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RANDOM WALK OVER A HYPERSPHERE

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ABSTRACT. In a recent paper the author had shown that a special case of S. M. Joshi transform (so named after the author's reverent father) of distributions

$$(S_b^a f)(X) = \left\langle f(y),\ {}_1F_1(a_0;b_0;ixy)\ \ {}_1F_1(a;b;-2ixy)\right\rangle$$

is a characteristic function of a spherical distribution. Using the methods developed in that paper; the problem of distribution of the distance CD, where C and D are points uniformly distributed in a hypersphere, has been discussed in the present paper. The form of characteristic function has also been obtained by the method of projected distributions. A generalization of Hammersley's result has also been developed. The main purpose of the paper is to show that although the use of characteristic functions, using the method of Bochner, is available in problems of random walk yet distributional S. M. Joshi transform can be used as a natural tool has been proved for the first time in the paper.

KEY WORDS AND PHRASES. Random walk, Distributional transform, Characteristic function, Hypersphere, S. M. Joshi transform, spherical, distribution.
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1. INTRODUCTION.

Bochner [1] observed that the Fourier transform of a radial function can be written as a Hankel transform. Though the use of characteristic functions in problems of addition of independent random vectors (using the results of Bochner) is available in the literature, it is not made clear that the Hankel transform is a natural tool. Further, it is not made clear that there is a theory for spherical distributions (i.e. when all directions are equally probable and the distribution of magnitude is independent of direction) in n dimensions which runs parallel to the theory of the addition of random variables in one dimension.

Moreover, in actual problems it is frequently necessary to evaluate the Hankel transform involved (in the formula for the probability function), but the ordinary methods of numerical integration are difficult to apply since the Bessel functions make the integrand oscillate irregularly between positive and negative values. When we try to evaluate the probability function) by expansion methods, the terms contain confluent hypergeometric functions.

A new account would, therefore, appear useful (involving confluent hypergeometric function kernel transform), especially as the use of Distributional Joshi transform

[2,3] and of its special case-the Distributional Fourier transform - in certain problems of mathematical physics is now known (See [4]).

In the present note we have tried to discuss the problems of the distribution of the distance CD, when C and D are points uniformly distributed in a (hyper) sphere of radius c in n dimensions. The treatment is based on general methods developed in one of my earlier papers [3].

Also a method other that that in [3] has been given here to obtain the form of the characteristic function. Hammersley's result [5] has been shown to be a natural corollary of our result.

2. PROBLEM OF DISTRIBUTION ON THE HYPERSPHERE.

Let C and D have the vector representation ξ_1 and ξ_2 in n-dimensional space. Then we require the distribution of $CD = \xi_1 - \xi_2$, where ξ_1 and ξ_2 are random vectors with identical uniform spherical distribution. But, because of the spherical symmetry ($-\xi_2$ has the same distribution as ξ_2), the problem is equivalent to finding the distribution of $\xi_1 + \xi_2$. This we shall attempt presently. In [3] I have shown that the characteristic function of a spherical distribution is an S.M. Joshi transform (so called after my reverent father, who initiated me into research outlook [2]

$$\tilde{S}(\zeta) = \int_0^\infty P(r)\ {}_1F_1(a_0;b_0;ir\zeta)\ {}_1F_1(\tfrac{1}{2}(n-1);\ n-1;\ -2ir\zeta)dr \tag{2.1}$$

with

$$P(r) = 2^{-n+2}\{\Gamma(\tfrac{1}{2}n)\}^{-2} \int_0^\infty \tilde{S}(\zeta)(r\zeta)^{n-1}\ {}_1F_1(a_0;b_0;\ -ir\zeta)\ \times$$
$$\times\ {}_1F_1(\tfrac{1}{2}(n-1);\ n-1;\ 2ir\zeta)\ d\zeta, \tag{2.2}$$

where

$$P(r) = P(r < |\xi| < r + dr); \tag{2.3}$$

$S(\ )$ being characteristic function of n-dimensional distribution of ξ and not of 1-dimensional $r = |\xi|$. Here $\zeta = |X|$, where X is variable and a_0, b_0 are the values of a and b at $\alpha = \mu = 0$ while $a = \alpha + \lambda + 1$, $b = a + \mu$. Now for a distribution uniform in a sphere of radius c we have

$$P(r) = nr^{n-1}c^{-n}, \qquad c \geq r \geq 0$$
$$= 0, \qquad r > c. \tag{2.4}$$

