifold technique of Maslov [2] and Arnold [3] determines the first term in the asymptotic series solution of linear partial differential equations near turning points. Through a slight modification of their technique, the full asymptotic series solution for second order "wave-type" equations can be obtained [4]. The Lagrange manifold formalism also applies to higher order linear differential equations, as we make explicit here illustrating with some 4th order equations of general interest.

2. BASIC FORMALISM.

We consider the differential equation

$$a_o(x)\,\frac{\partial^4\psi}{\partial x^4} + a_1(x)\,\frac{\partial^3\psi}{\partial x^3} + a_2(x)\,\frac{\partial^2\psi}{\partial x^2} + a_3(x)\,\frac{\partial\psi}{\partial x} + a_4(x)\psi + a_5(x)\,\frac{\partial^2\psi}{\partial t^2} = 0\ , \qquad (2.1)$$

where, for definiteness, x is a spatial variable, t is the time and the $a_i(x)$'s are analytic. With appropriate choice of coefficients and a harmonic time dependence, Equation (2.1) can be seen to include some physically significant equations as special cases. We assume Equation (2.1) has an asymptotic solution - near turning points of the highest order - of the form

$$\psi(x) - \exp\{i\tau^2 t\} \int A(x,p,\tau) \exp\{i\tau(xp-S(p))\}dp = O(\tau^{-\infty}) , \tag{2.2}$$

where $A(x,p,\tau)$ and its derivatives are bounded, p may be regarded as a momentum, τ is a large parameter and the stationary phase condition $[\frac{d}{dp}(xp-S(p))=0]$ determines the Lagrange manifold of Maslov near the turning point [5].

Following the procedure developed for second-order equations, the differentiation in Equation (2.1) is carried across the integral in Equation (2.2), obtaining

$$\int dp \exp\{i\tau(xp-S(p))\}\{(i\tau)^4[a_o(x)p^4-a_5(x)]A + (i\tau)^3[4a_o(x)p^3 \frac{\partial A}{\partial x} + a_1(x)p^3A]$$

$$+ (i\tau)^2[6a_o(x)p^2 \frac{\partial^2 A}{\partial x^2} + 3a_1(x)p^2 \frac{\partial A}{\partial x} + a_2(x)p^2A] + (i\tau)[4a_o(x)p \frac{\partial^3 A}{\partial x^3} + 3a_1(x)p \frac{\partial^2 A}{\partial x^2}$$

$$+ 2a_2(x)p \frac{\partial A}{\partial x} + a_3(x)pA] + [a_o(x) \frac{\partial^4 A}{\partial x^4} + a_1(x) \frac{\partial^3 A}{\partial x^3} + a_2(x) \frac{\partial^2 A}{\partial x^2} + a_3(x) \frac{\partial A}{\partial x} + a_4(x)A]\}$$

$$= O(\tau^{-\infty}) . \tag{2.3}$$

Maslov's Hamiltonian is the coefficient of the highest order $(i\tau)$ term, here $(i\tau)^4$,

$$H(x,p) = a_o(x)p^4 - a_5(x). \tag{2.4}$$

Then by invoking the stationary phase condition Maslov's Hamiltonian becomes an eikonal equation on the Lagrange manifold,

$$a_o(\frac{dS}{dp})p^4 - a_5(\frac{dS}{dp}) = 0 .$$

To obtain a transport equation, the Hamiltonian is expanded near the Lagrange manifold

$$a_o(x)p^4 - a_5(x) = a_o(\frac{dS}{dp})p^4 - a_5(\frac{dS}{dp}) + (x- \frac{dS}{dp})D = (x- \frac{dS}{dp})D \tag{2.5}$$

where

$$D = \int_0^1 \frac{\partial}{\partial x} H(t(x- \frac{dS}{dp}) + \frac{dS}{dp},p)\ dt,$$

i.e, the remainder of the Taylor series less a factor of $(x- \frac{dS}{dp})$. By substituting Equation (2.5) into Equation (2.3), noting

