TAUBERIAN CONDITIONS FOR CONULL SPACES

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ABSTRACT. The typical Tauberian theorem asserts that a particular summability method
cannot map any divergent member of a given set of sequences into a convergent sequence.
These sets of sequences are typically defined by an "order growth" or "gap" condition.
We establish that any conull space contains a bounded divergent member of such a set;
hence, such sets fail to generate Tauberian theorems for conull spaces.

1. INTRODUCTION

In this note we establish that a broad class of Tauberian conditions that hold
for regular matrix methods cannot hold for conull methods. In particular, we consider
"gap" and "order growth" conditions and show they are not Tauberian conditions for
conull spaces.

Before proceeding with the discussion, we pause to collect some definitions and
theorems. We let

ω = {the set of all sequences}

ϕ = {$x \in \omega$: x is finitely nonzero}

c_0 = {$x \in \omega$: $\text{limit}_n x_n = 0$}

c = {$x \in \omega$: $\text{limit}_n x_n$ exists}

ℓ^∞ = {$x \in \omega$: $\sup_n |x_n| < \infty$}

$e = (1,1,1,..)$

e^n = the nth unit vector, $\sigma^n = \sum\limits_{j=1}^{n} e^j$

and, if E is a Frechet space, E' denotes its continuous dual.

Recall that a locally convex Frechet space is FK-space if it is a vector subspace
of ω and the coordinate functionals are continuous. We call an FK-space a sequence
space if it contains ϕ .

Definition: A sequence space E is (a) conservative if $E \supseteq c$, (b) semi-conservative if (σ^n) is weakly Cauchy, and (c) conull if (σ^n) converges weakly to e.

Clealy conull spaces are semi-conservative. It is known that a sequence space E contains c_0 if and only if $\sum_{j=1}^{\infty} |f(e^j)| < \infty$ for all $f \in E'[1]$, hence conservative spaces are also semi-conservative.

Definition: An FK-space is pre-conull if every semi-conservative space containing it is conull.

We list some well-known facts in the following two theorems. ([2], [3])

Theorem: Let E , F be FK-spaces with $E \subseteq F$ (set theoretically) and Λ a matrix map. Then (a) the inclusion map from E into F is continuous; (b) $E_A = \{x \in \omega: Ax \in E\}$ is an FK-space; (c) if E is conull, F is conull; and (d) if $\phi \subseteq E$, E is closed in F and F is conull, then E is conull.

Theorem: The intersection of two conull spaces is conull.

We also use a characterization of conservative conull spaces. Let $r = (r_n)$ be be an increasing sequence of natural numbers with $r_1 = 1$. Define

$$O_n^r(x) = \max\{|x_u - x_v| : r_n \leq u < v \leq r_{n+1}\}$$

and set

$$\Omega(r) = \{x \in \omega: \lim_n O_n^r(x) = 0\}.$$

If we define $\|x\|_r = |x_1| + \sup_n O_n^r(x)$ for $x \in \Omega(r)$, then $(\Omega(r), \|\cdot\|_r)$ is a conservative conull space. In fact, we also have from [3]:

Theorem: Let E be conservative.   E is conull if and only if $E \supseteq \Omega(r)$ for some r.

Now we are ready to begin. Let E be an FK-space and $P \subseteq \ell^\infty$. We say that "P is a Tauberian condition for E" provided

(*) $x \in P \cap E$ implies $x \in c$.

Note that (*) is the general form of a (matrix summability) Tauberian theorem. The candidates for playing the part of P are defined as follows.

Definition: ("gap" conditions) Let $s = (s_n)$ be an increasing sequence of natural numbers; set

$$G(s) = \{x \in \omega: (\Delta x)_k = x_k - x_{k+1} \neq 0 \text{ only if } k = s_n \text{ for some } n \in N\}.$$

Definition: ("order growth" conditions) Let $\lambda = (\lambda_n)$ be a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \lambda_n = \infty$.

$$\eth(\lambda) = \{x \in \omega: |(\Delta x)_n| = o(\lambda_n)\}$$

and

$$\tilde{O}(\lambda) = \{x \in \omega: |(\Delta x)_n| = O(\lambda_n)\}.$$

Recall that if E is the FK-space of Caesaro summable sequences and we let $s = (s_n)$ satisfy $\lim_n s_{n+1}/s_n > 1$ and $\lambda_n = n^{-1}$, then $O(\lambda) \cap \ell^\infty$ and $G(s) \cap \ell^\infty$ are Tauberian conditions for E. Also, J. Fridy has shown that for any real regular matrix Λ there is always an s such that $G(s) \cap \ell^\infty$ is a Tauberian condition for the summability field of Λ[4].

2. PRE-CONULL SPACES.

Our arguments hinge on two properties of pre-conull spaces: that they are neces-
sarily "large" and that pre-conullity is preserved under intersection with a conull
space.   These properties are exposed in the next two lemmas.

Lemma 1:   If E is pre-conull and F is conull, then E ∩ F is pre-conull.

Proof: First observe that if H is a vector space containing E ∩ F, then (E + H ∩ F)
∩ F ⊆ H.   This follows from noting that if y ∈ E, z ∈ H ∩ F and y + z ∈ F, then
y ∈ E ∩ F, and clearly E ∩ F + H ∩ F ⊆ H.