Hence (2.1) gives

$$S(\zeta) = {}_1F_1(a_0;b_0;ic\zeta)\ F\ (\tfrac{1}{2}(n+1);\ n+1;\ -2ic\zeta) \tag{2.5}$$

But the C.F. of the distribution of $\xi_1 + \xi_2$ is the product of characteristic functions of ξ_1 and ξ_2, as is well known. Multiplying these characteristic functions and inverting we get the expression for the probability function for $\xi_1 + \xi_2$ as

$$p^{(2)}(r) = \frac{nr^{n-1}c^{-n}}{B(\frac{1}{2}(n+1),\frac{1}{2})}\{\tfrac{1}{2}(n+1)\}^{-1}\left(1 - \frac{r^2}{4c^2}\right)^{\frac{1}{2}(n+1)} \times$$
$$\times\ {}_2F_1\ \left[(\tfrac{1}{2}(n+1);\ \tfrac{1}{2}(n+3)/2;\ \left(1 - \frac{r^2}{4c^2}\right)\right], \tag{2.6}$$

where (2) in $p^{(2)}$ indicates that the sum of two random vectors is being considered.

3. PROOF BY THE METHOD OF PROJECTED DISTRIBUTIONS.

We could derive (2.6) in yet another way-by the method of projected distributions discussed in [3].

If we project from a spherical distribution of random vectors with origin as centre onto a space of lower dimension (vectors being projected orthogonally) we get another spherical distribution. We have the result (see [3]), in terms of SMJ operators,

$$\begin{aligned} P_n(r) &= S^{J^2}_{\frac{1}{2}n,k}\,[P_{n+2k}(r)] \\ &= \frac{2}{B(\frac{1}{2}n,k)}\, r^{n-1} \int_r^{\infty} P_{n+2k}(r)(\xi^2 - r^2)^{k-1}\, \xi^{-n-2k+2} d\xi \end{aligned} \tag{3.1}$$

where $P_{n+2k}(r)$ and $P_n(r)$ are probability functions of spherical distributions in $n+2k$ and n dimensions respectively.

In the distribution in $n+2k$ dimensions be uniform over the sphere of radius c, then (3.1) gives

$$\begin{aligned} P_n(r) &= \frac{2\,\Gamma(\frac{1}{2}n+k)}{\Gamma(\frac{1}{2}n)\,\Gamma(k)}\, c^{-n-2k+2}(c^2-r^2)^{k-1} r^{n-1}, \qquad c \geq r \\ &= 0, \qquad c < r. \end{aligned} \tag{3.2}$$

For $k=1$, (3.2) reduces to (2.4) and thus it is evident that a uniform distribution through the volume of an n-dimensional sphere can be obtained by a projection from a uniform distribution over the surface of an $(n+2)$-dimensional sphere (each having the same radius c). Further for the sum of two vectors we have, for the $(n+2)$-dimensional sphere,

$$\begin{aligned} P^{(2)}_{n+2}(r) &= \frac{2^{1-n}}{B(\frac{1}{2}(n+1),\frac{1}{2})}\, c^{-2n} r^n (4c^2-r^2)^{n-\frac{1}{2}}, \qquad r \leq 2c \\ &= 0, \qquad r > 2c. \end{aligned} \tag{3.3}$$

Using (3.3) and (3.1), with $k=1$ we get $P^{(2)}(r)$ as in (2.6).

In the general case of projection from the space of $(n+2k)$ dimensions we have

$$P^{(2)}(r) = C \int_r^{2a} (4c^2 - \xi^2)^{k+\frac{1}{2}(n-3)} (\xi^2 - r^2)^{k-1}\, \xi^{-2k+2} d\xi. \tag{3.4}$$

For $k=1$ this yields, again, (2.6).

4. GENERALIZATION OF HAMMERSLEY'S RESULT.

Before giving a generalization of Hammersley's result [5], we append a proof of the result (2.1) different from the one presented in [3]. The method of proof is relevant to the discussions in earlier sections in that it deals with the distribution of the sum of two independent random vectors.

The only defect of this method is that it does not bring out the importance of characteristic functions as such.

THEOREM 4.1.

