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FINITE DIMENSIONALITY IN SOCLE OF BANACH ALGEBRAS

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ABSTRACT. It is shown that if the socle soc(A) of a semisimple Banach algebra A is norm-closed, then soc(A) is already finite dimensional. The proof makes use of the Al-Moajil theorem. However it is remarked that our main theorem is an extension of the Al-Moajil's.

KEY WORDS AND PHRASES. Compactrum, socle, minimal idempotent, minimal left ideal, annihilator.
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1. INTRODUCTION AND MAIN THEOREM.

Throughout the note, we will refer to the notations and terminologies in the Bonsall-Duncan's book [3]. Let A be a (complex) Banach algebra and let comp(A) be the compactrum of A, that is the set of all x in A such that the mapping : a $\rightarrow$ xax is a compact operator of A into itself. A. H. Al-Moajil [2] gives some characterizations of a finite dimensionality of a semisimple Banach algebra in terms of its compactrum and socle, which generalizes a theorem of A. W. Tullo [4]. Indeed his characterization is essentially the following

Theorem A (A. H. Al-Moajil [2]). Let A be a semisimple Banach algebra with lan(comp(A)) = {0}. Then A is finite dimensional if and only if soc(A) is norm-closed. Here lan(comp(A)) denotes the left annihilator of comp(A).

However in case of a semisimple Banach algebra A which does not satisfy the Al-Moajil condition : lan(comp(A)) = {0}, the closeness of soc(A) is not necessarily equivalent to the finite dimensionality of A. There exists an easy counter example. In fact let A = C(Ω) be the algebra of all continuous complex-valued functions on = [0, 1] $\cup$ {2, 3,..., n}, with supremum norm. As it is well-known that comp(C([0, 1])) = {0}, we have comp(A) $\cong$ C({2,3,...n}) $\cong$ C^{n-1}, where C denotes the field of complex numbers. Then $\mathrm{lan(comp(A))} \cong C([0,1])$ and so $\mathrm{lan(comp(A))} \neq \{0\}$. Also notice that $\mathrm{soc(A)} \cong C(\{2,3,\dots,n\}) \cong C^{n-1}$, so that soc(A) is finite dimensional and hence norm-closed. But A is of course infinite dimensional.

On the other hand, this counter example suggests to us the following statement.

THEOREM. Let A be a semisimple Banach algebra with soc(A). Then soc(A) is finite dimensional if and only if soc(A) is norm-closed.

The purpose of this note is to prove the above theorem and to remark that our theorem is an extension of the Al-Moajil theorem. To do this, we will prepare some lemmas in the next section.

2. KNOWN RESULTS AND LEMMAS.

The next lemma can be seen in [3, pp. 155-156].

LEMMA 1. Let A be a semiprime algebra. Then L is a minimal left ideal of A if and only if $L = Ae$, where e is a minimal idempotent in A. The similar result holds for minimal right ideals. In particular, if A has minimal left ideals, then soc(A) exists.

The next lemma appears in A. H. Al-Moajil [2].

LEMMA 2. Let A be a semisimple Banach algebra. Then comp(A) is nonzero if and only if soc(A) exists, in this case $\mathrm{soc(A)} \subset \mathrm{comp(A)}$.

LEMMA 3. Let A be a semisimple Banach algebra and B a nonzero closed one-sided ideal of A. Then B is not a radical algebra.

PROOF. Suppose that B is radical and hence $B = \mathrm{rad}(B)$. Choose a nonzero element b of B, so that $r(b) = 0$ from [3, Proposition 25.1(i)]. If $AB \subset B$, then $r(ab) = 0$ for all $a \in A$ and hence $b \in \mathrm{rad}(A)$ from [3, Proposition 25.1(ii)]. The semisimplicity of A implies that $b = 0$, a contradiction. We therefore conclude that if B is a closed left ideal of A, then B is not radical. Of course, the same conclusion holds for closed right ideals.

