Internat. J. Math. & Math. Sci.
Vol. 8 No. 2 (1985) 257-266

PSEUDO-RIEMANNIAN MANIFOLDS ENDOWED WITH AN ALMOST PARA f-STRUCTURE

VLADISLAV V. GOLDBERG and RADU ROSCA

Department of Mathematics
New Jersey Institute of Technology
323 Dr. Martin Luther King Jr. Boulevard
Newark, N.J. 07102 U.S.A.

(Received December 27, 1983)

ABSTRACT. Let $\tilde{M}(U,\tilde{\Omega},\tilde{\eta},\xi,\tilde{g})$ be a pseudo-Riemannian manifold of signature $(n+1,n)$. One defines on $\tilde{M}$ an almost cosymplectic para f-structure and proves that a manifold $\tilde{M}$ endowed with such a structure is ξ-Ricci flat and is foliated by minimal hypersurfaces normal to ξ, which are of Otsuki's type. Further one considers on $\tilde{M}$ a $2(n-1)$-dimensional involutive distribution $P^{\perp}$ and a recurrent vector field $\tilde{V}$. It is proved that the maximal integral manifold $M^{\perp}$ of $P^{\perp}$ has V as the mean curvature vector (up to $1/2(n-1)$). If the complimentary orthogonal distribution P of $P^{\perp}$ is also involutive, then the whole manifold $\tilde{M}$ is foliate. Different other properties regarding the vector field $\tilde{V}$ are discussed.

KEY WORDS AND PHRASES. *Pseudo-Riemannian manifold, cosymplectic manifold, para f-structure, minimal hypersurface.*
1980 MATHEMATICS SUBJECT CLASSIFICATION CODE. 53C25.

1. INTRODUCTION.

Recently, many papers were devoted to *f-structures* or *para f-structures* (Ishichara and Yano [1]; Kiritchenko [2]; Yano and Kon [3]; Sinha [4]).

In this paper we consider a C^{∞}-pseudo-Riemannian manifold $(\tilde{M},\tilde{g})$ of dimension $2n+1$ and of inertia index $n+1$ and such that the $(1,1)$-tensor field f coincides with the *para-complex* operator U (Libermann [5]) of square $+1$. Furthermore we suppose that $\tilde{M}$ is equipped with a triple $(\tilde{\Omega},\tilde{\eta},\xi)$ where

1^{o}. $\tilde{\Omega}$ is a canonical 2-form of rank $2n$ exchangeable with the para-Hermitian component $\tilde{g}_{\eta}$ of the metric tensor $\tilde{g}$;

2^{o}. $\tilde{\eta}$ is a canonical 1-form such that $(\Lambda\tilde{\Omega})^{n}\Lambda\tilde{\eta} \neq 0$;

3^{o}. ξ is the canonical vector field such that

$$\left.\begin{aligned} &\tilde{\eta}(\xi) = 1,\ i_{\xi}\tilde{\Omega} = 0,\ U\xi = 0,\\ &d\tilde{\eta} = 0,\ \tilde{g}(\tilde{\nabla}_{\tilde{Z}}\xi,\tilde{Z}') = \tilde{g}(\tilde{\nabla}_{\tilde{Z}'}\xi,\tilde{Z}). \end{aligned}\right\} \tag{1.1}$$

In (1.1) $\tilde{\nabla}$ is the covariant differential operator on $\tilde{M}$ and $\tilde{Z}$, $\tilde{Z}'$ are any vector fields on M.

If the above conditions are satisfied, we say that $\tilde{M}$ is endowed with an *almost cosymplectic para f-structure* (abr. a.c.p. f-structure). In this case $\tilde{M}$ is called an a.c.p. f-manifold.

The differential distribution $D_\eta = \{\tilde{Z} \in T\tilde{M}, \tilde{\eta}(\tilde{Z}) = 0\}$ on $\tilde{M}$ is involutive and is called horizontal.

It is proved that an a.c.p. f-manifold is always ξ-Ricci flat and that it is foliated by minimal hypersurfaces M_η, tangent to D_η, which are of Otsuki's type (Otsuki [6]).

Suppose now that D and $D^\perp$ are two complementary orthogonal differential distributions in D_η and $\tilde{Z}$ is a vector field in D. If one has

$$\tilde{\nabla}\tilde{W} = \tilde{u} \otimes \tilde{X} + \tilde{u}^\perp \otimes \tilde{X}^\perp + \tilde{v} \otimes \xi \tag{1.2}$$

for all $\tilde{X} \in D$, $\tilde{X}^\perp \in D^\perp$, and $\tilde{u}, \tilde{u}^\perp, \tilde{v} \in \Lambda^1(\tilde{M})$, we say that D is *contact covariant decomposable* (abr. c.c.d.). Let P be a c.c.d. *hyperbolic* 2-plane of D_η. If the dual forms of two null vector fields which define P form an *exterior recurrent pairing* (Rosca [7]; Morvan and Rosca [8]), we say that the manifold $\tilde{M}$ admits a *strict* c.c.d. hyperbolic 2-plane.