$$\int dp\frac{d}{dp}[\exp\{i\tau(xp-S(p))\}AD] = \int dp \exp\{i\tau(xp-S(p))\}\{i\tau A(x - \frac{dS}{dp})D + D\frac{\partial A}{\partial p} + A\frac{\partial D}{\partial p}\} \tag{2.6}$$

and taking the surface integral over a sufficiently large radius that it vanishes, Equation (2.3) becomes

$$\int dp\ \exp\{i\tau(xp-S(p))\}(i\tau)^3[(4a_o p^3 \frac{\partial A}{\partial x} + a_1 p^3 A - A\frac{\partial D}{\partial p} - D\frac{\partial A}{\partial p} +$$

$$(i\tau)^{-1}(6a_o p^2 \frac{\partial^2 A}{\partial x^2} + 3a_1 p^2 \frac{\partial A}{\partial x} + a_2 p^2 A) + (i\tau)^{-2}(4a_o p \frac{\partial^3 A}{\partial x^3} + 3a_1 p \frac{\partial^2 A}{\partial x^2} + 2a_2 p \frac{\partial A}{\partial x} + a_3 pA)$$

$$+ (i\tau)^{-3}(a_o \frac{\partial^4 A}{\partial x^4} + a_1 \frac{\partial^3 A}{\partial x^3} + a_2 \frac{\partial^2 A}{\partial x^2} + a_3 \frac{\partial A}{\partial x} + a_4 A)] = O(\tau^{-\infty}) , \tag{2.7}$$

where, for clarity, the arguments of the $a_i(x)$'s have been deleted. Then requiring

$$4a_o p^3 \frac{\partial A}{\partial x} + a_1 p^3 A - A \frac{\partial D}{\partial p} - D \frac{\partial A}{\partial p} + (i\tau)^{-1}(6a_o p^2 \frac{\partial^2 A}{\partial x^2} + 3a_1 p^2 \frac{\partial A}{\partial x} + a_2 p^2 A) +$$

$$(i\tau)^{-2}(4a_o p\frac{\partial^3 A}{\partial x^3} + 3a_1 p\frac{\partial^2 A}{\partial x^2} + 2a_2 p\frac{\partial A}{\partial x} + a_3 pA) + (i\tau)^{-3}(a_o \frac{\partial^4 A}{\partial x^4} + a_1 \frac{\partial^3 A}{\partial x^3} + a_2 \frac{\partial^2 A}{\partial x^2}$$

$$+ a_3 \frac{\partial A}{\partial x} + a_4 A) = 0 \tag{2.8}$$

in a neighborhood of the Lagrange manifold leads to a transport equation if we introduce the non-Hamiltonian flow

$$\dot{x} = 4a_o(x)p^3 \frac{\partial A}{\partial x} \qquad \dot{p} = -D(x,p) \tag{2.9}$$

where the dots indicate time derivatives. That is, Equation (2.8) holds in a neighborhood of the Lagrange manifold if we allow the asymptotic series

$$A(x,p,\tau) = \sum_{k=0}^{\infty} A_k(x,p)(i\tau)^{-k}$$

to evolve along the transport equation

$$\frac{dA_k}{dt} + [a_1(x)p^3 - \frac{\partial D}{\partial p}]A_k + [6a_o(x)p^2 \frac{\partial^2}{\partial x^2} + 3a_1(x)p^2 \frac{\partial}{\partial x} + a_2(x)p^2]A_{k-1}$$

$$+ [4a_o(x)p\frac{\partial^3}{\partial x^3} + 3a_1(x)p \frac{\partial^2}{\partial x^2} + 2a_2(x)p \frac{\partial}{\partial x} + a_3(x)p]A_{k-2}$$

$$+ [a_o(x) \frac{\partial^4}{\partial x^4} + a_1(x) \frac{\partial^3}{\partial x^3} + a_2(x) \frac{\partial^2}{\partial x^2} + a_3(x) \frac{\partial}{\partial x} + a_4(x)]A_{k-3} = 0. \tag{2.10}$$