LEMMA 4. Let A be a semisimple Banach algebra with soc(A) and let min(A) be the set of all minimal idempotents of A. Then

$$\mathrm{lan(comp(A))} = \mathrm{lan(soc(A))} = \mathrm{lan(min(A))} = \mathrm{ran(min(A))}.$$

In particular, $\mathrm{lan(comp(A))} = \mathrm{ran(comp(A))}$. Here ran(min(A)) denotes the right annihilator of min(A).

PROOF. It is clear that $\mathrm{lan(comp(A))} \subset \mathrm{lan(soc(A))} = \mathrm{lan(min(A))}$ from Lemma 2, the definition of soc(A) and Lemma 1. Now in order to show that $\mathrm{lan(soc(A))} \subset \mathrm{ran(min(A))}$, assume, on the contrary, that $\mathrm{lan(soc(A))} \cap (A \setminus \mathrm{ran(min(A))}) \neq \emptyset$ and take an element from this set, say x. Then since $x \notin \mathrm{ran(min(A))}$, there is an element $e \in \mathrm{min(A)}$ with $ex \neq 0$ and hence $exA \neq \{0\}$ from the semisimplicity of A. Since $\{0\} \subsetneq exA \subset eA$ and eA is a minimal right ideal of A from Lemma 1, we have $exA = eA$, so that there exists an element $y \in A$ with $exy = e$. Note that $ye \in \mathrm{soc(A)}$, so that $xye = 0$ because $x \in \mathrm{lan(soc(A))}$. However $e = exy = exye = 0$, a contradiction. We therefore conclude that $\mathrm{lan(soc(A))} \subset \mathrm{ran(min(A))}$. Moreover the symmetric argument implies that $\mathrm{ran}(\min(A)) \subset \mathrm{lan}(\min(A))$ (consider the reversed algebra $\mathrm{rev}(A)$). In order to show that $\mathrm{lan}(\mathrm{soc}(A)) \subset \mathrm{lan}(\mathrm{comp}(A))$, we will apply the method which appears in the proof of [2, Lemma 3]. In fact suppose, on the contrary, that $\mathrm{lan}(\mathrm{soc}(A)) \cap (A \setminus \mathrm{lan}(\mathrm{comp}(A))) \neq \emptyset$ and take an element from this set, say x. Then since $x \notin \mathrm{lan}(\mathrm{comp}(A))$, there is an element $y \in \mathrm{comp}(A)$ with $xy \neq 0$. Set $J = \overline{xyA}$, the norm-closure of xyA. Then J is a nonzero closed right ideal of A. Also J is a compact Banach algebra from [2, Proposition 1] and it is not radical from Lemma 3. Then by [1, Theorem 4.3], J contains a nonzero idempotent e such that eJe is finite dimensional. Since $eAe = e(eA)e \subset eJe$, it follows that eAe is also finite dimensional and so $e \in \mathrm{soc}(A)$ from [1, Theorem 7.2]. Now since $e \in J$, we can write $e = \lim_{n\to\infty} xya_n$, $a_n \in A$ $(n = 1, 2, \ldots)$. But since each ya_ne belongs to $\mathrm{soc}(A)$ and $x \in \mathrm{lan}(\mathrm{soc}(A))$, we have that $e = e^2 = \lim_{n\to\infty} xya_ne = 0$, a contradiction. We therefore conclude that $\mathrm{lan}(\mathrm{soc}(A)) \subset \mathrm{lan}(\mathrm{comp}(A))$ and hence

$$\mathrm{lan}(\mathrm{comp}(A)) = \mathrm{lan}(\mathrm{soc}(A)) = \mathrm{lan}(\min(A)) = \mathrm{ran}(\min(A)).$$

In particular the symmetric argument implies again that $\mathrm{lan}(\mathrm{comp}(A)) = \mathrm{ran}(\mathrm{comp}(A))$.