With the paring $(P, P^\perp)$ are associated a vector field $\tilde{V} \in P$ (called the *recurrence* vector field) and two vector fields $\tilde{X}_n, \tilde{X}_{2n} \in P^\perp$ (called the *distinguished* vector fields).

In the present paper the following properties are proved:

(i) The $2(n-1)$-distribution $P^\perp$ is always involutive and the mean curvature vector field of its maximal integral manifold $M^\perp$ is (up to a constant factor) equal to the induced vector field of $\tilde{V}$.

(ii) The simple unit form $\tilde{\phi}$ of $P^\perp$ is exterior recurrent and $U\tilde{V}$ is a characteristic vector field of $\tilde{\phi}$.

(iii) The necessary and sufficient condition for $M^\perp$ to be quasi-minimal (Chen [9]) is that $\tilde{X}_n$ or $\tilde{X}_{2n}$ be a null vector field, and the necessary and sufficient condition for $M^\perp$ to be minimal is that both $\tilde{X}_n$ and $\tilde{X}_{2n}$ be null vector fields.

(iv) If $M^\perp$ is minimal, then the distribution P is also involutive and the integral surfaces of P are totally geodesic in M_η which in this case is foliate.

(v) Both vector fields $\tilde{X}_n$ and $\tilde{X}_{2n}$ are U-geodesic directions on $M^\perp$.

2. ALMOST COSYMPLECTIC PARA f-MANIFOLD $\tilde{M}(U, \tilde{\Omega}, \tilde{\eta}, \xi, \tilde{g})$.

Let $(\tilde{M}, \tilde{g})$ be a C^∞-pseudo-Riemannian manifold of dimension $2n+1$ and of inertia index $n+1$.

If $\tilde{M}$ is equipped with a non-zero tensor field f of type $(1,1)$ of constant rank and such that

$$f(f^2 - I) = 0 \tag{2.1}$$

(I is the identity tensor), then f is called a *para f-structure* (Sinha [4]).

In the following we suppose that f coincides with the para-complex operator U (Libermann [5]). In addition, we suppose that $\tilde{M}$ is equipped with the triple $(\tilde{\Omega}, \tilde{\eta}, \xi)$ where:

1°. $\tilde{\Omega}$ is a canonical 2-form of rank $2n$ exchangeable with the para-Hermitian component $\tilde{g}_\eta$ of the metric tensor $\tilde{g}$ (Buchner and Rosca [10]).

2^o. $\tilde{\eta}$ is a canonical 1-form such that $(\Lambda\tilde{\Omega})^n \Lambda \tilde{\eta} \neq 0$ everywhere.

3^o. ξ is the canonical vector field such that

$$\tilde{\eta}(\xi) = 1, \ i_\xi \tilde{\Omega} = 0; \quad \text{i: interior product.} \tag{2.2}$$

If one has

$$U^2 - 1 = -\tilde{\eta} \otimes \xi \Rightarrow U\xi = 0, \tag{2.3}$$

$$d\tilde{\eta} = 0, \tag{2.4}$$

$$\tilde{g}(\nabla_{\tilde{Z}}\xi, \tilde{Z}') = \tilde{g}(\tilde{\nabla}_{\tilde{Z}'}\xi, \tilde{Z}) \tag{2.5}$$

where $\tilde{\nabla}$ is the covariant differential operator on $\tilde{M}$ and $\tilde{Z}$, $\tilde{Z}'$ are vector fields in $\tilde{M}$, we say that $(U, \tilde{\Omega}, \tilde{\eta}, \xi, \tilde{g})$ defines on $\tilde{M}$ an *almost cosympletic para f-structure* (abr. a.c.p. f-structure) and $\tilde{M}(U, \tilde{\Omega}, \tilde{\eta}, \xi, \tilde{g})$ is called an a.c.p. f-manifold.

The differentiable distribution D_η on $\tilde{M}$ defined by

$$D_\eta = \{\tilde{X} \in T\tilde{M}, \ \tilde{\eta}(\tilde{X}) = 0\}$$

is called *horizontal.*

It is worthwhile to note that equations (2.3), (2.4) and (2.5) show that on $\tilde{M}$ the triple $(U, \tilde{\eta}, \xi)$ defines an *almost paracontact* structure (Sinha [4]), D_η defines a $(2n)$-foliation, and ξ is a gradient.