3. ODE APPLICATION

As a specific illustration, we consider the reference equations studied by Langer, Wasow and Lin and Rabenstein [6] respectively, in the transition from laminar to turbulent flow

$$\frac{d^4\Psi}{dx^4} + \tau^2(x \frac{d^2\Psi}{dx^2} + \alpha \frac{d\Psi}{dx}) = 0 \tag{3.1}$$

$$\frac{d^4\Psi}{dx^4} + \tau^2(x \frac{d^2\Psi}{dx^2} + \alpha\Psi) = 0 \tag{3.2}$$

$$\frac{d^4\Psi}{dx^4} + \tau^2(x\,\frac{d^2\Psi}{dx^2} + \alpha\frac{d\Psi}{dx} + \beta\Psi) = 0 \tag{3.3}$$

with α,β constants. Applying the above technique to these equations leads to the same Hamiltonian and flow (Equations (2.4) and (2.9)), explicitly

$$H(x.p) = p^4 - x$$

$$x = 2[(t+p_o)^2-1] \qquad p = t+p_o\ ,$$

but different transport equations

$$\frac{dA_k}{dt} - [p\alpha + \frac{\partial D}{\partial p}]\,A_k + [(6p^2-x)\,\frac{d^2}{dx^2} - \alpha\,\frac{d}{dx}]A_{k-1} + 4p\,\frac{d^3A_{k-2}}{dx^3} + \frac{d^4A_{k-3}}{dx^4} = 0 \tag{3.4}$$

$$\frac{dA_k}{dt} - \frac{\partial D}{\partial p}\,A_k + [(6p^2-x)\,\frac{d^2}{dx^2} - \alpha]A_{k-1} + 4p\,\frac{d^3A_{k-2}}{dx^3} + \frac{d^4A_{k-3}}{dx^4} = 0 \tag{3.5}$$

$$\frac{dA_k}{dt} - [p\alpha + \frac{\partial D}{\partial p}]\,A_k + [(6p^2-x)\,\frac{d^2}{dx^2} -\alpha\,\frac{d}{dx} -\beta]A_{k-1} + 4p\,\frac{d^3A_{k-2}}{dx^3} + \frac{d^4A_{k-3}}{dx^4} = 0. \tag{3.6}$$

Although these transport equations may have many different solutions depending on the Lagrange manifold at t=0, one interesting set of solutions is given by A_k = constant for all k leading to an asymptotic solution of the form

$$\Psi(x) = \int \exp\{i\tau(xp + \frac{1}{5}\,p^5)\}\tilde{A}(p)\,dp$$

where $\tilde{A}(p)$ is any smooth function with compact support.

As a second example we consider the dynamical equation for the inhomogeneous Euler-Bernoulli beam

$$\rho(x)a(x)\,\frac{\partial^2\psi}{\partial t^2} + \frac{\partial^2}{\partial x^2}\,\{EJ(x)\,\frac{\partial^2\psi}{\partial x^2}\} = 0, \tag{3.7}$$

whose WKB solution, away from turning points, has been detailed by Pierce [7]. In Equation (3.7), ψ represents the transverse deflection of the beam, $\rho(x)$ is the density, $a(x)$ is the cross-sectional area and $EJ(x)$ the bending modulus. Assuming a solution of the form given in Equation (2.2) leads to the Hamiltonian, flow and transport equations, respectively

$$H = EJ(x)p^4 - \rho(x)a(x) \tag{3.8}$$

$$\dot{x} = 4p^3EJ(x) \qquad \dot{p} = -D(x,p) \tag{3.9}$$

$$\frac{dA_k}{dt} + \left[2p^3\,\frac{d}{dx}\,\{EJ\} - \frac{\partial D}{\partial p}\right]\,A_k +$$

$$\left[6p^2EJ(x)\,\frac{d^2}{dx^2} + 6p^2\left(\frac{d}{dx}\,\{EJ\}\right)\,\frac{d^2}{dx^2} + 2p^2\left(\frac{d^2}{dx^2}\,\{EJ\}\right)\frac{d}{dx}\right]\,A_{k-1} +$$

$$\left[4pEJ(x)\frac{d^3}{dx^3} + 6p\left(\frac{d}{dx}\{EJ\}\right)\frac{d^2}{dx^2} + 2p^2\left(\frac{d^2}{dx^2}\{EJ\}\right)\frac{d}{dx}\right] A_{k-2} + \tag{3.10}$$

$$\left[EJ(x)\frac{d^4}{dx^4} + 2\left(\frac{d}{dx}\{EJ\}\right)\frac{d^3}{dx^3} + \left(\frac{d^2}{dx^2}\{EJ\}\right)\frac{d^2}{dx^2}\right] A_{k-3} = 0$$