Let $\xi = \xi_1 + \xi_2$ with probability functions $P_1(\xi_1)$ and $P_2(\xi_2)$ then the distribution function $D(r)$ of $|\xi|$ is given by

$$\begin{aligned} D(r) &= \frac{2^{1-n}\, r^n}{(\frac{1}{2}n)\,(\frac{1}{2}n+1)} \int_0^{\infty} \tilde{S}(\zeta)\, \zeta^{n-1}\, {}_1F_1(a_0;b_0;-ir\zeta)\, \times \\ &\quad \times {}_1F_1(\tfrac{1}{2}(n+1);\, n+1;\, 2ir\zeta)\, d\zeta, \end{aligned} \tag{4.1}$$

where $\tilde{S}(\zeta) = \tilde{S}_1(\zeta) S_2(\zeta)$.

PROOF.

We require the probability that

$|\xi| = |\xi_1 + \xi_2| < r$, or

$P = (r_1^2 + r_2^2 - 2r_1 r_2 \cos \emptyset)^{\frac{1}{2}} < r$, where

$|\xi| = P$, $|\xi_1| = r_1$, $|\xi_2| = r_2$ and $\emptyset$ is the angle between ξ_1 and ξ_2.

Then it is known that the desired probability is given by

$$\int p_1(\xi_1)\, p_2(\xi_2)\, \Theta_A(\xi_1, \xi_2) d\xi_1 d\xi_2, \tag{4.2}$$

where $\Theta_A(x)$ is any (mathematical) characteristic function of the set A viz.([4] p.1)

$$\begin{aligned} \Theta_A(x) &= 1, \qquad x \in A \\ &= 0, \qquad x \bar{\in} A, \end{aligned} \tag{4.3}$$

where A is the set in which the desired condition is satisfied.

Now if we let the characteristic function Θ_A be the integral

$$\begin{aligned} I = \frac{r^n}{2^{n+1}} \Gamma(\tfrac{1}{2}n)\, \Gamma(\tfrac{1}{2}n+1) \int_0^\infty \zeta^{n-1} e^{(n-P)i\zeta}\, X \\ X\ {}_1F_1(\tfrac{1}{2}(n+1);\, n+1;\, -2ir\zeta)\, {}_1F_1(\tfrac{1}{2}(n-1);\, n-1;\, 2iP\zeta) d\zeta, \end{aligned} \tag{4.4}$$

(which is equal to zero when $P > r$ and equal to one when $P < r$) and evaluate the inner integral in (4.2) with the present value of $\Theta_A(\xi_1, \xi_2)$, we obtain the value of the inner integral for a fixed value of ξ_1 as

$$\begin{aligned} r^n\, \Gamma(\tfrac{1}{2}n)\, \Gamma(\tfrac{1}{2}(n+1)) \int_0^\infty \tilde{S}_2(\zeta)\, \zeta^{n-1} e^{i(r_1 + r_2)\zeta}\, X \\ X\ {}_1F_1(\tfrac{1}{2}(n-1);\, n-1;\, -2ir_1\zeta)\, {}_1F_1(\tfrac{1}{2}(n+1);\, n+1;\, -2ir_2\zeta) d\zeta, \end{aligned} \tag{4.5}$$

where $\tilde{S}_2(\zeta) = \int_0^\infty {}_1F_1(a_0; b_0; (i\xi_2; r_2) p_2(r_2) dr_2$. Here, as is evident from R.H.S. of (4.5), $\tilde{S}_2(\zeta)$ is the (statistical) characteristic function of the probability function $P_2(r_2)$ corresponding to the distribution of ξ_2 (n-dimensional), while P(r) in (2.1) refers to the probability function of the distribution of $|\xi| = r$ which is one dimensional. The two are related by the well-known result

$P(r) = p(r)\, r^{n-1}\Omega_n = 2\Pi^{\frac{1}{2}n} r^{n-1} p(r)/\Gamma(\frac{1}{2}n)$ (Ω_n is the surface of the unit sphere in n-dimensions). (4.6)

In this evaluation we have used the fact

$$P_2(\xi_2) d\xi_2 = p_2(r_2) r_2^{n-1} dr_2 d\Omega_n = p_2 r_2^{n-1} \sin^{n-2}\emptyset d\emptyset d\Omega_{n-1}, \tag{4.7}$$

where we have used $\emptyset$ as a co-latitude in defining the position of ξ_2. (See [6] p. 76).