Let $W = \text{vect}\,\{h_a, h_{a^*}, h_0 = \xi;\ a = 1, \ldots, n,\ a^* = a+n\}$ be a local field of Witt frames (Vranceanu and Rosca [11]).

One has (Libermann [5]):

$$Uh_a = h_a, \ Uh_{a^*} = -h_{a^*}, \ U\xi = 0 \tag{2.6}$$

and at each point $\tilde{p} \in \tilde{M}$ one has the splitting

$$(D_\eta)_{\tilde{p}} = S_{\tilde{p}} \oplus S^*_{\tilde{p}} \tag{2.7}$$

where $\tilde{S}_{\tilde{p}}$ and $\tilde{S}^*_{\tilde{p}}$ are two *self-orthogonal* vector n-spaces spanned by $\{h_a\}$ and $\{h_{a^*}\}$ respectively.

Since the null vector fields h_a and h_{a^*} are normed, one may write

$$\left.\begin{aligned} &\tilde{g}(h_a, h_A) = 0, \quad \tilde{g}(h_{a^*}, h_B) = 0, \\ &\tilde{g}(h_a, h_{a^*}) = 1, \quad \tilde{g}(\xi, \xi) = 1 \end{aligned}\right\} \tag{2.8}$$

where $A, B = 0, 1, \ldots, 2n;\ A \neq a^*,\ B \neq a$.

Now let $\{\tilde{\omega}^A\}$ be the dual basis of W and $\tilde{\theta}^A_B = \tilde{\gamma}^A_{BC}\tilde{\omega}^C$ $(\tilde{\gamma}^A_{BC} \in C^\infty(\tilde{M}))$ be the connection forms on $\tilde{M}$. Then the line element $d\tilde{p}$ of $\tilde{M}$ ($d\tilde{p}$ is a canonical vectorial 1-form) and the connection equations are expressed by

$$d\tilde{p} = \tilde{\omega}^a \otimes h_a + \tilde{\omega}^{a^*} \otimes h_{a^*} + \tilde{\eta} \otimes \xi \tag{2.9}$$

and

$$\tilde{\nabla} h_A = \tilde{\theta}^B_A \otimes h_B$$

where $\tilde{\nabla}$ is the covariant differentiation operator on $\tilde{M}$. By (2.8) and (2.10) one finds

$$\left.\begin{array}{lll}\tilde{\theta}^a_{a^*} = \tilde{\theta}^{a^*}_a = 0, & \tilde{\theta}^a_a = 0, & \\ \tilde{\theta}^a_b + \tilde{\theta}^{b^*}_{a^*} = 0, & \tilde{\theta}^{a^*}_b + \tilde{\theta}^{b^*}_a = 0, & \tilde{\theta}^a_{b^*} = 0, \\ \tilde{\theta}^o_a + \tilde{\theta}^{a^*}_o = 0, & \tilde{\theta}^o_{a^*} + \tilde{\theta}^a_o = 0, & \end{array}\right\} \tag{2.11}$$

and the structure equations (E. Cartan) may be written in the following symbolic form:

$$d\tilde{\omega} = -\tilde{\theta}\wedge\tilde{\omega} \tag{2.12}$$

and

$$d\tilde{\theta} = -\tilde{\theta}\wedge\tilde{\theta} + \tilde{\Theta} \tag{2.13}$$

where $\tilde{\Theta} \equiv \tilde{\Theta}^A_B$ are the curvature 2-forms.

Further taking into account (2.4), we may set

$$\tilde{\theta}^o_a = \tilde{\omega}^a, \ \tilde{\theta}^o_{a^*} = -\tilde{\omega}^{a^*} . \tag{2.14}$$

Now by means of (2.10), (2.11) and (2.14) one gets

$$\tilde{\nabla}\xi = \tilde{\omega}^{a^*} \otimes h_a - \tilde{\omega}^a \otimes h_{a^*}. \tag{2.15}$$

In addition it follows from (2.15) that ,

$$\tilde{\nabla}_\xi \xi = 0 \tag{2.16}$$

which proves that ξ is a *geodesic* direction.

From (2.9) and (2.8) one gets

$$\tilde{g} = \langle d\tilde{p}, d\tilde{p}\rangle = 2\sum_a \tilde{\omega}^a \otimes \tilde{\omega}^{a^*} + \tilde{\eta}\otimes\tilde{\eta} \tag{2.17}$$

where $\tilde{g}_\eta = 2\sum \tilde{\omega}^a \otimes \tilde{\omega}^{a^*}$ is the *para-Hermitian* (Buchner and Rosca [10]) component of the metric tensor $\tilde{g}$.