For completeness, assuming the analyticity near the turning point of the derivative coefficients in Equation (3.7), the phase may be determined by noting

$$x = f^{-1}(p^4) = \frac{dS}{dp}$$

$$S(p) = \int_{p_o}^{p} f^{-1}(u^4)du$$

$$\phi(x,p) = xp - S(p)$$

where $f(x) = \rho(x)a(x) - EJ(x)$ [8]. With the phase above and the A_k from Equation (3.10) the integrals

$$\int A_k(x,p)\exp\{i\tau\phi(x,p)\}dp$$

can now be evaluated asymptotically using standard techniques [9].

4. PDE APPLICATION

The application of the classical WKB technique to some physically significant 4th order partial differential equations in two spatial variables and one time variable (in which case the approach is often referred to as the geometrical optics formalism) has been considered by Krasil´nikov [10] and Germogenova [11] as well as Pierce. To apply the Lagrange manifold technique to the most general such 4th order equation is quite cumbersome. So we restrict our attention to the dynamical equation for the transverse motion of an inhomogeneous thin Euler-Bernoulli plate, specifically considered by both Pierce and Germogenova

$$Z(\bar{r})\left(\frac{\partial^4\psi}{\partial x^4} + \frac{\partial^4\psi}{\partial y^4} + 2\frac{\partial^4\psi}{\partial x^2\partial y^2}\right) + Z_1(\bar{r})\left(\frac{\partial^3\psi}{\partial x^3} + \frac{\partial^3\psi}{\partial y^2\partial x}\right) + Z_2(\bar{r})\left(\frac{\partial^3\psi}{\partial y^3} + \frac{\partial^3\psi}{\partial x^2\partial y}\right) +$$

$$\tilde{Z}(\bar{r})\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) + (1-\sigma)\left(\hat{Z}(\bar{r})\frac{\partial^2\psi}{\partial x\partial y} - Z_3(\bar{r})\frac{\partial^2\psi}{\partial y^2} - Z_4(\bar{r})\frac{\partial^2\psi}{\partial x^2}\right) + \rho_S(\bar{r})\frac{\partial^2\psi}{\partial t^2} = 0. \tag{4.1}$$

In Equation (4.1), $\bar{r} = (x,y)$, $Z(\bar{r})$ is the stiffness of the plate, $Z_1 = 2\frac{\partial Z}{\partial x}$, $Z_2 = 2\frac{\partial Z}{\partial y}$, $\tilde{Z} = \Delta Z$, $\hat{Z} = \frac{\partial^2 Z}{\partial x\partial y}$, $Z_3 = \frac{\partial^2 Z}{\partial x^2}$, $Z_4 = \frac{\partial^2 Z}{\partial y^2}$, σ is the Poisson ratio and $\rho_S(\bar{r})$ is the surface density of the plate. Analogous to the solution assumed in Equation (2.2), here we assume a solution of the form

$$\psi(\bar{r}) - \exp\{i\tau^2 t\}\int A(\bar{r},\bar{p},\tau)\exp\{i\tau(\bar{r}\cdot\bar{p}-S(\bar{p}))\}d\bar{p} = O(\tau^{-\infty}) \tag{4.2}$$

where $\bar{p} = (p_x,p_y)$. Carrying the differentiation in Equation (4.1) across the integral in Equation (4.2) - and temporarily, for clarity, retaining only the $(i\tau)^4$ and $(i\tau)^3$ terms - we have

$$\int d\bar{p}\ \exp\{i\tau(\bar{r}\cdot\bar{p}-S(\bar{p}))\}[(i\tau)^4 Z(\bar{r})(p_x^4+p_y^4+2p_x^2p_y^2) - \rho_S(\bar{r})\}A + \tag{4.3}$$

$$(i\tau)^3\{4Z(\bar{r})(p_x(p_x^2+p_y^2)\frac{\partial A}{\partial x} + p_y(p_x^2+p_y^2)\frac{\partial A}{\partial y} + (Z_1(\bar{r})p_x(p_x^2+p_y^2) + Z_2(\bar{r})p_y(p_x^2+p_y^2))A\}] = O(\tau^{-2}).$$