Now suppose H is a semi-conservative space containing E ∩ F.   Since F is conull,
F ∩ H is semi-conservative (i.e., (σ^n) is weakly Cauchy in F ∩ H), and consequently
E + F ∩ H is semi-conservative.   Now E ⊆ E + F ∩ H implies that E + F ∩ H is conull,
and since F is conull, (E + F ∩ H) ∩ F is conull.   Consequently H is also conull and
we have established the lemma.

Lemma 2:   If E is pre-conull, then $(E \cap \ell^\infty) + c \supset \Omega(r) \cap \ell^\infty$ for some r.   In particular,
E contains a bounded divergent sequence.

Proof: It is easy to check that $(E \cap \ell^\infty) + c = (E + c) \cap \ell^\infty$.

Since E + c is conservative, hence semi-conservative, and E is pre-conull, E + c
is a conservative conull space.   Consequently E + c contains an $\Omega(r)$ for some r and
$(E \cap \ell^\infty) + c \supset \Omega(r) \cap \ell^\infty$.

Now E contains a bounded divergent sequence since $\Omega(r)$ does, i.e., $\Omega(r)$ contains
a bounded divergent sequence of the form y + z where $y \in E \cap \ell^\infty$ and $z \in c$ and y must
be divergent (otherwise, y + z ∈ c).

We also give a sufficient condition for a space to be pre-conull.

Lemma 3:   E is pre-conull if there is a sequence $(z^n) \subset \alpha$ such that z^n converges to e
in E and $\sup_n \sum_{k=1}^{\infty} |(\Delta z^n)_k| < \infty$.

Proof: This follows from the fact that semi-conservative space F is conull   if (and
only if) there is a sequence $(z^n) \subseteq \phi$ such that z^n converges to e in F and $\sup_n \sum_{k=1}^{\infty}$
$|(\Delta z^n)_k| < \infty$ [5].

3. THE MAIN RESULT.

Lemma 4:   If G(s), $\tilde{\sigma}(\lambda)$ and $\delta(\lambda)$ are defined as in section 1, they are all pre-conull
spaces.

Proof: First we establish that G(s) is a pre-conull space.   Observe that G(s) is a
closed subspace of ω when ω is given the topology of coordinatewise convergence (ω's
FK-topology), hence it is an FK-space.

Now let E be any semi-conservative space containing G(s) and set

$$\sigma^{n_k} = \sum_{j=1}^{s_k} e^j, \quad k = 1, 2, \ldots .$$

Since (σ^{n_k}) converges to e in G(s), it converges to e in E.   Now we have that (σ^n)
is weakly Cauchy in E and has a subsequence which converges weakly to e, hence (σ^n) is
weakly convergent to e in E.   Thus E is conull, and consequently, G(s) is pre-conull.

We now turn our attention to $\delta(\lambda)$ and $\tilde{O}(\lambda)$. Consider the natrix $B = (b_{n,k})$ defined by $b_{1,1} = 1$, $b_{n,n-1} = \lambda_{n-1}^{-1}$, $b_{n,n} = -\lambda_{n-1}^{-1}$ and $b_{n,k} = 0$ otherwise. Observe that $\tilde{O}(\lambda) = \ell_B^{\infty}$ and $\delta(\lambda) = (c_0)_B$, hence both are FK-spaces with the metric topology being given by the norm $||x||_B = \max\{|x_1|, \sup_k \lambda_k^{-1}|(\Delta x)_k|\}$ in both cases. Also note that, since $\delta(\lambda) \subseteq \tilde{O}(\lambda)$, it suffices to show that $\delta(\lambda)$ is pre-conull to show that $\tilde{O}(\lambda)$ is pre-conull.

We now construct a sequence in $\delta(\lambda)$ that satisfies the hypothesis of Lemma 3. Let (β_k) be a sequence of reals such that $0 < \beta_k < 1$ for all $k \in N$, $\sum_{k=1}^{\infty} \beta_k = \infty$ and $\text{limit}_k \lambda_k^{-1} \beta_k = 0$ (this is possible by one of Abel's results [6, p. 125]), and set $I(n) = \max\{k: \sum_{i=n}^{k-1} \beta_i < 1\}$. Note that $n < I(n) < \infty$ for all $n \in N$. Define z^n by $z_k^n = $ for $k \le n$, $z_k^n = 1 - \sum_{i=n}^{k-1} \beta_i$ for $n < k \le I(n)$ and $z_k^n = 0$ for $k > I(n)$. It's easy to check that $||z^n - e||_B \le \max\{|\lambda_k^{-1}\beta_k|: n \le k \le I(n)\}$, hence $\text{limit}_n z^n = e$ in $\delta(\lambda)$.

It's also clear that $(z^n) \subseteq \phi$ and $\sum_{k=1}^{\infty} |(\Delta z^n)_k| = 1$ for all $n \in N$, hence, by Lemma 3, $\delta(\lambda)$ is pre-conull.

We can now show that neither $\delta(\lambda)$, $\tilde{O}(\lambda)$ nor $G(s)$ can generate Tauberian conditions for a conull space.

Theorem: If E is a conull space, then $\delta(\lambda) \cap E$, $\tilde{O}(\lambda) \cap E$ and $G(s) \cap E$ all contain a bounded divergent sequence.

Proof: Since $\tilde{O}(\lambda)$, $\delta(\lambda)$ and $G(s)$ are all pre-conull, $\tilde{O}(\lambda) \cap E$, $\delta(\lambda) \cap E$ and $G(s) \cap E$ are all pre-conull, hence each contain a bounded divergent sequence.

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