The 2-form $\tilde{\Omega}$ which is exchangeable with $\tilde{g}_\eta$ is then expressed by

$$\tilde{\Omega} = \sum_a \tilde{\omega}^a \wedge \tilde{\omega}^{a^*} . \tag{2.18}$$

Using (2.15), we can find the following expression of the quadratic differential form $\langle\tilde{\nabla}\xi,\tilde{\nabla}\xi\rangle$:

$$\langle\tilde{\nabla}\xi,\tilde{\nabla}\xi\rangle = -2\sum_a \tilde{\omega}^a \otimes \tilde{\omega}^{a^*} = -\tilde{g}_\eta. \tag{2.19}$$

Denote by

$$\tilde{\pi} = \tilde{\omega}^1 \wedge\ldots\wedge \tilde{\omega}^{2n} \tag{2.20}$$

the simple unit form corresponding to D_η. One may write the volume element $\tilde{\sigma}$ of $\tilde{M}$ as

$$\tilde{\sigma} = \tilde{\pi}\wedge\tilde{\eta}. \tag{2.21}$$

If $L_{\tilde{Z}}$ means the Lie derivative in the direction $\tilde{Z}$, then by a simple argument one can find

$$L_\xi\tilde{\sigma} = d\tilde{\pi} = (\mathrm{div}\ \xi)\tilde{\sigma}. \tag{2.22}$$

Using (2.12) and (2.13), one gets $d\tilde{\pi} = 0$, and this yields

$$\mathrm{div}\ \xi = 0. \tag{2.23}$$

But on a Riemannian or pseudo-Riemannian manifold the following Yano integral formula holds (Yano and Kon [12]):

$$\begin{aligned}&\operatorname{div}(\tilde{\nabla}_{\tilde{Z}}\tilde{Z}) - \operatorname{div}(\operatorname{div}\tilde{Z})\tilde{Z}\\ &= \operatorname{Ric}(\tilde{Z}) + \sum_{A,B} g(\tilde{\nabla}_{e_A}\tilde{Z},e_B)\tilde{g}(e_A,\tilde{\nabla}_{e_B}\tilde{Z}) - (\operatorname{div}\tilde{Z})^2.\end{aligned} \tag{2.24}$$

In (2.24) $\tilde{Z}$, Ric and $\{e_A\}$ are arbitrary vector fields on $\tilde{M}$, the Ricci tensor of $\tilde{M}$ and a vectorial basis respectively.

Continuing the consideration, one finds (2.24) and (2.15) by means of (2.5). Taking into account (2.8), a short computation gives $\operatorname{Ric}(\xi) = 2n$.

Hence $\tilde{M}$ is *Ricci constant* in the direction of the structure vector ξ (or ξ-Ricci constant).

On the other hand, by means of (2.19) and (2.4) one sees that $\tilde{\pi}$ is *coclosed*, i.e. $\partial\tilde{\pi} \equiv 0$. Hence since $d\tilde{\pi} = 0$, it follows that $\tilde{\pi}$ is *harmonic*. Then if we denote by M_η the leaf of D_η, it follows from the theorem of Tachibana [13] that M_η is *minimal*. This property can also be verified by a direct computation.

Since the induced value $\Omega = \tilde{\Omega}|_{M_\eta}$ of the almost sympletic form $\tilde{\Omega}$ is also almost symplectic, the submanifold M_η is an example of a minimal submanifold having an almost symplectic structure Ω.

If $\tilde{M}$ is endowed with a para co-Kaehlerian structure (Buchner and Rosca [10]), then Ω is a symplectic form.

Denote now by III the induced value on M_η of the quadratic differential form given by (2.19). Since ξ is normal to M_η, then, as is known, III represents the *third* fundamental form of M_η.

Thus according to (2.19) III is conformal to the metric of M_η. Taking into account of the para-Hermitian form of $\tilde{g}_\eta$ and (2.15), it is easy to see that M_η possesses principal curvatures equal to $+1$ and principal curvatures equal to -1. Therefore referring to Otsuki [6], we may say that M_η is a minimal hypersurface of Otsuki's type.

THEOREM 1. Let $\tilde{M}(U,\tilde{\Omega},\tilde{\eta},\xi,\tilde{g})$ be a pseudo-Riemannian manifold endowed with an a.c.p. f-structure. Such a manifold is ξ-Ricci constant and is foliated by minimal hypersurfaces M_η of Otsuki's type which are orthogonal to the structure vector field ξ.