As above the Hamiltonian is the coefficient of $(i\tau)^4$

$$H(\bar{r},\bar{p}) = Z(\bar{r})(p_x^4 + p_y^4 + 2p_x^2p_y^2) - \rho_S(\bar{r}) \tag{4.4}$$

Then using the same device as in Equations (2.3) and (2.4), Equation (4.3) becomes

$$\int d\bar{p}\ \exp\{i\tau(\bar{r}\cdot\bar{p}-S(\bar{p}))\}[(i\tau)^3\ 4Z(\bar{r})\{p_x(p_x^2+p_y^2)\ \frac{\partial A}{\partial x} + p_y(p_x^2+p_y^2)\ \frac{\partial A}{\partial y} - \bar{D}\cdot\nabla_p A$$

$$+ A\{Z_1(r)p_x(p_x^2+p_y^2) + Z_2(r)p_y(p_x^2+p_y^2) - \nabla_p\cdot D\} = O(\tau^{-2}), \tag{4.5}$$

where

$$\bar{D}(\bar{r},\bar{p}) = \int_0^1 \nabla_r\ H(t(\bar{r} - \nabla_p S(\bar{p})) + \nabla_p S(\bar{p}),\bar{p})\ dt.$$

Then introducing the flow

$$\dot{x} = 4Z(\bar{r})p_x(p_x^2 + p_y^2) \qquad \dot{y} = 4Z(\bar{r})p_y(p_x^2 + p_y^2)$$

$$\dot{p}_x = -D_x \qquad \dot{p}_y = -D_y \tag{4.6}$$

into Equation (4.5) - and including the higher order terms in it neglected earlier - leads to the transport equation

$$\frac{dA_k}{dt} + \{Z_1(\bar{r})p_x(p_x^2+p_y^2) + Z_2(\bar{r})p_y(p_x^2+p_y^2) - \nabla_p\cdot\bar{D}\}\ A_k +$$

$$\{Z(\bar{r})[(6p_x^2+2p_y^2)\ \frac{\partial^2}{\partial x^2} + (6p_y^2+2p_x^2)\ \frac{\partial^2}{\partial y^2} + 8p_xp_y\ \frac{\partial^2}{\partial x\partial y}] + [Z_1(\bar{r})(3p_x^2+p_y^2) +$$

$$2Z_2(\bar{r})\ p_xp_y]\ \frac{\partial}{\partial x} + [Z_2(\bar{r})(3p_y^2+p_x^2)+ 2Z_1(\bar{r})p_xp_y]\ \frac{\partial}{\partial y} + [\tilde{Z}(r) - (1-\sigma)Z_4(\bar{r}))p_x^2 +$$

$$(\tilde{Z}(\bar{r})-(1-\sigma)Z_3(\bar{r}))p_y^2 + (1-\sigma)\hat{Z}(r)p_xp_y]\}A_{k-1} + \{4Z(\bar{r})(p_x\ \frac{\partial^3}{\partial x^3} + p_y\ \frac{\partial^3}{\partial y^3} + p_x\ \frac{\partial^3}{\partial x^2\partial y} +$$

$$p_y\ \frac{\partial^3}{\partial y^2\partial x}) + (3Z_1(\bar{r})p_x+Z_2(\bar{r})p_y)\ \frac{\partial^2}{\partial x^2} + (3Z_2(\bar{r})p_y + Z_1(\bar{r})p_x)\ \frac{\partial^2}{\partial y^2} +$$

$$2(Z_1(\bar{r})p_y + Z_2(\bar{r})p_x)\ \frac{\partial^2}{\partial x\partial y} + (1-\sigma)[\hat{Z}(\bar{r})p_y - 2Z_4(r)p_x)\ \frac{\partial}{\partial x} + (\hat{Z}(\bar{r})p_x -$$