Evaluating the final integral in (4.2), using

$$p_1(\xi_1) d\xi_1 = p_1(r) r_1^{n-1} d\Omega_n, \tag{4.8}$$

we obtain the distribution function (or so called probability integral, $D(r) = \int_0^r P(r)dr$) as

$$\begin{aligned} D(r) = \frac{r^n}{2^{n-1}} \{\Gamma(\tfrac{1}{2}n)\, \Gamma(\tfrac{1}{2}(n+1))\}^{-1} \int p_1(r_1) \tilde{S}_2(\zeta) r_1^{n-1} \zeta^{n-1} \\ e^{i(r_1 + r_2)\zeta}\, {}_1F_1(\tfrac{1}{2}(n-1);\, n-1;\, -2ir_1\zeta)\, {}_1F_1(\tfrac{1}{2}(n+1);\, n+1;\, -2ir\zeta) d\zeta\, dr_1 d\Omega_n \end{aligned} \tag{4.9}$$

Integration over Ω_n gives rise to the factor Ω_n and that with respect to r_1 yields

$$\frac{\{\Gamma(\tfrac{1}{2}n)\}^{-1}}{2^{\frac{1}{2}n-1}} \int_0^\infty p_1(r_1) r_1^{n-1} \zeta^{\frac{1}{2}n-1} \; {}_1F_1(\tfrac{1}{2}(n-1);n-1;-2ir_1\zeta)dr_1$$

$$= (2\Pi)^{-\frac{1}{2}n} \zeta^{(\frac{1}{2}n-1} S_1(\zeta). \tag{4.10}$$

Thus, finally, we have $D(r)$ as in (4.1) which is (2.2.12) of [5]; with $\tilde{S}(\zeta)$ replaced by $\tilde{S}_1(\zeta)\tilde{S}_2(\zeta)$.

This incidentally proves

$$S(X) = S_1(X)\ S_2(X) \tag{4.11}$$

i.e. the characteristic function $S(X)$ of the sum of two independent random vectors ξ_1 and ξ_2 is equal to the product of the characteristic functions $S_1(X)$ of ξ_1 and $S_2(X)$ of ξ_2.

Inverting and using $D'(r) = P(r)$ we get (2.1).

5. HAMMERSLEY'S RESULT.

By showing $|\xi_1 + \xi_2|$ is asymptotically distributed in a normal distribution with mean $2^{\frac{1}{2}}c$ and variance $c^2/2n$ as n tends to infinity, Hammersley [5] has proved that for large values of n the distance between two points in a sphere is nearly always equal to $2^{\frac{1}{2}}c$-the diagonal of the rectangle determined by the orthogonal radii.

Now as shown in [3] the distribution (3.2) has the characteristic function

$${}_1F_1(a_0;b_0;ic\zeta)\ {}_1F_1(\tfrac{1}{2}n+k-\tfrac{1}{2};\ n-1 + 2k;\ -2ic\zeta). \tag{5.1}$$

But by [7] (4.11) is asymptotially equivalent to

$$e^{-\frac{c^2\zeta^1}{2(n+2k)}} \tag{5.2}$$

But again, we know [3] that a normal distribution has the probability

$$P(r) = Ar^{n-1}\ e^{-\frac{nr^2}{2\mu_2}} \tag{5.3}$$

and has the characteristic function

$$\tilde{S}\ (\zeta) = e^{\frac{1}{2}\mu_2\zeta^{2/n}} \tag{5.4}$$

Thus the distribution, as is clear from (5.1), is asymptotically normal with $\mu_2 = \frac{nc^2}{n+2k}$.

Also, taking k=1 (3.2) we get (2.4) and thus (2.4) is asymptotically normal with variance $\frac{nc^2}{(n+2)}$. The distribution of $\xi_1 + \xi_2$ is, therefore, asymptotically, normal with variance double that of the above.

Again taking $k = 0$, we see that the distribution uniform over the surface of the sphere with radius d is asymtotic to normal distribution with variance d^2. Comparing this variance with the variance just obtained for $\xi_1 + \xi_2$, we notice that the distributation of $\xi_1 + \xi_2$ is asymptotic to the distribution over the surface of the sphere of radius $(2n)^{\frac{1}{2}}c/(n+2)^{\frac{1}{2}}$, which again is asymptotically equivalent to $2^{\frac{1}{2}}c(1 + \frac{1}{n})$. Thus we have Hamersley's result.

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