3. CONTACT COVARIANT DECOMPOSABLE DISTRIBUTIONS ON $\tilde{M}(U,\tilde{\Omega},\tilde{\eta},\xi,\tilde{g})$.

Referring to the definition given by Rosca [7], we give now the following

DEFINITION. Let $\tilde{M}$ be an odd-dimensional C^∞-Riemannian (resp. C^∞-pseudo-Riemannian) manifold equipped with an almost contact (resp. almost para contact) structure defined by a structure 1-form $\tilde{\eta}$ and a structure vector field ξ. Let D_η, D and $\tilde{\nabla}$ be the horizontal distribution defined by $\tilde{\eta} = 0$, a differentiable distribution of D_η and the covariant differentiation operator on $\tilde{M}$. Let $D^\perp$ be the complementary orthogonal distribution of D in D_η and $\tilde{W}$ be a vector field of D. Then if one has

$$\tilde{\nabla}\tilde{W} = \tilde{u}\otimes\tilde{X} + \tilde{u}^\perp\otimes\tilde{X}^\perp + \tilde{v}\otimes\xi \tag{3.1}$$

where $\tilde{X}\in D$, $\tilde{X}^\perp\in D^\perp$ and $\tilde{u},\tilde{u}^\perp,\tilde{v}\in\Lambda^1(\tilde{M})$, we say that the distribution D is *contact covariant decomposable* (abr. c.c.d.).

As is known, the null vectorial basis $\{h_a,h_{a^*}\}$ of D_η admits the orthogonal decomposition

$$D_\eta = P_1 \perp \dots \perp P_a \perp \dots \perp P_n \tag{3.2}$$

where $P_a = (h_a, h_{a^*})$ is a *hyperbolic* 2-plane.

We say that the a.c.p. f-manifold $\tilde{M}(U,\tilde{\Omega},\tilde{\eta},\xi,\tilde{g})$ defined in Section 2, carries a *strict contact covariant decomposable hyperbolic plane* P (abr. s.c.c.d. hyperbolic plane) if:

1° the distribution P is contact convariant decomposable;

2° the dual forms of the null vectors which define P form an *exterior recurrent pairing* (in the sense of Rosca [7]).

Without loss of generality, one may suppose that P is defined by h_n and $h_{n^*} = h_{2n}$.

In the first place, using (2.10) and (3.1), one finds

$$\left.\begin{aligned} \tilde{\theta}^\alpha_n &= \tilde{x}^\alpha_n \tilde{\pi}_n , \\ \tilde{\theta}^\alpha_{2n} &= \tilde{x}^\alpha_{2n} \tilde{\pi}_{2n} \end{aligned}\right\} \tag{3.3}$$

where $\tilde{\pi}_n, \tilde{\pi}_{2n} \in \Lambda^1(\tilde{M})$; $\tilde{x}^\alpha_n, \tilde{x}^\alpha_{2n} \in C^\infty(\tilde{M})$ and $\alpha \in \{i, i^*;\ i = 1,\dots,n;\ i^* = i+n\}$.

Denote by $P^\perp$ the complementary orthogonal distribution of P in D_η.

Obviously one has $P^\perp = \{h_a\}$, and we set

$$\left.\begin{aligned} \tilde{X}_n &= \tilde{x}^\alpha_n h_\alpha \in P^\perp , \\ \tilde{X}_{2n} &= \tilde{x}^\alpha_{2n} h_\alpha \in P^\perp . \end{aligned}\right\} \tag{3.4}$$

Secondly, according to Rosca [7]; Morvan and Rosca [8], the dual forms $\tilde{\omega}^n, \tilde{\omega}^{2n}$ corresponding to $P = (h_n, h_{2n})$ define an exterior recurrent pairing if one has

$$\left.\begin{aligned} d\tilde{\omega}^n &= \tilde{\gamma}^n \wedge \tilde{\omega}^n + \tilde{\gamma}^{2n} \wedge \tilde{\omega}^{2n} , \\ d\tilde{\omega}^{2n} &= \tilde{\nu}^n \wedge \tilde{\omega}^n + \tilde{\nu}^{2n} \wedge \tilde{\omega}^{2n} \end{aligned}\right\} \tag{3.5}$$

where $\tilde{\gamma}^n, \tilde{\gamma}^{2n}, \tilde{\nu}^n, \tilde{\nu}^{2n} \in \Lambda^1(\tilde{M})$.