$$2Z_3(\bar{r})p_y)\ \frac{\partial}{\partial y}]\}\ A_{k-2} + \{Z(\bar{r})(\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} + 2\ \frac{\partial^4}{\partial x^2\partial y^2}) + Z_1(\bar{r})(\frac{\partial^3}{\partial x^3} + \frac{\partial^3}{\partial y^2\partial x}) +$$

$$Z_2(\bar{r})(\frac{\partial^3}{\partial y^3} + \frac{\partial^3}{\partial x^2\partial y}) + (\tilde{Z}(\bar{r}) - (1-\sigma)Z_4(\bar{r}))\ \frac{\partial^2}{\partial x^2} + (\tilde{Z}(\bar{r}) - (1-\sigma)Z_3(\bar{r}))\ \frac{\partial^2}{\partial y^2} +$$

$$(1-\sigma)\hat{Z}(\bar{r}) \frac{\partial^2}{\partial x \partial y}]\} A_{k-3} = 0 \tag{4.7}$$

In this case, the phase $\phi(\bar{r},\bar{p}) = \bar{r}\cdot\bar{p} - S(\bar{p})$ cannot be determined by direct integration as in the one-dimensional case above. The phase can be obtained parametrically, however, from the eikonal equation, i.e., the Hamiltonian on the Lagrange manifold

$$Z(\nabla_p S(\bar{p}))(p_x^4 + p_y^4 + 2p_x^2 p_y^2) - \rho_S(\nabla_p S(\bar{p})) = 0 ,$$

using the classical method of characteristics [7] in $\bar{p}$-space to find $S(\bar{p})$ and thus $\phi(\bar{r},\bar{p})$. Explicitly, applying Hamilton's equations to Equation (4.4) leads to

$$\dot{x} = 2p_x \qquad \dot{p}_x = -\frac{\partial H}{\partial x}$$

$$\dot{y} = 2p_y \qquad \dot{p}_y = -\frac{\partial H}{\partial y}$$

which obtains the map

$$x = x(t,\theta) \qquad p_x = p_x(t,\theta)$$

$$y = y(t,\theta) \qquad p_y = p_y(t,\theta)$$

where t is the time and θ is a parametrized initial condition. Then inverting the momentum $(\bar{p})$ equations yields

$$t = t(p_x,p_y) \qquad \theta = \theta(p_x,p_y).$$

Substituting for t and θ in the configuration space equations determines

$$x = x[t(p_x,p_y),\ \theta(p_x,p_y)] = \underline{\bar{X}}(p_x,p_y) = \frac{\partial S}{\partial p_x}$$

$$y = y[t(p_x,p_y),\ \theta(p_x,p_y)] = \underline{\bar{Y}}(p_x,p_y) = \frac{\partial S}{\partial p_y}$$

where $\underline{\bar{X}}(p_x,p_y)$ and $\underline{\bar{Y}}(p_x,p_y)$ are explicit functions of p_x and p_y. An integration then obtains the phase, cf. Berry [12]. A simpler approach applies for Hamiltonians cylic in either x or y [13].

It is interesting to note that once $S(\bar{p})$ is determined, the caustic curve - the higher dimensional analog of turning points - is determined as well. Setting

$$\det\left(\frac{\partial^2\phi}{\partial p_x \partial p_y}\right) = 0$$

determines the caustic curve in $\bar{p}$-space. Each $\bar{p}_c = (p_x,p_y)$ on this curve corresponds to a point on the caustic in (x,y) space obtained from the Lagrange manifold

$$x = \frac{\partial S(\bar{p}_c)}{\partial p_x} \qquad y = \frac{\partial S(\bar{p}_c)}{\partial p_y}$$

The locus of these points is the caustic curve. With the phase determined as above and the A_k from Equation (4.6), the integrals

$$\int A_k(\bar{r},\bar{p})\ \exp\{i\tau\phi(\bar{r},\bar{p})\}d\bar{p}$$

can now be evaluated asymptotically using standard techniques [9]. While this analysis has focused on a particular 4th order equation in two spatial variables and one time variable, the procedure applies to more general such equations equally well.

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