NOTE. By the above lemma, if $\mathrm{lan}(\mathrm{comp}(A)) = \{0\}$, then $\mathrm{ran}(\mathrm{comp}(A)) = \{0\}$. We then see that the conclusion of [2, Lemma 3] holds certainly for closed left ideals.

The next result is known in the structure theorem of a finite dimensional complex algebra (cf. [3, Proposition 26.7]), but we here give an alternative proof.

LEMMA 5. If A is a semisimple finite dimensional Banach algebra, then it has an identity element.

PROOF. Note that $A = \mathrm{comp}(A)$ from the finite dimensionality of A and $\mathrm{lan}(A) = \{0\}$ from the semisimplicity of A, so that $\mathrm{lan}(\mathrm{comp}(A)) = \{0\}$. Therefore the second half of the proof of [2, Theorem] implies directly the desired conclusion.

3. PROOF OF MAIN THEOREM.

Let denote by B the norm-closure of $\mathrm{soc}(A)$ in A. Then B is a closed two-sided ideal of A. Since A is semisimple, it follows that B is also semisimple. We first claim that $\mathrm{soc}(A)$ exists and $\mathrm{soc}(A) = \mathrm{soc}(B)$. Actually, choose a minimal left ideal L of A arbitrarily. Then by Lemma 1, we can write $L = Ae$, where e is a minimal idempotent in A. But $\{0\} \subsetneq eBe \subset eAe$ and eAe is one-dimensional and so $eBe = eAe$. In other words, e is also a minimal idempotent in B. Again by Lemma 1, Be is a minimal left ideal of B and so $\mathrm{soc}(B)$ exists. Also since $e \in \mathrm{soc}(A)$ and $Ae \subset B$, $L = Ae = (Ae)e \subset \mathrm{soc}(B)$. We thus obtain that $\mathrm{soc}(A) \subset \mathrm{soc}(B)$. Conversely choose a minimal left ideal M of B and write $M = Bf$ for some minimal idempotent f in B. But since B is a two-sided ideal of A, $fAf \subset B$ and so $fAf = fBf$. Thus f is also a minimal idempotent in A. Then $f \in \mathrm{soc}(A)$ and hence $M = Bf \subset \mathrm{soc}(A)$. In other words, $\mathrm{soc}(B) \subset \mathrm{soc}(A)$ and so $\mathrm{soc}(A) = \mathrm{soc}(B)$.

Now it is clear that if $\mathrm{soc}(A)$ is finite dimensional, then it is norm-closed, and hence assume conversely that $\mathrm{soc}(A)$ is norm-closed. Since $\mathrm{soc}(B)$ exists from the above argument, it follows from Lemma 2 that $\mathrm{soc}(B) \subset \mathrm{comp}(B)$ and so

$$B = \overline{\mathrm{soc}(A)} = \mathrm{soc}(A) = \mathrm{soc}(B) \subset \mathrm{comp}(B) \subset B,$$

where the bar denotes the norm-closure in A. Then B = soc(B) = comp(B). But since B is semisimple, lan(B) = {0}, so that lan(comp(B)) = {0}. Therefore Theorem A implies that B is finite dimensional and so is soc(A). The proof is complete.

4. REMARK.

In this section we see that our theorem implies the Al-Moajil theorem. Indeed if A is finite dimensional, then by [1, Theorem 7.2], A = soc(A) and hence soc(A) is norm-closed. Then it is sufficient to show that if the socle of a semisimple Banach algebra A with lan(comp(A)) = {0} is norm-closed, then A is already finite dimensional. Then assume that soc(A) is norm-closed and lan(comp(A)) = {0}. By our theorem, soc(A) is a semisimple finite dimensional Banach algebra and so it has an identity element e from Lemma 5. Let x be any element of A. Then x(1 - e) belongs to lan(soc(A)) and hence lan(comp(A)) from Lemma 4. Hence we obtain that x(1 - e) = 0. Also since soc(A) is a two-sided ideal of A, it follows that e is a central element of A. Then e is an identity element of A and so A = soc(A). Therefore A is finite dimensional.

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