As a consequence of (3.5), using (2.12), (2.11), (2.14), and (3.3), we find:

$$\left.\begin{aligned} \tilde{\pi}_n &= \tilde{f}_n (\sum_\alpha \tilde{x}^\alpha_n \tilde{\omega}^\alpha), \\ \tilde{\pi}_{2n} &= \tilde{f}_{2n} (\sum_\alpha \tilde{x}^\alpha_{2n} \tilde{\omega}^\alpha) \end{aligned}\right\} \tag{3.6}$$

where $\tilde{f}_n, \tilde{f}_{2n} \in C^\infty(\tilde{M})$ vanish nowhere on M. Therefore (3.5) become of the form

$$\left.\begin{aligned} d\tilde{\omega}^n &= \tilde{\omega}^n \wedge \tilde{\gamma} + \tilde{\eta} \wedge \tilde{\omega}^{2n} , \\ d\tilde{\omega}^{2n} &= -\tilde{\omega}^{2n} \wedge \tilde{\gamma} + \tilde{\eta} \wedge \tilde{\omega}^n \end{aligned}\right\} \tag{3.7}$$

where we have set

$$\tilde{\gamma} = \tilde{\theta}^n_n = -\tilde{\theta}^{2n}_{2n} . \tag{3.8}$$

Denote now by

$$\tilde{\psi} = \tilde{\omega}^n \wedge \tilde{\omega}^{2n} \tag{3.9}$$

the simple unit form which corresponds to P. It follows from (3.7) that

$$d\tilde{\psi} = 0. \tag{3.10}$$

Since $\dim(\operatorname{Ker} \tilde{\psi}) \neq 0$, we may also say that $\tilde{\psi}$ is a *presymplectic* form (Souriau [14]).

Further taking the exterior derivative of equations (3.7) and referring to (2.4), one gets by an easy argument that

$$d\tilde{\gamma} = \tilde{\ell}\tilde{\psi} \rightarrow \tilde{\gamma} = \tilde{\ell}\tilde{\eta}; \quad \tilde{\ell} \in C^{\infty}(\tilde{M}). \tag{3.11}$$

It follows from (3.11) that

$$d\tilde{\gamma} = (d\tilde{\ell}/\tilde{\ell}) \wedge \tilde{\gamma}, \tag{3.12}$$

i.e. $\tilde{\gamma}$ is exterior recurrent and has the exact form $d\tilde{\ell}/\tilde{\ell}$ as the *recurrence* 1-form.

Denote now by $I(P^{\perp}) = \{\tilde{\omega} \in \Lambda(\tilde{M}) : \tilde{\omega}$ annihilates $P^{\perp}\}$ the ideal in $\Lambda(\tilde{M})$ of the distribution $P^{\perp}$. Obviously $\tilde{\psi}$ belongs to this ideal and by means of (3.10) we may say that $I(P^{\perp})$ is a *differentiable* ideal $(dI(P^{\perp}) \subset I(P^{\perp}))$.

It follows as is known, that the distribution $P^{\perp}$ is involutive (this can be also checked by a direct computation with the help of (3.3) and (3.6)).

Let us now denote

$$\tilde{\phi} = \tilde{\omega}^{1} \wedge \dots \wedge \tilde{\omega}^{n-1} \wedge \tilde{\omega}^{1*} \wedge \dots \wedge \tilde{\omega}^{n*-1} \tag{3.13}$$

the simple unit form corresponding to the distribution $P^{\perp}$. Then by means of (2.12), (2.11), (2.14), (3.3), (3.4) and (3.6), a straightforward calculation gives

$$d\tilde{\phi} = (\tilde{f}_{n}\tilde{g}(\tilde{X}_{n},\tilde{X}_{n})\tilde{\omega}^{n} + \tilde{f}_{2n}\tilde{g}(\tilde{X}_{2n},\tilde{X}_{2n})\tilde{\omega}^{2n}) \wedge \tilde{\phi}. \tag{3.14}$$

Hence the $2(n-1)$ form $\tilde{\phi}$ is *exterior recurrent* and has the form

$$\tilde{\alpha} = \tilde{f}_{n}\tilde{g}(\tilde{X}_{n},\tilde{X}_{n})\tilde{\omega}^{n} + \tilde{f}_{2n}\tilde{g}(\tilde{X}_{2n},\tilde{X}_{2n})\tilde{\omega}^{2n} \tag{3.15}$$

as a *recurrence* form (Datta [15]).

In the following we will call the vector field

$$\tilde{V} = \tilde{f}_{n}\tilde{g}(\tilde{X}_{n},\tilde{X}_{n})h_{2n} + \tilde{f}_{2n}\tilde{g}(\tilde{X}_{2n},\tilde{X}_{2n})h_{n} \tag{3.16}$$

the *recurrence vector field* on $\tilde{M}$ $(\tilde{\alpha}(\tilde{V}) = \tilde{g}(\tilde{V},\tilde{V}))$ and $\tilde{X}_{n},\tilde{X}_{2n}$ the *distinguished vectors* (abr. d.v.) of the distribution $P^{\perp}$.

By means of (2.6) one has

$$U\tilde{V} = \tilde{f}_{2n}\tilde{g}(\tilde{X}_{2n},\tilde{X}_{2n})h_{n} - \tilde{f}_{n}\tilde{g}(\tilde{X}_{n},\tilde{X}_{n})h_{2n} \tag{3.17}$$

and according to (2.8) this implies

$$\tilde{\alpha}(U\tilde{V}) = 0. \tag{3.18}$$

Since $U\tilde{V} \in P$, we have from (3.13), (3.14), and (3.18)

$$\left.\begin{aligned} i_{U\tilde{V}}\tilde{\phi} &= 0, \\ i_{U\tilde{V}}d\tilde{\phi} &= 0 \end{aligned}\right\} \tag{3.19}$$

and the above equations proved that $U\tilde{V}$ is a *characteristic vector field* of $\tilde{\phi}$. Moreover, if $\tilde{X} \in P$ is any vector field of P, one gets instantly $L_{\tilde{X}}\tilde{\phi} = \tilde{\alpha}(\tilde{X})\tilde{\phi}$, i.e. $\tilde{X}$ is an *infinitesimal conformal transformation* of $\tilde{\phi}$. Next the *Ricci 2-form* corresponding to P is $\tilde{\Theta}^{n}_{n}$ $(\equiv -\tilde{\Theta}^{2n}_{2n})$, and it can be found by means of (2.14), (3.3) and (3.12):

$$\frac{d\tilde{\ell}}{\tilde{\ell}} \wedge \tilde{\gamma} = \tilde{\Theta}^{n}_{n} + \tilde{\phi} + \tilde{g}(\tilde{X}_{n},\tilde{X}_{2n})\tilde{\pi}_{2n} \wedge \tilde{\pi}_{n}. \tag{3.20}$$

Hence equations (3.12) and (3.10) show that the necessary and sufficient condition for $\tilde{\Theta}^n_n$ to be closed is that the vector fields $\tilde{X}_n$ and $\tilde{X}_{2n}$ are orthogonal. Using now (3.11) and (3.9), one gets

$$\tilde{\Theta}^n_n(\tilde{X}_n,\tilde{X}_{2n}) = \tilde{g}(\tilde{X}_n,\tilde{X}_{2n})<\tilde{X}_n \wedge \tilde{X}_{2n}, \tilde{\pi}_n \wedge \tilde{\pi}_{2n}> . \tag{3.21}$$

Therefore, if $\tilde{X}_n$ and $\tilde{X}_{2n}$ are orthogonal, then $\tilde{\Theta}^n_n(\tilde{X}_n,\tilde{X}_{2n})$ vanishes.

Denote now by $M^{\perp}$ the maximal connected integral manifold of $P^{\perp}$ and let $\textcircled{H}$ be the *mean curvature* (2n-3)-*form* of $M^{\perp}$. Then $\textcircled{H}$ is defined by

$$\begin{aligned}\textcircled{H} = &\sum_i (-1)^{i-1}\omega^1 \wedge\ldots\wedge \hat{\omega}^i \wedge\ldots\wedge \omega^{n-1}\wedge\omega^{1^*} \wedge\ldots\wedge \omega^{n^*-1} \otimes h_{i^*} \\ &+ \sum_i (-1)^{i^*-1}\omega^1 \wedge\ldots\wedge \omega^{n-1}\wedge\hat{\omega}^{1^*} \wedge\ldots\wedge \hat{\omega}^{i^*} \wedge\ldots\wedge \omega^{n^*-1} \otimes h_i\end{aligned} \tag{3.22}$$

(the roofs indicate the missing terms and we denote the induced elements on $M^{\perp}$ by supressing $\sim$). Since ϕ is the volume element of $M^{\perp}$, one has (see Chen [9])

$$d^{\nabla}\textcircled{H} = 2(n-1)\phi \otimes H \tag{3.23}$$

where H is the *mean curvature vector field* of $M^{\perp}$, $\phi = \tilde{\pi}|_{M^{\perp}}$, and d^{∇} is the exterior covariant differentiation with respect to $\nabla = \tilde{\nabla}|_{M^{\perp}}$ (Poor [18]). Using (2.10), (2.12) and taking into account (2.14), (3.3), (3.6), and (3.16), one finds after some calculations

$$H = \frac{1}{2(n-1)} V; \quad V = \tilde{V}|_{M^{\perp}} . \tag{3.24}$$

Hence the mean curvature vector is, up to the factor $\frac{1}{2(n-1)}$, equal to the induced value of the recurrence vector field $\tilde{V}$ in $\tilde{M}$. Using the definition given by Rosca [16], [17], we obtain the following results:

1^o. The necessary and sufficient condition for $M^{\perp}$ to be *quasi-minimal* i.e., H be a null vector field, is that one of the d.v. of the distribution P be a null vector.

2^o. The necessary and sufficient condition for $M^{\perp}$ to be minimal is that both d.v. of P be null vectors.

We shall now make the following consideration. According to (2.21), (3.9) and (3.13) the volume element of the hypersurface M_η defined by $\eta = 0$ may be written as:

$$\sigma = \phi \wedge \psi \tag{3.25}$$

In (3.25) ϕ and ψ are the restrictions of $\tilde{\phi}$ and $\tilde{\psi}$ on M_η.

It follows from (3.10) that if one has $g(X_n,X_n) = g(X_{2n},X_{2n}) = 0$, one may write $\Delta\phi = 0$ where $\Delta = d\circ\delta + \delta\circ d$ is the harmonic operator. Therefore we are in the situation of Tashibana's theorem (Tashibana [13]) and M_η is covered by two families of minimal submanifolds, $M^{\perp}$ and M, tangent to $P^{\perp}$ and P respectively.

Equations (2.6) shows that $UP^{\perp} = P^{\perp}$ and $UP = P$. Hence we may say that if both d.v. X_n and X_{2n} are null vectors, then M_η is foliated by two families of invariant submanifolds tangent to $P^{\perp}$ and P, and therefore the whole manifold $\tilde{M}$ is foliate. Moreover, if we consider the immersion of M in M_η, then the 1-forms θ^α_n, θ^α_{2n} given by (3.3) define the normal vector quadratic form II (it is known that II is independent of the normal connection). But by means of (3.6) we can see that II vanishes, and therefore M is totally geodesic in M_η.

We shall give now the following

DEFINITION. Let M be an invariant submanifold of a manifold $\tilde{M}$ endowed with a para f-structure and II be the normal vector quadratic form of M. Then any tangent vector field X of M such that $II(X,fX) = 0$ is called an f-*geodesic* direction on M.

Let us consider now the immersion $x: M^\perp \to M$. Denote by $\ell_n = \langle dp, \nabla h_n \rangle$ and $\ell_{2n} = \langle dp, \nabla h_{2n} \rangle$ the second quadratic forms associated with x.

By means of (2.9), (2.10), (3.3), and (3.6) one finds after some calculation

$$\left.\begin{aligned} \ell_n &= \frac{1}{f_n}\pi_n \otimes \pi_n = f_n\Big(\sum_\alpha x_n^\alpha \omega^\alpha\Big)^2 , \\ \ell_{2n} &= \frac{1}{f_{2n}}\pi_{2n} \otimes \pi_{2n} = f_{2n}\Big(\sum_\alpha x_{2n}^\alpha \omega^\alpha\Big)^2 . \end{aligned}\right\} \tag{3.26}$$

Therefore the normal vector quadratic form $II \in (T^*\otimes T^*) \otimes (T^\perp M^\perp)$ is given by

$$II = \frac{1}{f_n}(\pi_n \otimes \pi_n) \otimes h_n + \frac{1}{f_{2n}}(\pi_{2n}\otimes\pi_{2n}) \otimes h_{2n} . \tag{3.27}$$

Referring now to (2.4), one gets by means of (2.26) and (2.27)

$$II(X_n, UX_n) = 0, \qquad II(X_{2n}, UX_{2n}) = 0 .$$

Therefore the d.v. fields on $M^\perp$ are both *U-geodesic.*

THEOREM 2. Let $\tilde{M}(U,\tilde{\Omega},\tilde{\eta},\tilde{\xi},\tilde{g})$ be an a.c.p. f-manifold admitting a strict contact covariant decomposable hyperbolic plane P and $P^\perp$ be the orthogonal component of P in the horizontal distribution D_η. Further let $\tilde{V} \in P$ and $\tilde{X}_n, \tilde{X}_{2n} \in P^\perp$ be the recurrence vector field and the distinguished vector fields associated with the pairing $(P, P^\perp)$.

Then the following properties hold:

(i) The distribution $P^\perp$ is always involutive and the mean curvature vector field of the maximal integral manifold $M^\perp$ of $P^\perp$ is (up to a constant factor) equal to the induced vector field of $\tilde{V}$.

(ii) The simple unit form $\tilde{\phi}$ of $P^\perp$ is exterior recurrent and $U\tilde{V}$ is a characteristic vector field of $\tilde{\phi}$.

(iii) The necessary and sufficient condition for $M^\perp$ to be quasi-minimal is that one of the d.v. fields of $M^\perp$ be a null vector and the necessary and sufficient condition for $M^\perp$ to be minimal is that both d.v. fields of $M^\perp$ be null vectors.

(iv) If $M^\perp$ is minimal, then the distribution P is also involutive and the integral surfaces of P are totally geodesic in M_η which in this case is foliate.

(v) Both d.v. fields on $\tilde{M}$ are U-geodesic directions on $M^\perp$.

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